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Research Article

On the permutation cycle structures for almost Moore digraphs of degrees 4 and 5

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Abstract

Almost Moore digraphs of degree d and diameter k, for short (d, k)-digraphs, appear in the degree/diameter problem as a class of extremal directed graphs. Their order is one less than the unattainable Moore bound $M(d, k) = 1 + d + \cdots + d^k$. So far, their existence has only been shown for $k = 2$. Their nonexistence has been proven for $k = 3, 4$ and for $d = 2, 3$, when $k \geq 3$. In this paper, we study the possible cycle structures of the permutation of repeats for the $(4, k)$ and $(5, k)$-digraphs.

Keywords: digraphs; Moore bound; degree/diameter problem.

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1. Introduction

In the design of interconnection networks, it is useful to find digraphs with the largest number of vertices N for given values of the maximum [out-]degree d and diameter k. This is the so-called degree/diameter problem. For a broad and deep view of the problem see the survey given by Miller and Širáň in [19]. Plesník and Znám in [20] and later Bridges and Toueg in [5] proved that the number of vertices is less than the Moore bound, $M(d, k) = 1 + d + \cdots + d^k$ unless $d = 1$ or $k = 1$. Thus, it is an interesting problem to find for which values of $d > 1$ and $k > 1$, there exist digraphs of order

$$N = M(d, k) - 1.$$

Digraphs with maximum out-degree at most $d > 1$, diameter $k > 1$ and order $N = M(d, k) - 1$ are called *almost Moore* (d, k)*-digraphs* or (d, k)-digraphs with *defect* 1 (for short (d, k)*-digraphs*). These digraphs turn out to be diregular (both in- and out-regular) of degree d [17].

In general, the existence of (d, k)-digraphs remains a difficult open problem. Fiol et al. showed in [12] that $(d, 2)$-digraphs do exist for any degree $d > 1$ and all $(d, 2)$-digraphs were classified by Gimbert in [14]. For the remaining cases, it seems they do not exist. Nevertheless, their nonexistence has only been proved in a few cases. Miller and Fris in [16] proved that there are no $(2, k)$-digraphs with $k \geq 3$ and Baskoro et al. showed in [4] the nonexistence of $(3, k)$-digraphs for $k \geq 3$. Concerning the diameter, Conde et al. showed in [10, 11] the nonexistence of $(d, 3)$ and $(d, 4)$-digraphs.

Moreover, there exist two conjectures such that, assuming either of them is true, the nonexistence of (d, k)-digraphs for any $d \geq 4$ and $k \geq 5$ is proven. Cholily proves the nonexistence [6] based on the conjecture of the structure of the out-neighbours of a k-type vertex (introduced in Section 2) [1, 8]. Gimbert in [13] gave the other conjecture, which is related to the factorization in $\mathbb{Q}[x]$ of the polynomials $F_{n,k}(x) = \Phi_n(1 + x + \cdots + x^k)$, $\Phi_n(x)$ being the nth cyclotomic polynomial. In [9] the nonexistence is also proven modulo this conjecture.

Another interesting problem is, for each value $d > 1$ and $k > 1$, finding digraphs with minimum out-degree d and order $M(d, k) + 1$, that is, with excess 1. These digraphs, also called $(d, k; +1)$-digraphs, are k-geodetic, that is, for every pair of vertices u and v there is at most one walk of length $\leq k$ from u to v. Sillasen showed in [22] that the (d, k)-digraphs with excess 1 have diameter $k + 1$ and in [18] that they are diregular. Comparing their existence with respect to the case of defect 1, so far we know $(d, k; +1)$-digraphs exist neither for $d = 2$ [18] nor for $k = 2$ [18, 24].

In this paper, we study the cycle structure of the permutation of repeats (introduced in Section 2) given by Sillasen in [23] for almost Moore digraphs with degrees 4 and 5. In the case of having self-repeats, López et al. showed in [15] they do not exist. Then, we focus on the case without self-repeats, reducing the number of possible structures such digraphs can have.

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2. Almost Moore digraphs

An almost Moore digraph G of degree d and diameter k has the property that each vertex $v \in V(G)$ has a unique vertex $u \in V(G)$, called the *repeat* of v and denoted by $r(v)$, such that there are exactly two walks from v to $r(v)$ of length $\leq k$, at least one of which must have length exactly k. If $r(v) = v$, the vertex v is called a *self-repeat* of G. It is also said that v is a k-*type vertex* if $d(v, r(v)) = k$. Otherwise, v is a 0-*type vertex*.

The map r, which assigns to each $v \in V(G)$ its repeat $r(v)$, is a permutation of $V(G)$ and also an automorphism of G [3]. The smallest integer $t \geq 1$ such that $r^t(v) = v$ is called the order of v.

Given a (d, k)-digraph G, its adjacency matrix $\boldsymbol{A}$ satisfies the equation

$$\boldsymbol{I} + \boldsymbol{A} + \cdots + \boldsymbol{A}^k = \boldsymbol{J} + \boldsymbol{P} \tag{1}$$

where $\boldsymbol{J}$ denotes the all-one matrix and $\boldsymbol{P} = (p_{ij})$ is the (0,1)-matrix associated with the permutation r of the set of vertices $V(G) = \{v_1, \ldots, v_N\}$, which is defined by $p_{ij} = 1$ if and only if $r(v_i) = v_j$. The permutation r has a cycle structure that corresponds to its unique decomposition in disjoint cycles. The number of permutation cycles of G of each length $n \leq N$, will be denoted by m_n and the vector

$$(m_1, m_2, \ldots, m_N), \quad \sum_{i=1}^{N} i m_i = N,$$

will be referred to as the *permutation cycle structure* of G.

Since G has no cycles of length less than k, its adjacency matrix $\boldsymbol{A}$ satisfies

$$\operatorname{Tr} \boldsymbol{A}^i = 0 \text{ for } i = 1, 2, \ldots, k-1.$$

Furthermore, in [3] it is proved that $m_1 \in \{0, k\}$ if $k \geq 3$. Hence, for digraphs without self-repeats, $\operatorname{Tr} \boldsymbol{A}^k = \operatorname{Tr} \boldsymbol{P} = 0$.

On the factors of the characteristic polynomial

Let G be a (d, k)-digraph with permutation cycle structure $(m_1, m_2, \ldots, m_N)$. The characteristic polynomial of the matrix $\boldsymbol{J} + \boldsymbol{P}$, $\phi(\boldsymbol{J} + \boldsymbol{P}, x) = \det(x\boldsymbol{I} - (\boldsymbol{J} + \boldsymbol{P}))$, $\boldsymbol{P}$ being the matrix associated to the permutation of repeats, has this expression

$$\phi(\boldsymbol{J} + \boldsymbol{P}, x) = (x - (N+1))(x-1)^{m_1 - 1} \prod_{n=2}^{N} (x^n - 1)^{m_n}.$$

Its factorization in $\mathbb{Q}[x]$ in terms of cyclotomic polynomials $\Phi_n(x)$, since $x^n - 1 = \prod_{d|n} \Phi_d(x)$, is:

$$\phi(\boldsymbol{J} + \boldsymbol{P}, x) = (x - (N+1))(x-1)^{m(1)-1} \prod_{n=2}^{N} \Phi_n(x)^{m(n)}, \tag{2}$$

where $m(n) = \sum_{n|i} m_i$ represents the total number of permutation cycles of order a multiple of n.

From Equation (1), the characteristic polynomial of G, $\phi(G, x) = \det(x\boldsymbol{I} - \boldsymbol{A})$, can be derived from $\phi(\boldsymbol{J} + \boldsymbol{P}, x)$ by substituting x by $1 + x + \ldots + x^k$ and taking irreducible factors in $\mathbb{Q}[x]$ with appropriate multiplicities. Then, the problem of the factorization of the characteristic polynomial of G, $\phi(G, x) = \det(x\boldsymbol{I} - \boldsymbol{A})$ in $\mathbb{Q}[x]$ is related to the study of factorization in $\mathbb{Q}[x]$ of the polynomials:

$$F_{n,k}(x) = \Phi_n(1 + x + \cdots + x^k), \ n \geq 2.$$

In [13] Gimbert gave the following result.

Lemma 2.1. *If $F_{n,k}(x)$ is irreducible in $\mathbb{Q}[x]$, then $F_{n,k}(x)$ is a factor of $\phi(G, x)$ and its multiplicity is $m(n)/k$.*

The “cyclotomic conjecture” proposed by Gimbert in [13] gives the factorization in $\mathbb{Q}[x]$ of the polynomials $F_{n,k}(x)$. In particular, it is known that:

Lemma 2.2. *The polynomial $F_{2,k}(x) = \Phi_2(1 + x + \cdots + x^k) = 2 + x + \cdots + x^k$ is irreducible in $\mathbb{Q}[x]$. Hence k divides $m(2)$.*

If the “cyclotomic conjecture” were true the nonexistence of (d, k)-digraphs for $k \geq 3$ would be proven [9].

3. Permutation cycle structures of (d, k)-digraphs with $d = 4$ and $d = 5$

We will consider $(4, k)$ and $(5, k)$-digraphs with $k \geq 3$. For these cases, Sillasen [23] gives the possible cycle structures of the permutation of repeats. The structures containing self-repeats have also been deduced in [2, 8] by Baskoro et al. studying properties of vertex orders of such digraphs. However, according to [15], (d, k)-digraphs, $d = 4$ or $d = 5$, with permutation cycle structures with $m_1 = k$, that is, with self-repeats, do not exist. The following proposition summarises the possible permutation cycle structures for degrees 4 and 5.

Proposition 3.1. *The permutation cycle structures for a (d, k)-digraph G with $d = 4$ or $d = 5$ must be one of these forms, where $N = d + d^2 + \cdots + d^k$.*

- *If $d = 4$:*

$$(0, \ldots, 0, m_i, 0, \ldots, 0), \quad im_i = N, \ 2 \leq i \leq N.$$

- *If $d = 5$:*

$$(0, \ldots, 0, m_i, 0, \ldots, 0), \quad im_i = N, \ 2 \leq i \leq N,$$

$$(0, \ldots, 0, m_i, 0, \ldots, m_{2i}, 0, \ldots, 0), \quad im_i + 2im_{2i} = N, \textit{ with either}$$

$k + 2$ vertices of order i and $N - k - 2$ of order $2i$, or

$M(3, k) + 1$ vertices of order i and $N - M(3, k) - 1$ of order $2i$.

Thus, from now on, we will study these permutation cycle structures with the aim of proving that some of these permutation structures are not possible for $d = 4$ or $d = 5$.

Permutation cycle structures of $(4, k)$-digraphs

We will show some of the permutation cycle structures given in Proposition 3.1 for $(4, k)$-digraphs, $k \geq 3$, are impossible.

Proposition 3.2. *Almost Moore digraphs of degree $d = 4$ and diameter k with permutation cycle structures*

$$(0, \ldots, 0, m_N), \ (0, \ldots, 0, m_{N/2}, 0 \ldots, 0) \textit{ and } (0, \ldots, 0, m_{N/3}, 0 \ldots, 0)$$

do not exist for $k \geq 3$.

Proof. Clearly, for these permutation cycle structures $m_N = 1$, $m_{N/2} = 2$ and $m_{N/3} = 3$, respectively. For the first structure, since N is even for $d = 4$, it turns out that $m(2) = \sum_{2|n} m_n = m_N = 1$. But, according to Lemma 2.2, the multiplicity of $F_{2,k}(x)$ as an irreducible factor of its characteristic polynomial is $m(2)/k$, which is a contradiction. For the second structure, taking into account that N is a multiple of 4, we derive $m(2) = m_{N/2} = 2$. As before, the multiplicity of $F_{2,k}(x)$ would be $m(2)/k$, which is not an integer since $k \geq 3$. Finally, for the third structure, if k is not a multiple of 3, then neither is N. Otherwise, $N/3$ is multiple of 4 and we deduce $m(2) = m_{N/3} = 3$. Hence, k would divide 3, which does not make sense. □

Then, the possible permutation cycle structures for degree $d = 4$ and diameter $k \geq 3$ are reduced to $(0, \ldots, 0, m_i, 0 \ldots, 0)$, with $2 \leq i \leq N/4$.

Permutation cycle structures of $(5, k)$-digraphs

This subsection showcases the nonexistence of some permutation cycle structures given in Proposition 3.1 for $(5, k)$-digraphs, $k \geq 3$.

Theorem 3.1. *Almost Moore digraphs of degree $d = 5$ and diameter k with permutation cycle structure*

$$(0, \ldots, 0, m_i, 0, \ldots, 0), \ im_i = N, \ i \textit{ even}$$

do not exist for $k \geq 3$.

Proof. Regarding this structure, since i is even, we have $m(2) = m_i$. Hence $k \mid m_i$. Taking $h = m_i/k$, we can write $5(5^k - 1)/4 = ikh$, but the powers of 2 on either side are different. Indeed, denoting the 2-adic valuation of a natural number x by $v_2(x)$, that is, $v_2(x) = r$ if $x = 2^r x'$ with $2 \nmid x'$, we have $v_2(5(5^k - 1)/4) = v_2(k)$ whereas $v_2(ikh) \geq 1 + v_2(k)$, which is a contradiction. □

Theorem 3.2. *Almost Moore digraphs of degree $d = 5$ and diameter k with permutation cycle structure*

$$(0, \ldots, 0, m_i, 0, \ldots, 0, m_{2i}, 0, \ldots, 0),\ im_i + 2im_{2i} = N,\ \text{with } k+2 \text{ vertices of order } i \text{ and } N-k-2 \text{ vertices of order } 2i$$

do not exist for $k \geq 3$.

Proof. Assume G is a $(5, k)$-digraph with such a permutation cycle structure. Theorem 1 in [7] can be generalized by allowing $im_i > 2$ and $i \geq 2$. The proof of the theorem uses the lemma from the same paper, which states that all the vertices in a walk between two vertices v, v' of length $l < k$, or $l = k$ and $v' \neq r(v)$, must have order a multiple of the least common multiple of the order of the first vertex and the last one of the walk. Therefore, if there are more than three vertices of the same minimal order, there is a path in which all the vertices have the minimum order. Cholily proves the theorem by having an order larger than 2 but it is sufficient to have more than two vertices of the minimal order without self-repeats.

With the previous generalization, there are only two possible options for the vertices of order i:

- More than one outgoing vertex of order i for some vertex.
- All vertices are adjacent to exactly one vertex of order i.

In the case that there is a vertex with more than one outgoing vertex of order i, we call v_0 the vertex that has more than two outgoing vertices, and enumerate all the vertices sequentially (for example, all the vertices of order i are $v_0, v_1, v_2, \ldots, v_{k+1}$). Then, one of the outgoing paths has to have at least length k, by following [7] Theorem 1, and knowing that there are no self-repeats in the graph by definition, which means that a path to return to the same vertex has to have at least length $k+1$. As such, the construction has to be Figure 1.

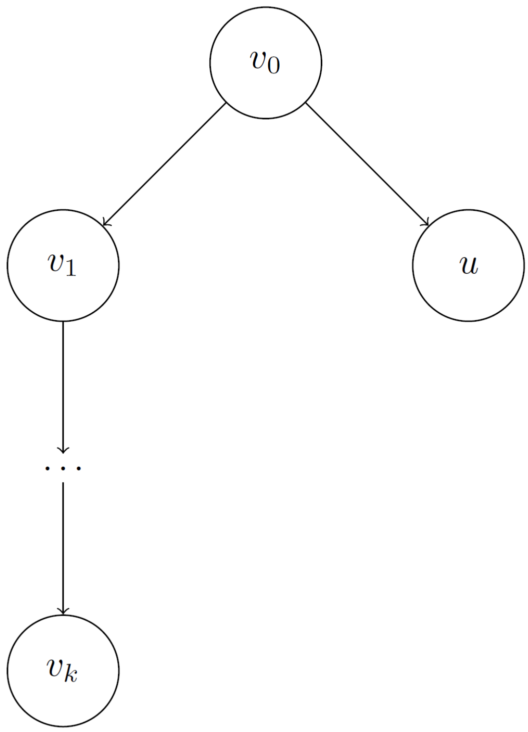

Figure 1: Construction with two outgoing vertices of order i.

With this construction, note that:

- There is one vertex left to use (v_{k+1}).
- Vertex u must either be $u = v_{k+1}$ and then $u \to v_k$, or $u = v_k$. Otherwise, there would be more than one repeat for vertex v_0, being $(v_0, v_1, ..., v_{k-1}, v_k)$ and $(v_0, u..., v_{k-1}, v_k)$, with v_{k-1} and v_k both as repeats for v_0.
- If $u = v_k$, then v_k will inevitably go to another vertex of order i following [7] Theorem 1. Either $v_k \to v_{k+1} \to v_i$, $i \in \{0, \ldots, k\}$, or $v_k \to v_i$, $i \in \{0, \ldots, k-1\}$, which will make two paths from v_0, as $v_0, v_1, ..., v_i$, or $v_0, v_k, [v_{k+1}], v_i$, making multiple repeats of v_0.

- If $u = v_{k+1}$, and $v_{k+1} \to v_k$, then $r(v_0) = v_k$. But, given the theorem of [7], this vertex must have a path to another vertex of order i, which would make two repeats for the vertex v_0. As such, this construction is not possible.

If all the vertices have only one outgoing vertex of order i, knowing the three following statements:

- If a path between vertices of order i has vertices of order $2i$, then using Lemma A from [6] either:
  - Length is k, which means it is the repeat of the vertex.
  - Length is greater than k.
- At $k + 2$ it will close to a cycle, as shown in Figure 2, since there are no more vertices left to use.
- To connect v_0 and v_{k+1}, the vertex v_{k+1} must be $r(v_0)$, and both paths have all the vertices with order $2i$.

Then for the construction to be valid, the only possibility is shown in Figure 2.

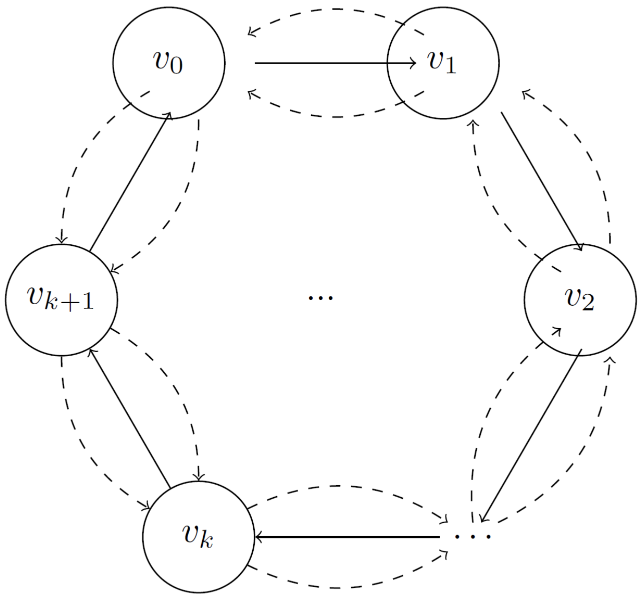

Figure 2: Construction with one outgoing vertex of order i.

For the graph to remain connected, then the vertex $r(v_i) = v_j$, where $j \equiv i - 1 \pmod{k+2}$, with two paths with vertices of order $2i$, and length exactly k. Figure 2 represents this with a dashed path. Hence

$$i = k + 2 \quad \text{and} \quad m_i = 1.$$

Therefore $M(5, k) - 1 = im_i + 2im_{2i} = (k + 2)(1 + 2m_{2i})$, that is

$$5(5^k - 1) = 4(k + 2)(1 + 2m_{2i}). \tag{3}$$

In the case k is even with $v_2(k) = \alpha \geq 1$, we can see by induction that $v_2(5^k - 1) = \alpha + 2$. Thus, when $v_2(k) \geq 2$, it turns out that $v_2(k + 2) = 1$, so the 2-adic valuation of the expressions of both sides in (3) is different. Besides, if $v_2(k) = 1$ we have $v_2(k + 2) \geq 2$, obtaining $v_2(5^k - 1) = 3 < v_2(4(k + 2))$. Hence, there is no even integer k satisfying (3).

When k is odd, $m(2) = \sum_{2|n} m_n = m_{2i} = N - (k + 2)$. Then, according to Lemma 2.2 the multiplicity of $F_{2,k}(x)$ as an irreducible factor of its characteristic polynomial is $m(2)/k$. Therefore, k must divide $N - (k + 2)$, that is, $k \mid (N - 2)$. However, we get a contradiction since $(k + 2) \mid N$. □

We shall now consider $(5, k)$-digraphs G with permutation cycle structures

$$(0, \ldots, 0, m_i, 0, \ldots, 0, m_{2i}, 0, \ldots, 0), \; im_i + 2im_{2i} = N,$$

with $M(3, k) + 1$ vertices of order i and $N - M(3, k) - 1$ vertices of order $2i$.

For these permutation patterns, the subdigraph G' induced by the vertices of order i is a $(3, k; +1)$-digraph, which means with excess one. Recall that a $(d, k; +1)$-digraph [22] has diameter $k + 1$ and for each vertex u there is exactly one vertex, the outlier $o(u)$, such that $d(u, o(u)) = k + 1$. The outlier map is an automorphism of G. Note that in our case the outlier map of G' coincides with the repeat map, so its permutation cycle structure is $(0, \ldots, 0, m_i, 0, \ldots, 0)$ with $im_i = M(3, k) + 1$.

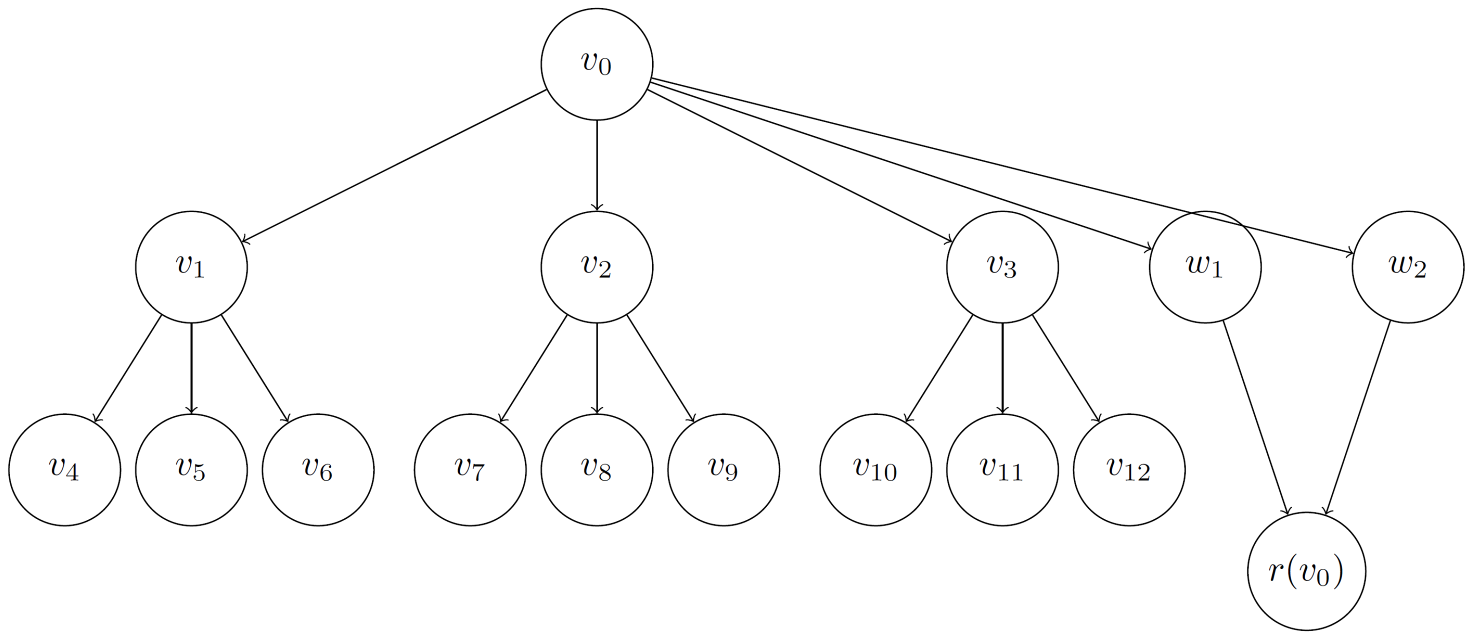

Figure 3: Case $k = 2$: subdigraph induced by vertices of order i.

For the case $k = 2$, we show in Figure 3 the induced subdigraph, representing the set of its vertices (those with order i) by $\{v_0, \ldots, v_{12}, r(v_0)\}$. We have added two vertices of order $2i$ to represent the walks of length k from v_0 to its repeat $r(v_0)$.

However, J. Tuite has proven in [24] that $(3, 2; +1)$-digraphs do not exist. Hence, neither do $(5, 2)$-digraphs with this permutation cycle structure. Moreover, in [18] the nonexistence of $(3, 3; +1)$ and $(3, 4; +1)$-digraphs is shown. Thus, as we already know, $(5, 3)$ and $(5, 4)$-digraphs with this structure do not exist.

We will see for a general k, in certain conditions, $(5, k)$-digraphs do not exist either.

Proposition 3.3. *Almost Moore digraphs of degree $d = 5$ and diameter k with permutation cycle structure*

$(0, \ldots, 0, m_i, 0, \ldots, 0, m_{2i}, 0, \ldots, 0)$, $im_i + 2im_{2i} = N$, $M(3, k) + 1$ *vertices of order i and $N - M(3, k) - 1$ vertices of order $2i$*

do not exist for $k \geq 3$, i even and $v_2(k) > 1$.

Proof. Assume G is a $(5, k)$-digraph with such a permutation cycle structure. From $im_i = 2 + 3 + \ldots + 3^k$, we have k even. Since i is even, it turns out that $m(2) = \sum_{2|n} m_n = m_i + m_{2i}$. But, according to Lemma 2.2, the multiplicity of $F_{2,k}(x)$ as an irreducible factor of its characteristic polynomial is $m(2)/k$. Therefore, k divides $m_2 + m_{2i}$, that is $m_i + m_{2i} = km'$. Then, from $N = im_i + 2im_{2i} = 2i(m_i + m_{2i}) - im_i$ we have

$$5(5^k - 1) + 6(3^k - 1) = 8ikm' - 8.$$

Considering 2-adic valuations, since $v_2(5^k - 1) = v_2(3^k - 1) = 2 + v_2(k)$, we get $v_2(5(5^k - 1) + 6(3^k - 1)) = 2 + v_2(k)$, which is greater than $v_2(8ikm' - 8) = 3$ when $v_2(k) > 1$. □

In [24] (also in Sillasen's thesis [21]), vertices in the subdigraph G' with excess one are classified according to two types. A vertex u is Type I if there is an arc from $o(u)$ to u, $o(u)$ being the outlier of u. Otherwise, u is Type II.

Proposition 3.4. *There are no (d, k)-digraphs of degree $d = 5$ and odd diameter k with permutation cycle structure $(0, \ldots, 0, m_i, 0, \ldots, 0, m_{2i}, 0, \ldots, 0)$, with $M(3, k) + 1$ vertices of order i and $N - M(3, k) - 1$ vertices of order $2i$ if all vertices in G' are Type I.*

Proof. From Corollary 1 in [24], since G' is a $(3, k; +1)$-digraph with all vertices of Type I, it turns out that

$$(k + 1) \mid 3(M(3, k) + 1).$$

But $k + 1$ is even, and $3(M(3, k) + 1)$ is odd, which is a contradiction. □

An open problem

To summarize, we can see in the following corollary the possible permutation cycle structures for the (d, k)-digraphs with $d = 4, 5$ and $k \geq 3$.

Corollary 3.1. *The possible permutation cycle structures for a (d, k)-digraph with $d = 4$ or $d = 5$ are the following:*

- *If $d = 4$:*

$$(0, \ldots, 0, m_i, 0, \ldots, 0), \quad im_i = N,\ 2 \leq i \leq \tfrac{N}{4}.$$

- *If $d = 5$:*

$$(0, \ldots, 0, m_i, 0, \ldots, 0), \quad im_i = N,\ 2 \leq i \leq N, i \text{ odd},$$

$$(0, \ldots, 0, m_i, 0, \ldots, m_{2i}, 0, \ldots, 0),\ im_i + 2im_{2i} = N, \text{with } M(3, k)+1 \text{ vertices of order } i \text{ and } N - M(3, k) - 1 \text{ of order } 2i.$$

Proof. The claim for $d = 4$ follows from Proposition 3.2, since the only divisors i of N greater than $N/4$ are not possible according to this proposition. Concerning $d = 5$, Theorems 3.1 and 3.2 show some of the permutation structures given in Proposition 3.1 are not possible. □

Therefore, an open problem for almost Moore digraphs of degrees 4 and 5 is to prove that the permutation cycle structures given in Corollary 3.1 are not possible.

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