Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **12** (2023) 173–180
DOI: 10.47443/dml.2023.098

Research Article

The b_q-coloring of graphs

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(Received: 22 May 2023. Received in revised form: 24 July 2023. Accepted: 9 November 2023. Published online: 27 November 2023.)

Abstract

In this article, a new graph coloring, called the b_q-coloring, is introduced. A b_q-coloring of a graph G is a proper vertex coloring of G with k colors such that every color class c admits a set of vertices S of size at most q provided that every color except c appears in the neighborhood of S. The aim of this coloring is to generalize the domination constraint given in the b-coloring of a graph where every color admits only one dominating vertex (adjacent to every other color). The largest positive integer k for which a graph has a b_q-coloring using k colors is the b_q-chromatic number. Some classes of graphs for which the b_q-chromatic number has maximum value are presented. Also, the exact values of this parameter for paths and cycles are given. Furthermore, some bounds for Cartesian products of graphs are presented.

Keywords: vertex coloring; dominating coloring; complete coloring.

2020 Mathematics Subject Classification: 05C15.

1. Introduction

We consider graphs without loops or multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$ where $|V(G)|$ is the *order* of G and $|E(G)|$ is the *size* of G. In the graph G, the *neighborhood* of a vertex $x \in V(G)$ is the set of vertices adjacent to x and it is denoted $N_G(x)$. By extension, the neighborhood of a subset of vertices $V' \subseteq V(G)$ is the set of vertices in $V(G)\backslash V'$ adjacent to a vertex of V' (*i.e.* $N_G(V') = \bigcup_{v \in V'} N_G(v)\backslash V'$). The *degree* of a vertex x is then defined by $d_G(x) = |N_G(x)|$ and the *maximum degree* of the graph is $\Delta(G) = \max\{d_G(x)|x \in V(G)\}$. If there is no ambiguity, parameters $N_G(x)$, $N_G(V)$ and $d_G(x)$ are denoted respectively by $N(x)$, $N(V)$ and $d(x)$.

A *proper* k*-coloring* $c : V(G) \to \{1, 2, \ldots, k\}$ of G is an assignment of colors $1, 2, \ldots, k$ to the vertices of G such that $c(u) \neq c(v)$ for all adjacent vertices u and v. Note that if a vertex u has a neighbor colored by c, we say that u and c are adjacent. By extension, a subset of vertices V' is adjacent to a set of colors C if every color of C is adjacent to at least one vertex of V'.

The smallest number of colors needed to have a proper coloring of a graph G is called its *chromatic number* and is denoted $\chi(G)$. To determine the chromatic number is an NP-complete problem for non-bipartite graphs, *i.e.* $\chi(G) \geq 3$. The parameter was intensively studied (recently [4, 11, 17]) and many derived parameters were defined. An *achromatic coloring* of a graph G is a proper vertex coloring of G where every pair of colors appears on at least one pair of adjacent vertices. The *achromatic number* of G, denoted $\psi(G)$, is the maximum number of colors in any achromatic coloring of G. The parameter was introduced by Harary and Hedetniemi in [8] and determining the achromatic number of a graph was also proved as an NP-complete problem [20].

In a proper k-coloring, a subset of vertices $D_i \subseteq V(G)$ such that $N(D_i)$ contains a vertex of each color $\{1, 2, \ldots, k\}\backslash\{i\}$ is called a *dominating set* for the color i. Each vertex in such a subset is a *dominating vertex*. A *b-coloring* of a graph G is a proper vertex k-coloring of G where every color i, $1 \leq i \leq k$, admits a dominating set of size one. Such a coloring is clearly an achromatic coloring. The maximum number of colors within a b-coloring of a graph G is called the *b-chromatic number* and denoted $\varphi(G)$. This coloring was introduced by Irving and Manlove [10] and they proved the NP-completeness to determine the b-chromatic number of a graph. This parameter was studied for many classes of graphs [1, 3, 5–7, 12, 14–16, 18, 19] (see [13] for a survey on this parameter).

This property of domination is very interesting because there is a privileged relation between the color classes through these dominating vertices. However, having only one dominating vertex for every color makes them very vulnerable to structure modifications of the graph. The constraint, for a vertex, to be a dominating vertex is strong and the deletion

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of a dominating vertex or one of its neighbors can impact the entire coloring (since determining a new dominating vertex is generally hard). In the following, we present a relaxed b-coloring of graphs where the size of dominating sets can be larger than one. Thus in Section 2 we define the *b_q-coloring* of a graph. We also introduce the *b_q-chromatic number* as the largest positive integer k for which a graph admits a b_q-coloring using k colors. In Section 3 we propose some results on this coloring. In particular, we present classes of graphs for which the b_q-chromatic number has maximum value and we prove exact values of the parameter for cycles and paths. We also propose a Nordhaus-Gaddum type inequality and some bounds of the b_q-chromatic number for Cartesian products of graphs. Finally in Section 4 we conclude and suggest further work.

2. The b_q-coloring of graphs

As described above, in a b-coloring of a graph the domination constraint is focused on only one vertex. We propose to generalize this coloring by relaxing the domination property to larger dominating sets. A *b_q-coloring* of a graph G is then defined as a proper vertex coloring of G where each color class admits a dominating set of size at most q (and at least one). The *b_q-chromatic number* of G, denoted $\varphi_q(G)$, is the maximum number of colors such that G admits a b_q-coloring. In a b_q-coloring, we denote the dominating set for every color i by D_i.

Definition 2.1. *Let G be a graph. A b_q-coloring of G is a proper vertex k-coloring of G such that every color i, $1 \leq i \leq k$, admits a set of dominating vertices denoted D_i where $1 \leq |D_i| \leq q$ and D_i is adjacent to colors $\{1, 2, \dots, k\}\backslash\{i\}$.*

Note that if we required at least q dominating vertices for every color, with $q > 1$, in the definition of a b_q-coloring, then such a coloring would not necessarily exist. Therefore, it is better to assume $1 \leq |D_i| \leq q$ and not $|D_i| \geq q$.

From this definition, we can see that such a coloring always exists since a b-coloring has dominating sets of size 1 (*i.e.* $\varphi_q(G) \geq \varphi_1(G) = \varphi(G)$ for $q > 1$). More generally, we have the following relation.

Claim 2.1. *For any graph G, if $q > 1$, then $\varphi_q(G) \geq \varphi_{q-1}(G)$.*

Proof. By Definition 2.1, a b_{q-1}-coloring of G is also a b_q-coloring of G. □

By definition, the b_q-coloring of a graph is an achromatic coloring. The b_q-chromatic number can then be bounded by the achromatic and b-chromatic numbers.

Property 2.1. *Let G be a graph. Then*

$$\chi(G) \leq \varphi(G) \leq \varphi_q(G) \leq \psi(G).$$

More particularly, the b_q-chromatic number of a graph can be upper bounded by the maximum degree of this graph.

Theorem 2.1. *For a graph G we have $\varphi_q(G) \leq q\Delta(G) + 1$.*

Proof. Each dominating set has at most q vertices of degree $\Delta(G)$. Therefore $\varphi_q(G) \leq q\Delta(G) + 1$. □

A relation between the b_q-chromatic number of a graph and its stability number can also be established.

Theorem 2.2. *Let G be a graph with stability number α. Then $\varphi_q(G) \leq n - \alpha + 1$.*

Proof. Let S be a stable set of G of order α. In a b_q-coloring $\mathscr{C}$ of G, p colors are used in S (colors $C = \{1, 2, \dots, p\}$ with $1 \leq p \leq \alpha$).
Consider $p = 1$. Since S is a maximum stable set, every vertex of $G\backslash S$ is adjacent to a vertex of S colored by 1. Thus $\varphi_q(G\backslash S) \leq n - \alpha$ and $\varphi_q(G) \leq n - \alpha + 1$.
Consider $p \geq 2$. We distinguish two subcases. If none of the vertices in S is a dominating vertex in $\mathscr{C}$, then at least p vertices of $G\backslash S$ are colored with the colors of C. Otherwise, let c be a color of C with a dominating vertex in S. Since S is stable, then the $p - 1$ other colors of C are on vertices of $G\backslash S$. Thus from both subcases, at least $p - 1$ vertices of $G\backslash S$ are colored with colors of C. The number of other colors is then at most $(n - \alpha) - (p - 1)$ and we deduce $\varphi_q(G) \leq n - \alpha + 1$. □

3. Some results on the b_q-chromatic number of graphs

In this section, we present some results for the b_q-chromatic number of some graphs. We present some cases for which the upper bound of the parameter is reached, in particular for regular graphs and graphs with large independent sets. Then we propose an inequality of Nordhaus-Gaddum type and we prove the exact value of the b_q-chromatic number for paths and cycles. Finally, we propose some results on the Cartesian product of graphs. We start the section with some simple graphs.

Theorem 3.1. *Let K_n, S_n and $K_{n,n'}$ be respectively a complete graph of order n, a stable graph of order n, and a complete bipartite graph with partitions of sizes n and n'. Then we have $\varphi_q(K_n) = n$, $\varphi_q(S_n) = 1$, and $\varphi_q(K_{n,n'}) = 2$ for any $q \geq 2$.*

Proof. The graph K_n cannot be properly colored if two vertices are in the same color class. Thus we have n dominating sets of size 1 and $\varphi_q(K_n) = n$.

A coloring of S_n cannot have a dominating vertex for each color if it is colored with more than one color. Thus $\varphi_q(S_n) = 1$.

Finally, if a proper coloring of $K_{n,n'}$ has more than one color in a partition, these colors cannot have dominating vertices. Thus at most one color appears in each partition and we have $\varphi_q(K_{n,n'}) = 2$. □

We recall that $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$; see [2]. Now, we prove the following result.

Proposition 3.1. *Let $K_{q,s}$ be a complete bipartite graph. Let G be a connected $K_{q,s}$-free graph with $q \geq 2$, then $\varphi_q(G) \leq (q+s-1)\chi(G) + 1 - q$.*

Proof. For a graph G, we consider a b$_q$-coloring of k colors where at least one dominating set D has size $|D| = q$ (otherwise $\varphi_q(G) = \varphi_{q-1}(G)$). Thus there exists a set S of $k-1$ neighbors of D with distinct colors. Let H be the subgraph induced by $D \cup S$. By hypothesis on G we have $\alpha(H) \leq q+s-1$. Moreover, it is clear that $\chi(H) \leq \chi(G)$. Thus $k - 1 + q = |V(H)| \leq \alpha(H).\chi(H) \leq (q+s-1)\chi(G)$ and we deduce $k \leq (q+s-1)\chi(G) + 1 - q$. □

Note that under conditions, if a graph is partially colored with a b$_q$-coloring on k colors, this coloring can be extended to the whole graph without introducing new colors.

Property 3.1. *Let G be a graph. If an induced subgraph G' of G is b$_q$-colored with k colors, and every vertex of $V(G)\backslash V(G')$ has a degree lower than k, then G admits a b$_q$-coloring with k colors.*

Proof. For any non-colored vertex $u \in V(G)$, since its degree is $d(u) < k$, then there exists a color c to properly color u, with $1 \leq c \leq k$. □

Regular graphs

We can note that for graphs with a sufficiently large independent set, the upper bound of the b$_q$-chromatic number is reachable. In a graph G, we denote by $dist(u, v)$ the minimum distance between vertices u and v.

Theorem 3.2. *Let G be a graph of maximum degree Δ. Consider integers $q \geq 2$ and $k = q^2\Delta + q$. If G contains vertices $S = \{u_1, u_2, \ldots, u_k\}$ such that $d(u_i) = \Delta$ and $dist(u_i, u_j) \geq 4$, then $\varphi_q(G) = q\Delta + 1$, for every $i, j \in \{1, \ldots, k\}, i \neq j$.*

Proof. Consider the set of colors $\mathscr{C} = \{1, 2, \ldots, q\Delta+1\}$. Let S_i be the subset of S defined by $S_i = \{x_{(i-1)q+1}, x_{(i-1)q+2}, \ldots, x_{iq}\}$, with $1 \leq i \leq q\Delta+1$ (note that $\bigcup_{i=1}^{q\Delta+1} S_i = S$). We have $|N(S_i)| = q\Delta$ since the distance between two vertices of S_i is at least four. Color the vertices of S_i with color i and the vertices of $N(S_i)$ with the $q\Delta$ other colors of $\mathscr{C}$. Due to the distance between vertices of S, the partial coloring of G on $(q\Delta + 1)$ colors is proper, and set S is a dominating set of this partial coloring. Since every non-colored vertex has a degree at most Δ it can be colored properly by Property 3.1 to have a b$_q$-coloring of G where S is the dominating set. Therefore $\varphi_q(G) \geq q\Delta + 1$ and by Theorem 2.1 the result holds. □

This property allows us to find some characteristics for regular graphs to get the upper bound for the b$_q$-chromatic number.

Theorem 3.3. *Let G be a d-regular graph of order n. If $n \geq q^2d^4$, then $\varphi_q(G) = qd + 1$, with $d \geq 3$ and $q \geq 2$ or $d = 2$ and $q \geq 4$.*

Proof. Choose an arbitrary vertex u_1. Remove it and its neighbors at distance at most three. The number of removed vertices is at most $1 + d + d(d-1) + d(d-1)^2 = d^3 - d^2 + d + 1$. Repeat the operation $q(qd+1)$ times. Thus the number of removed vertices is at most $(q^2d + q)(d^3 - d^2 + d + 1) = q^2d^4 + (q - q^2)d^3 + (q^2 - q)d^2 + (q^2 + q)d + q \leq q^2d^4$ if $d \geq 3$ and $q \geq 2$ or $d = 2$ and $q \geq 4$. In the stable set given by the chosen vertices (each of degree $d = \Delta(G)$), the distance between each vertex is at least four and by Theorem 3.2 we deduce $\varphi_q(G) = qd + 1$. □

Corollary 3.1. *Let G be a d-regular graph of order n. If $n \geq q^2d^4 + 3$ with $d = 2$ and $q = 3$ (respectively if $n \geq q^2d^4 + 6$ with $d = q = 2$), then $\varphi_q(G) = qd + 1$.*

Proof. Use the same proof as for Theorem 3.3. Remove an arbitrary vertex and its neighbors at distance at most three ($q(qd+1)$ times) to finally remove $n_r = q^2d^4 + (q - q^2)d^3 + (q^2 - q)d^2 + (q^2 + q)d + q$ vertices. Since $n_r \leq q^2d^4 + 3$ if $d = 2$ and $q = 3$, and $n_r \leq q^2d^4 + 6$ if $d = q = 2$, then $n_r \leq n$ in both cases. And since the distance between two arbitrary chosen vertices is at least four, then Theorem 3.2 gives $\varphi_q(G) = qd + 1$. □

Inequality of Nordhaus-Gaddum type

Theorem 3.4. *Let G be a graph of order n and $\overline{G}$ its complement. For $q \geq 2$ we have $\varphi_q(G) + \varphi_q(\overline{G}) \leq n$.*

Proof. Consider b$_q$-colorings $\mathscr{C}$ and $\overline{\mathscr{C}}$ of respectively G and $\overline{G}$ with respectively φ_q and $\overline{\varphi_q}$ colors. The dominating set of the color i is denoted D_i in $\mathscr{C}$ and $\overline{D_i}$ in $\mathscr{C}$. Moreover note that every class of color C_i in $\mathscr{C}$ gives a complete graph K_{n_i} in $\overline{G}$, with $1 \leq i \leq \varphi_q$. And remark that for any dominating set D_i in $\mathscr{C}$ we have $d_G(D_i) \geq \varphi_q - 1$ and $d_{\overline{G}}(D_i) \leq n - \varphi_q - q + 1$. We distinguish two cases.
Suppose there exists a color i such that a vertex is a dominating vertex in $\mathscr{C}$ and in $\overline{\mathscr{C}}$ (*i.e.* $D_i \cap \overline{D_i} \neq \emptyset$). Thus $\overline{\varphi_q} - 1 \leq d_{\overline{G}}(D_i) \leq n - \varphi_q - q + 1$. Therefore $\varphi_q + \overline{\varphi_q} \leq n - q + 2 \leq n$ since $q \geq 2$.
If for every i, $1 \leq i \leq \varphi_q$, we have $D_i \cap \overline{D_i} = \emptyset$, then the number of vertices able to be dominating vertices in $\overline{\mathscr{C}}$ is at most $n' = \sum_{i=1}^{\varphi_q}(n_i - |D_i|) \leq \sum_{i=1}^{\varphi_q}(n_i - 1) \leq n - \varphi_q$, where n_i is the number of vertices colored with color i. Thus $\overline{\varphi_q} \leq n' \leq n - \varphi_q$ and the result holds. □

Cycles and paths

We now focus on the b$_q$-coloring of cycles. We start by presenting the following results on large cycles.

Lemma 3.1. *If $q \geq 2$, $k = 2q + 1$ and $n = qk$, then $\varphi_q(C_n) = 2q + 1$.*

Proof. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. Consider the complete graph K_k and let $V(K_k) = \{x_1, x_2, \ldots, x_k\}$. A shortest closed walk in K_k that contains every of its edges is given by solving the Chinese postman problem and this walk has length $\ell = \frac{k^2-k}{2}$ because k is odd. Moreover in this walk, each vertex of K_k appears $\frac{\ell}{k} = q$ times. Thus by traversing the ℓ edges of such a walk, each edge $(x_j, x_{j'})$, with $1 \leq j, j' \leq k$, allows to color C_n by $c(v_i) = j$ where $1 \leq i \leq \ell = qk$. Every vertex colored by j is then adjacent to two different colors and we have a b$_q$-coloring with k colors where every vertex is a dominating vertex. Thus $\varphi_q(C_n) \geq 2q + 1$. Moreover Theorem 2.1 gives $\varphi_q(C_n) \leq 2q + 1$ and the equality holds. □

Corollary 3.2. *If $q \geq 2$, $k = 2q + 1$ and $n \geq qk + 2$, then $\varphi_q(C_n) = 2q + 1$.*

Proof. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. By Lemma 3.1 there exists a b$_q$-coloring of C_{qk} with k colors, denoted $\mathscr{C}$. Copy the coloring $\mathscr{C}$ on the qk first vertices of C_n, then put $c(v_{qk+1}) = c(v_1)$ and $c(v_n) = c(v_{qk})$ (note that $c(v_1) \neq c(v_{qk})$ in $\mathscr{C}$). Dominating vertices in $\mathscr{C}$ keep dominating in the partial coloring of C_n and non-colored vertices have degree 2. By Property 3.1 we obtain a b$_q$-coloring of C_n on $2q+1$ colors. Since Theorem 2.1 gives $\varphi_q(C_n) \leq 2q+1$, the equality holds. As in Lemma 3.1, note that every dominating vertex is adjacent to two different colors. □

Then we propose a construction of a b$_q$-coloring for smaller cycles.

Lemma 3.2. *If $q \geq 2$, $k = 2q + 1$ and $qk - q \leq n < qk$, then $\varphi_q(C_n) \geq 2q$.*

Proof. Consider the cycle C_{qk} and its coloring $\mathscr{C}$ on k colors given by Lemma 3.1. Note that in this coloring every dominating set has size q and all the vertices are dominating vertices. Thus every vertex is adjacent to two different colors. Do the following operations:

1. Remove the color k from $\mathscr{C}$,
2. Remove $qk - n$ non-colored vertices (and connect their two neighbors, which have different colors),
3. Color remaining non colored vertices by Property 3.1 (they have degree 2).

Thus remaining dominating vertices given in $\mathscr{C}$ keep dominating and we have a b$_q$-coloring of C_n with $k-1$ colors. Therefore $\varphi_q(C_n) \geq 2q$. □

Corollary 3.3. *If $q \geq 2$, $k = 2q + 1$ and $n = qk + 1$, then $\varphi_q(C_n) \geq 2q$.*

Proof. Use the construction given in Lemma 3.2 to transform the coloring of C_{qk+2} on $2q + 1$ colors (Corollary 3.2) into a coloring for C_n by removing one color and one dominating vertex of this color. As in Lemma 3.2 we see that the remaining dominating vertices from the coloring of C_{n+1} keep dominating and Property 3.1 completes the coloring to have a b$_q$-coloring of C_n with $2q$ colors. □

Before presenting the result on cycles, we recall the following result of Harary et al. [9] on the achromatic number of cycles:

Theorem 3.5. [9] *If* $k\left\lceil\frac{k-1}{2}\right\rceil \le n < (k+1)\left\lceil\frac{k}{2}\right\rceil$, *then*

$$\psi(C_n) = \begin{cases} k-1 & \text{if } n = k\left\lceil\frac{k-1}{2}\right\rceil + 1 \text{ and } k \text{ odd,} \\ k & \text{otherwise.} \end{cases}$$

Thus we present the exact value of the b_q-chromatic number for cycles.

Theorem 3.6. *Let C_n be a cycle of order $n \ge 3$. If $q \ge 2$ and $k = 2q+1$, then*

$$\varphi_q(C_n) = \begin{cases} 2q+1 & \text{if } n \ge qk \text{ and } n \ne qk+1, & (a) \\ 2q & \text{if } n = qk+1 \text{ or } qk-q \le n < qk, & (b) \\ \varphi_{q-1}(C_n) & \text{if } n < qk-q. & (c) \end{cases}$$

Proof. Let $V(C_n) = \{v_1, v_2, \dots, v_n\}$.

(a) Given by Lemma 3.1 and Corollary 3.2.

(b) Consider $n = q(2q+1)+1$. Let the integer $s = 2q+1$. Since s is odd and $n = s\frac{s-1}{2}+1$, then Theorem 3.5 gives that $\psi(C_n) = s-1 = 2q$. By Property 2.1 we deduce $\varphi_q(C_n) \le \psi(C_n) = 2q$. Moreover Corollary 3.3 shows $\varphi_q(C_n) \ge 2q$. Thus the equality holds.
If $qk-q \le n < qk$, then Lemma 3.2 shows $\varphi_q(C_n) \ge 2q$. Suppose there exists a b_q-coloring of C_n with $k' \ge 2q+1 = k$ colors. Since $\Delta(C_n) = 2$, then dominating sets have at least q vertices and $n \ge qk' \ge qk$ which is a contradiction. Therefore $\varphi_q(C_n) \le 2q$ and the result holds.

(c) Consider $n < qk-q$. Claim 2.1 shows $\varphi_q(C_n) \ge \varphi_{q-1}(C_n)$. Suppose $\varphi_q(C_n) > \varphi_{q-1}(C_n)$. Thus there exists a dominating set D such that $|D| = q$, otherwise we have a contradiction since $\varphi_q(C_n) = \varphi_{q-1}(C_n)$. Moreover every dominating vertex $x \in D$ is adjacent to at least one color not present in the neighborhood of the other dominating vertices of D. Indeed if every neighboring color of x is also in the neighborhood of another dominating vertex of D, then $D' = D\backslash\{x\}$ is also a dominating set of size $|D'| < |D| = q$ and we have a contradiction too. Thus since $|D| = q$ and every dominating vertex is adjacent to a particular color, the coloring has at least $2q = k-1$ colors. Since $\Delta(C_n) = 2$, then dominating sets have at least q vertices (otherwise a dominating set is adjacent to at most $2q-2$ colors). Thus we have $n \ge q(k-1)$, a contradiction. Therefore $\varphi_q(C_n) = \varphi_{q-1}(C_n)$. □

We now present the exact value of the b_q-chromatic number for paths.

Theorem 3.7. *Let P_n be a path of order $n \ge 3$. If $q \ge 2$ and $k = 2q+1$, then*

$$\varphi_q(P_n) = \begin{cases} 2q+1 & \text{if } n \ge qk+2 & (a) \\ 2q & \text{if } qk-q \le n < qk+2, & (b) \\ \varphi_{q-1}(P_n) & \text{if } n < qk-q. & (c) \end{cases}$$

Proof. Let $V(P_n) = \{v_1, v_2, \dots, v_n\}$.

(a) Consider the cycle C_n and its b_q-coloring on k colors given by Corollary 3.2. Then remove the edge (v_{n-2}, v_{n-1}) to obtain the graph P_n. It is clear that the coloring of C_n keeps a b_q-coloring of P_n since neither v_{n-1} nor v_{n-2} is a dominating vertex. Therefore $k \le \varphi_q(P_n) \le 2q+1$ (by Theorem 2.1) and the result holds.

(b) We have $qk-q \le n < qk+2$. Consider the path P_{qk+2} and its coloring given in case (a) where every dominating vertex is adjacent to two different colors. Uncolored vertices of color k. Remove $qk+2-n$ non-colored vertices (and connect the two neighbors of every removed vertex if it is not and endvertex of the path). Note that the remaining dominating vertices given by the initial coloring keep dominating in the partial coloring of P_n. Finally color remaining non-colored vertices (which have degree at most 2) by Property 3.1 to obtain a b_q-coloring on $k-1 = 2q$ colors for P_n. Thus $\varphi_q(P_n) \ge 2q$.
Suppose there exists a b_q-coloring of P_n with $k' \ge 2q+1$ colors. By Theorem 2.1 we have $\varphi_q(P_n) \le 2q+1$, thus $k' = k = 2q+1$. Since $k' = 2q+1$, each dominating set D_i has size q, $1 \le i \le k'$ (due to the maximum degree 2 of vertices). If all dominating

vertices are internal vertices of P_n, then we have $n \geq k'|D_i|+2 \geq (2q+1)q+2$, a contradiction. Thus at least one endvertex of P_n is a dominating vertex. Let D be the dominating set containing this endvertex. Since $|D| = q$, D contains also either $q-1$ internal vertices and $d(D) = 2(q-1)+1 = 2q-1 < k'-1$ (a contradiction to be a dominating set), or $q-2$ internal vertices and the second endvertex and $d(D) = 2(q-2)+2 = 2q-2 < k'-1$ (a contradiction too). Therefore no b_q-coloring on k' colors exists and $\varphi_q(P_n) \leq 2q$.

(c) Consider $n < qk - q$. We use the same reasoning as in Theorem 3.6(c). Thus Claim 2.1 shows $\varphi_q(P_n) \geq \varphi_{q-1}(P_n)$ and we suppose $\varphi_q(P_n) > \varphi_{q-1}(P_n)$. We observe that at least one dominating set D has size $|D| = q$ and every dominating vertex $x \in D$ is adjacent to a unique color not present in the neighborhoods of the other dominating vertices of D. Thus the b_q-coloring of P_n has at least $2q$ colors. Since $\Delta(P_n) = 2$, then dominating sets have at least q vertices and $n \geq q(k-1)$, which is a contradiction. Therefore $\varphi_q(C_n) = \varphi_{q-1}(C_n)$. □

Cartesian product of graphs

The Cartesian product of two graphs G and H, denoted $G\Box H$, has the vertex set $V(G) \times V(H)$ and the neighborhood of every vertex (x, y) is $N_{G\Box H}((x, y)) = (x \times N_H(y)) \cup (N_G(x) \times y)$, where $x \in V(G)$ ad $y \in V(H)$. Thus in the graph $G\Box H$ we find several copies of graphs G and H denoted respectively by G^i and H^j, with $1 \leq i \leq |V(H)|$ and $1 \leq j \leq |V(G)|$.

Proposition 3.2. *Let G and H be two graphs, then $\varphi_q(G\Box H) \geq \max\{\varphi_q(G), \varphi_q(H)\}$.*

Proof. Suppose $\varphi_q(G) \geq \varphi_q(H)$. Color the vertex (x, y) of $G\Box H$ with $c(x) + c(y) - 1$ modulo $\varphi_q(G)$. For an edge $e = ((x, y_1), (x, y_2))$ we have $(y_1, y_2) \in E(H)$ and y_1 and y_2 have different colors in a b_q-coloring of H. Thus edge e is properly colored. Similarly we can show that edges $((x_1, y), (x_2, y))$ are also properly colored. Thus the coloring of $G\Box H$ is proper. Moreover, choose a copy of G corresponding to a vertex $y \in V(H)$ such that $c(y) = 1$. The coloring of this copy is similar to the coloring of G and the dominating sets are found. Therefore the coloring is a b_q-coloring of $G\Box H$. □

If the graphs G and H are bipartite, one more color can be used in a b_q-coloring of $G\Box H$.

Proposition 3.3. *Let G be a bipartite graph. Let P_3 be a path of order 3. Then $\varphi_q(G\Box P_3) \geq \varphi_q(G) + 1$.*

Proof. Suppose that G admits a b_q-coloring on k colors. The proof is given by construction. Let X and Y be the two vertex partitions of G (and respectively X^i and Y^i the two vertex partitions of the copy G^i in $G\Box H$, with $1 \leq i \leq 3$).
Color the copy G^2 with the coloring of G. This gives dominating sets for colors 1 to k denoted D_i in the partial coloring, with $1 \leq i \leq k$. Color vertices of Y^1 and X^3 with color $k+1$. Then color vertices of X^1 and Y^3 with the coloring of respectively X and Y with these modifications:

- use a cyclic permutation of colors $\{1, 2, \ldots, k-2, k-1\}$ to use respectively colors $\{2, 3, \ldots, k-1, 1\}$,
- color k is replaced by color $k+1$.

In such a coloring, note that vertices of D_i, $1 \leq i \leq k$, are adjacent to color $k+1$ either in Y^1 or in X^3. Thus every set D_i is a dominating set in the final coloring. It remains to determine a dominating set for the color $k+1$. Consider the dominating set D_k in G^2. Choose the same set called D' in the graph $G' = X^3 \cup Y^1$. By construction, every vertex of D' is adjacent to color k in G^2. Moreover, since vertices of D_k are adjacent to colors 1 to $k-1$ in G^2, then vertices of D' are adjacent to colors 1 to k in G'. Thus D' is the dominating set D_{k+1} for the coloring of $G\Box P_3$ and $\varphi_q(G\Box P_3) \geq \varphi_q(G) + 1$. □

Theorem 3.8. *Let G and H be two bipartite graphs with vertex partitions respectively X_G, Y_G and X_H, Y_H. If H is not a disjoint union of K_2, then $\varphi_q(G\Box H) \geq \varphi_q(G) + 1$.*

Proof. Let x be a vertex of X_H and y_1, y_2 be two neighbors of x in Y_H. The induced subgraph given by vertices $\{y_1, x, y_2\}$ is a path P_3. Proposition 3.3 shows that $\varphi_q(G\Box P_3) \geq \varphi_q(G) + 1$ and gives a coloring where $\mathscr{C}_x$ and $\mathscr{C}_y$ are the colorings of copies of G corresponding to vertices x and y_1 in H. Then for every vertex $v \in X_G$ ($v \neq x$) color the corresponding copy of G in $G\Box H$ by $\mathscr{C}_x$ and for every vertex $w \in Y_G$ ($w \neq y_1, y_2$) color the corresponding copy of G in $G\Box H$ by $\mathscr{C}_y$. Since H is bipartite, the coloring is proper and $\varphi_q(G\Box H) \geq \varphi_q(G) + 1$. □

Finally, if the induced subgraphs given by dominating sets of G and H are stable sets with a dominating set of size one, the b_q-chromatic number of $G\Box H$ depends on $\varphi(G)$ and $\varphi(H)$.

Theorem 3.9. *Let G and H be two graphs of orders respectively n_G and n_H. For $q \geq 2$, G and H have b_q-colorings of k_G and k_H colors respectively ($k_G \geq k_H$) such that the subgraphs induced by their dominating sets are stable. In the coloring of G (respectively H) if a dominating set has size 1, then $\varphi_q(G\Box H) \geq \varphi_q(G) + \varphi_q(H) - 1$.*

Proof. Let $S_G = \{x_1, x_2, \ldots, x_n\}$ (respectively $S_H = \{y_1, y_2, \ldots, y_{n'}\}$) be the subset of dominating vertices in the coloring of G (respectively H). The subset of remaining vertices is denoted $U_G = \{x_{n+1}, x_{n+2}, \ldots, x_{n_G}\}$ (respectively $U_H = \{y_{n'+1}, y_{n'+2}, \ldots, y_{n_H}\}$). Note that since S_G and S_H are stable, $S_G \square S_H$ is also stable. By hypothesis, S_G (respectively S_H) contains a unique dominating vertex for a color c (respectively c'), denoted x_p with $1 \leq p \leq n$ (respectively y_q with $1 \leq q \leq n'$). Wlog, consider in each coloring that $c = 1$ (respectively $c' = 1$).
We present a coloring for the graph $G \square H$. Let $C_1 = \{1, 2, \ldots, k_G\}$ be a set of colors and $C_{k_G+1}, C_{k_G+2}, \ldots, C_{k_G+k_H-1}$ be $k_H - 1$ different circular permutations of C_1 (possible since $k_G \geq k_H$). Start by coloring the copy G^q with the coloring of G. Then for every vertex y_i, with $1 \leq i \neq q \leq n'$, a dominating vertex for color c ($1 \leq c \leq k_G$), color the copy G^i as follow:

- the vertices of S_G^i with the color $k_G + c - 1$,
- the vertices of U_G^i as the set U_G with the color set C_{k_G+c-1} (each color class used in the coloring of U_G is replaced by the corresponding color class given by C_{k_G+c-1}).

Note that since S_H is stable, copies G^j are disjointed and the partial coloring is currently proper.
From the construction sets S_H^i, $1 \leq i \leq n$, are already colored as in the initial coloring of H where the initial set of colors $\{1, 2, \ldots, k_H\}$ is replaced by $\{c(x_i), k_G + 1, k_G + 2, \ldots, k_G + k_H - 1\}$. We complete the coloring of H^i by coloring the sets U_H^i with the same coloring as in H by using the set of colors $\{c(x_i), k_G + 1, k_G + 2, \ldots, k_G + k_H - 1\}$ (color classes in the initial coloring are replaced by the new color classes). Since the copies H_j are disjointed (because S_G is stable), the partial coloring is always proper. Note that H^p contains colors $\{1, k_G + 1, k_G + 2, \ldots, k_G + k_H - 1\}$.
It remains to color the vertices of $U_G \square U_H$. From the coloring, vertices of U_H^p are colored with colors $\{1, k_G+1, k_G+2, \ldots, k_G+k_H - 1\}$. For each vertex $y_i \in U_H^p$, colored by $c \in \{1, k_G + 1, k_G + 2, \ldots, k_G + k_H - 1\}$, color the vertices of U_G^i as the set U_G with the color set C_c. Thus, if $c((x_p, y_i)) = 1$, then copy G^i has the same coloring as G and when $c((x_p, y_i)) \geq k_G + 1$, vertices of U_G^i have colors 1 to k_G. Thus copies G^i, $n' \leq i \leq n_H$, are properly colored. Moreover, since two distinct color classes of U_H^p are associated with two distinct circular permutations of C_1, then copies H^j, $n \leq j \leq n_G$, are also properly colored. Therefore the coloring is proper. Figure 1 illustrates the above coloring.

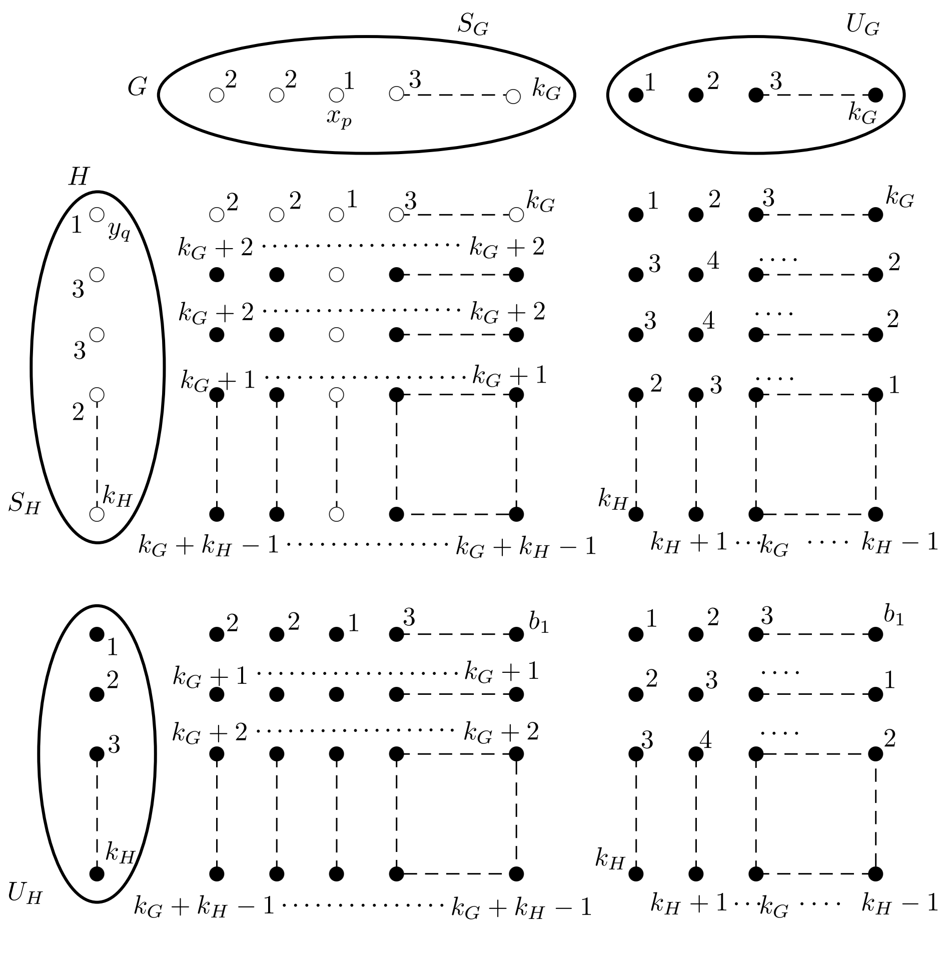

Figure 1: A b_q-coloring of G, H and $G \square H$, where white vertices are dominating vertices.

We now prove that the coloring is a b_q-coloring. Consider the set of vertices given by $D = S_G^q \cup S_H^p$. From the above construction, colors $1, 2, \ldots, k_G + k_H - 1$ appear D (colors $1, 2, \ldots, k_G$ on S_G^q and colors $k_G + 1, k_G + 2, \ldots, k_G + k_H - 1$ on S_H^p). Furthermore, every color appears at most q times in D because the coloring is based on the dominating sets of G and H. Then we check that every dominating vertex is adjacent to colors 1 to k_G. Since G^q is colored as G, then vertices of S_G^q are adjacent to these colors. Moreover, since x_p is the unique dominating set for color 1 in the coloring of G and colors 1 to k_G appear on U_G^i, $1 \leq i \leq n'$, then every vertex of S_H^p is also adjacent to these colors. Similarly, we check that vertices of D are adjacent to colors $k_G + 1$ to $k_G + k_H - 1$. Since H^p is colored as H with colors $\{1, k_G + 1, k_G + 2, \ldots, k_G + k_H - 1\}$, then vertices of S_H^p are adjacent to these colors. Moreover, since y_q is the unique dominating set for color 1 in the coloring of H and colors $k_G + 1$ to $k_G + k_H - 1$ appear on U_H^j, $1 \leq j \leq n$, then every vertex of S_G^q is also adjacent to these colors. Therefore we have a b_q-coloring of $G \Box H$ where D is a dominating set and $\varphi_q(G \Box H) \geq \varphi_q(G) + \varphi_q(H) - 1$. □

Note that the inequality of Theorem 3.9 is not obvious and we can find counterexamples. Thus we have $\varphi_1(P_2) = \varphi_2(P_2) = 2$. Then a b_1-coloring of $K_n \Box P_2$ cannot have more than n colors otherwise some colors have no dominating vertex. Thus $\varphi_1(K_n \Box P_2) = \varphi_1(K_n)$ with $n \geq 3$. We can also see that $P_2 \Box P_2 \equiv C_4$ and $\varphi_2(P_2 \Box P_2) = \varphi_2(C_4) = 2 = \varphi_2(P_2)$.

4. Conclusion

A b-coloring of a graph allows to determine a dominating vertex for each color, adjacent to every other color used in the coloring. In this article, we introduced a relaxed b-coloring of graphs, called b_q-coloring, in order to reduce the domination constraints carried by the single dominating vertex of each color. Finding a vertex with strong constraints would be more difficult than finding several vertices with the lowest constraints to replace it. Thus the b_q-coloring is a proper vertex coloring whose aim is to maximize the number of colors used to color a graph by determining a dominating set of size at most q for every color. We positioned the b_q-chromatic number relatively to other graph parameters (stability number, chromatic, and achromatic numbers). We proposed some bounds and exact values for some classical classes of graphs in particular for cycles, paths, regular graphs, and Cartesian product of graphs. Interesting questions could be to identify the values of q to have $\varphi_q(G) > \varphi(G)$ or to determine the minimum integer q to have $\varphi_q(G) = \psi(G)$.

Replace a unique dominating vertex with a dominating set to reduce the constraints may imply dominating sets with very different sizes. An extension of this coloring could be proposed. The *equitable b_q-coloring* of a graph could be introduced as a b_q-coloring for which two dominating sets would differ in size by at most one. This allows to homogenize the sizes of the dominating sets. If we denote by $\varphi_q^e(G)$ the maximum number of colors to have an equitable b_q-coloring of a graph G, we have clearly $\varphi_q(G) \geq \varphi_q^e(G) \geq \varphi(G)$. This second parameter could also be studied for different classes of graphs.

Acknowledgment

The author thanks the referees for their useful suggestions that helped to improve this article.

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