Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **12** (2023) 166–172
DOI: 10.47443/dml.2023.106

Research Article

Extremal trees with fixed degree sequence for σ-irregularity

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(Received: 28 May 2023. Received in revised form: 3 November 2023. Accepted: 6 November 2023. Published online: 27 November 2023.)

Abstract

A natural extension of the well-known Albertson irregularity of graphs is the so-called σ-irregularity. For a simple graph G, it is defined as $\sigma(G) = \sum_{uv\in E(G)}(d_G(u) - d_G(v))^2$, where $d_G(v)$ denotes the degree of a vertex v of G. In this study, we characterize trees with minimal and maximal σ-irregularity among trees with a given degree sequence. Specifically, we show that greedy trees minimize σ-irregularity, while adopting trees maximize it among trees with a prescribed degree sequence.

Keywords: irregularity (of graph); irregularity measures; σ-irregularity; extremal trees.

2020 Mathematics Subject Classification: 05C05, 05C07, 05C35, 05C75.

1. Introduction

Let G be a simple undirected connected graph with $|V(G)| = n$ vertices and $|E(G)| = m$ edges. The *degree* of a vertex v in G is the number of edges incident with v and it is denoted by $d_G(v)$. A graph G is *regular* if all its vertices have the same degree, otherwise, it is *irregular*. Knowing how irregular a given graph is important in many applications and problems.

The *imbalance* of an edge $e = uv \in E$, defined as $\text{imb}(e) = |d_G(u) - d_G(v)|$, appears implicitly in the context of Ramsey problems with repeat degrees [9], and later in the work of Chen, Erdős, Rousseau, and Schlep [12], where 2-colorings of edges of a complete graph were considered. In [8], Albertson defined the *irregularity* of G as the sum of imbalances of all edges of a graph, i.e.,

$$\text{irr}(G) = \sum_{e\in E(G)} \text{imb}(e) = \sum_{uv\in E(G)} |d_G(u) - d_G(v)|.$$

For results on the Albertson irregularity, we refer the readers to [2,4,8,18,19].

To overcome certain shortcomings of the Albertson irregularity, as pointed out in [1], a new measure of irregularity of a graph, so-called the *total irregularity* of a graph, was defined as

$$\text{irr}_t(G) = \frac{1}{2} \sum_{(u,v)\in V(G)\times V(G)} |d_G(u) - d_G(v)|.$$

Results of the total irregularity as well as a comparison between the irregularity and the total irregularity of a graph was studied in [3,5,6,15,16,23]. Trying to avoid the absolute value calculation in the Albertson irregularity, one naturally arrived at the irregularity measure $\sigma(G)$ introduced in [20] and defined as

$$\sigma(G) = \sum_{uv\in E(G)} (d_G(u) - d_G(v))^2.$$

In [7] graphs with maximal σ-irregularity were characterized. The inverse σ-irregularity problem was solved in [20]. k-cyclic graphs with maximal σ-irregularity were determined in [10]. For complete split-like graphs, which represent a broad subclass of bidegreed connected graphs, in [22] it was shown that for these graphs the equality $\sigma(G) = n^2\text{Var}(G)$ holds. Here $\text{Var}(G)$ is the variance of the vertex degrees of a graph G

$$\text{Var}(G) = \frac{1}{n}\sum_{v\in V(G)} d_G(v)^2 - \left(\frac{1}{n}\sum_{v\in V(G)} d_G(v)\right)^2, \tag{1}$$

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another well-established irregularity measure also known as Bell's irregularity measure [11].

A sequence $\mathcal{D} = (d_1, d_2, \ldots, d_n)$ is *graphical* if there is a graph whose vertex degrees are d_i, $i = 1, \ldots, n$. If in addition $d_1 \geq d_2 \geq \cdots \geq d_n$, then $\mathcal{D}$ is a *degree sequence*. In the rest of the paper we assume that the degrees of the pendent vertices are excluded from the degree sequence.

For a tree T and $i = 0, 1, \ldots$, let $L_i = L_i(T)$ be the set of vertices in T, whose minimum distance from to the set of pendent vertices of T is i. Clearly, L_0 is exactly the set of pendent vertices in T.

Lin, Gao, Chen, and Lin in [21], and Gan, Liu, and You in [17] introduced the so-called *switching transformation* when they studied the atom-bond connectivity index. Next, we present it in the context of σ-irregularity.

Proposition 1.1. *Let $G = (V, E)$ be a connected graph with $uv, xy \in E(G)$ and $uy, xv \notin E(G)$. Let $H = G - uv - xy + uy + xv$. If $d(u) \geq d(x)$ and $d(v) \leq d(y)$, then $\sigma(H) \leq \sigma(G)$, with the equality if and only if $d(u) = d(x)$ or $d(v) = d(y)$.*

Proof. The change of the σ-irregularity after the above transformation is

$$\begin{aligned} \Delta\sigma(H, G) &= \sigma(H) - \sigma(G) \\ &= (d(u) - d(y))^2 + (d(x) - d(v))^2 - (d(u) - d(v))^2 - (d(x) - d(y))^2 \\ &= 2(d(u) - d(x))(d(v) - d(y)) \leq 0. \end{aligned}$$

□

Observe that H in the above proposition is not necessarily connected. As it was pointed out in [13], if G is for example the path $vuwyx$, then H is a disjoint union of K_3 and K_2. H is disconnected only if in $G - uv - xy$ there is no $v - u$, $v - y$, $x - u$, or $x - y$ paths.

In the following two sections, we consider the trees with minimal and maximal σ-irregularity among trees with a given degree sequence. The results, and their corresponding proofs, presented there are close to those presented in [14], where trees of a given degree sequence that maximize the sum of the products of the degrees of adjacent vertices were determined. Related results concerning the Randić index, atom-bond connectivity index and Wiener index were presented in [24–26].

2. Trees with fixed degree sequence and minimal σ-irregularity

First, we prove some properties of the trees with minimum σ-irregularity. These lead to an algorithm that constructs a tree with minimum σ-irregularity, also known as a greedy tree.

Lemma 2.1. *Let T be a tree with minimum σ-irregularity among the trees with fixed degree sequence. Let $P = v_1 v_2 \ldots v_t$ be a path in T, where $t \geq 4$ and $d(v_1) < d(v_t)$. Then, $d(v_2) \leq d(v_{t-1})$.*

Proof. Assume that the claim of the proposition is not true and that $d(v_2) > d(v_{t-1})$. Let T' be the tree obtained by deleting the edges $v_1 v_2$ and $v_{t-1} v_t$, and adding the edges $v_1 v_{t-1}$ and $v_2 v_t$ to T. Notice that T and T' have the same degree sequence. With the above relation of the degrees $d(v_1), d(v_2), d(v_{t-1}), d(v_t)$, applying Proposition 1.1, we get that $\sigma(T') - \sigma(T) < 0$. This is a contradiction to the initial assumption that T is a tree with minimum σ-irregularity. □

As a consequence of Lemma 2.1, one can obtain the following three corollaries. The argument of the proof of the next corollary is adopted from [14].

Corollary 2.1. *Let T be a tree with minimum σ-irregularity among the trees with fixed degree sequence. Then there is no path $P = v_1 v_2 \ldots v_t$ in T with $t \geq 3$ such that $d(v_1) > d(v_i)$ and $d(v_t) > d(v_i)$ for some $2 \leq i \leq t - 1$.*

Proof. We assume that the above claim is false and that there is a path $P = v_1 v_2 \ldots v_t$ in T with $t \geq 3$, such that for some $2 \leq i \leq t - 1$, the relations $d(v_1) > d(v_i)$ and $d(v_t) > d(v_i)$ hold. Also, it holds that $2 \leq d(v_i)$.

Firstly, consider the case when $d(v_i) < d(v_{i+1})$. Let $P' = v_{-k} v_{-k+1} \ldots v_0 v_1 \ldots v_i v_{i+1}$ be a path such that $d(v_{-k}) = 1$. Note that $k \geq 0$, since $d(v_1) > d(v_i)$. By Lemma 2.1, $d(v_{-k}) < d(v_{i+1})$ implies $d(v_{-k+1}) \leq d(v_i)$, and thus, $d(v_{-k+1}) < d(v_i)$. Again applying Lemma 2.1, $d(v_{-k+1}) < d(v_i)$ implies $d(v_{-k+2}) \leq d(v_i)$, and consequently $d(v_{-k+2}) < d(v_i)$. Repeating this argument, we obtain $d(v_1) \leq d(v_i)$, which is a contradiction to the initial assumption $d(v_1) > d(v_i)$.

Secondly, consider the case when $d(v_i) > d(v_{i+1})$. Now, let $P' = v_i v_{i+1} \ldots v_t v_{t+1} \ldots v_{t+k-1} v_{t+k}$ be a path such that $d(v_{t+k}) = 1$. Applying repeatedly Lemma 2.1 as in the previous case, we obtain that $d(v_{i+1}) \geq d(v_{t+k-1}), \ldots, d(v_{i+1}) \geq d(v_t)$, and thus, $d(v_i) > d(v_t)$, a contradiction to the assumption $d(v_i) < d(v_t)$.

Finally, assume that $d(v_i) = d(v_{i+1})$. Let p be the smallest index larger than $i+1$ such that $d(v_i) > d(v_p)$ or $d(v_i) < d(v_p)$ is satisfied. Notice that if not before this is satisfied when $p = t$, namely then by the initial assumption $d(v_i) < d(v_t)$. Then, $d(v_i) = d(v_{p-1}) < d(v_p)$ or $d(v_i) = d(v_{p-1}) > d(v_p)$ and we can proceed as in one of the two previous cases, to obtain again a contradiction. □

Corollary 2.2. *Let T be a tree with minimum σ-irregularity among the trees with fixed degree sequence. For every positive integer d, the vertices with degrees at least d induce a subtree of T.*

Proof. Let T' be the graph induced by the vertices of degree at least d. Assume that T' is not a tree. Let v_i and v_j be two vertices that belong to different components of T'. Consider the path P between v_i and v_j in T. Since v_i and v_j belong to different components of T', there must exist a vertex v_k in P with $d(v_k) < d$. However, due to Corollary 2.1, it is not possible. □

Corollary 2.3. *Let T be a tree with minimum σ-irregularity among the trees with fixed degree sequence. Then there are no two non-adjacent edges v_1v_2 and v_3v_4 such that $d(v_1) < d(v_3) \leq d(v_4) < d(v_2)$.*

Proof. Having two edges v_1v_2 and v_3v_4, let consider the possible paths in T, which contain all vertices v_1, v_2, v_3, and v_4 and begin and end with one them. There are four such possibilities: $P_1 = v_1v_2 \dots v_3v_4$, $P_2 = v_1v_2 \dots v_4v_3$, $P_3 = v_2v_1 \dots v_3v_4$, and $P_4 = v_2v_1 \dots v_4, v_3$.

If there exist the path P_1, by the claim of the corollary, we have that $d(v_1) < d(v_4)$ and $d(v_3) < d(v_2)$. On the other hand, applying Lemma 2.1, it follows that $d(v_2) \leq d(v_3)$, which is a contradiction.

Similarly, contradictions can be shown if there exist paths P_2, P_3, and P_4. □

By Corollary 2.2 the degrees of vertices of T at level L_i are not larger than the degrees of vertices at L_{i+1} for all $i = 0, 1, 2, \dots$. Thus the vertices of larger degrees have farther distances from L_0 than the vertices of smaller degrees. It is not difficult to see that the tree T with minimal σ-irregularity is not always uniquely determined up to isomorphism (see Figure 1 for an example). However, having the above properties one can efficiently construct a tree with minimal σ-irregularity, with the algorithm first proposed by Delorme et al. [10] and later generalized by Wang [24] and named the *greedy algorithm*. Now, by this algorithm, an extremal tree T that achieves the minimum σ-irregularity among the trees with fixed degree sequence $\mathcal{D} = \{d_1, d_2, \dots, d_m\}$ can be constructed as:

1. Label the vertex with the largest degree as v (the root).
2. Label the neighbors of v as $v_1, v_2, \dots$, assign the largest degree available to them such that $d(v_1) \geq d(v_2) \geq \dots$.
3. Label the neighbors of v_1 (except v) as $v_{11}, v_{12}, \dots$ such that they take all the largest degrees available and that $d(v_{11}) \geq d(v_{12}) \geq \dots$, then do the same for $v_2, v_3, \dots$.
4. Repeat 3. for all newly labeled vertices, always starting with the neighbors of the labeled vertex with the largest degree whose neighbors are not labeled yet.

Theorem 2.1. *Given the degree sequence, the greedy tree minimizes the σ-irregularity.*

Proof. The greedy tree obviously satisfies Lemma 2.1 and Corollaries 2.1–2.3. However, there could be other trees for which these conditions hold (see Example 2.1). Now, we only show that the σ-irregularity of the greedy tree achieves the minimum among these trees. Let denote the greedy tree by T. Assume that T does not minimize the σ-irregularity. Let T' be a rooted tree that has minimum σ-irregularity. Since T' is not a greedy tree, there are two vertices v_i and v_k, with $d(v_i) \geq d(v_k)$ and $i > k$, such that v_i has a child v_j and v_k has a child v_l, with $d(v_k) \leq d(v_l)$. We apply the switching transformation from Proposition 1.1 by deleting edges v_iv_j and v_kv_l and adding edges v_iv_l and v_kv_i. Observe that after this transformation, the resulting tree remains connected since there is a path between v_i and v_k. After this transformation, the σ-irregularity does not increases. We apply as many times the switching transformation as above until we obtain a greedy tree T. After all this transformations the σ-irregularity does not increase. Since T' is a tree with minimum σ-irregularity and the obtained greedy tree T does not have larger σ-irregularity, it follows that the greedy tree has also minimum σ-irregularity. □

Example 2.1. *In Figure 1 two trees, which have maximum σ-irregularity among all trees with degree sequence $\mathcal{D} = (5, 5, 5, 4, 3, 3, 3, 2, 2)$, are presented. The tree T is obtained by the "greedy algorithm".*

We would like to note that for a given degree sequence, the greedy tree, which achieves the minimum σ-irregularity also archived the minimum (general) Randić index for $\alpha < 0$ [24], the minimum atom-bond connectivity index [17, 21, 25] and the minimum Wiener index [26].

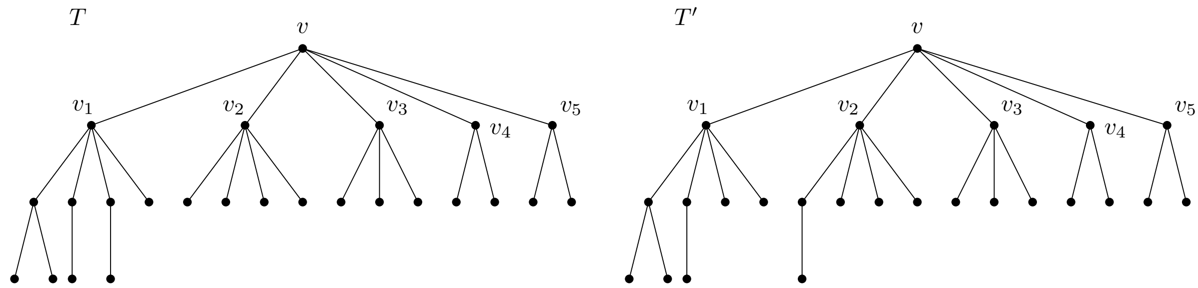

Figure 1: Two extremal trees of degree sequence $\mathcal{D} = (5, 5, 5, 4, 3, 3, 3, 2, 2)$ with the minimum σ-irregularity. Observe that only T is obtained by the "greedy algorithm".

3. Trees with fixed degree sequence and maximal σ-irregularity

As in the previous section, here also several results and definition will be adapted from [17, 21, 24, 25]. The next result is crucial for characterizing and constructing trees with maximum σ-irregularity and a given degree sequence.

Lemma 3.1. *Let T be a tree with maximum σ-irregularity. Then, every path with end-vertices of degree 1, can be enumerated as $v_0v_1 \dots v_tv_{t+1}$ in T, where $d(v_0) = d(v_{t+1}) = 1$ and $1 \le i \le (t+1)/2$, such that the following properties hold:*

(a) if i is odd, then $d(v_i) \ge d(v_{t+1-i}) \ge d(v_k)$ for $i < k < t + 1 - i$;

(b) if i is even, then $d(v_i) \le d(v_{t+1-i}) \le d(v_k)$ for $i < k < t + 1 - i$.

Proof. We prove the above claims by induction on i.

For $i = 1$, we have to show that $d(v_1) \ge d(v_t) \ge d(v_k)$, for $2 \le k \le t - 1$.

Firstly, if $d(v_1) \ge d(v_t)$ does not hold, then we enumerate the vertices in the considered path in the reversed order. Then, we stay by this enumeration.

Secondly, we show that $d(v_1) \ge d(v_k)$. We assume that it is not true and that $d(v_1) < d(v_k)$. We obtain a tree T' from T by deleting the edges v_0v_1 and v_kv_{k+1} and adding edges v_0v_k and v_1v_{k+1}. With the constraints, $d(v_{k+1}) > d(v_0) = 1$ and $d(v_k) > d(v_1)$, by Proposition 1.1, we have that $\sigma(T') - \sigma(T) > 0$. This contradicts the claim that T is a tree with maximum σ-irregularity, and thus, $d(v_1) \ge d(v_k)$, $2 \le k \le t - 1$, holds.

The relation $d(v_t) \ge d(v_k)$, $2 \le k \le t - 1$, we prove similarly. Now, we delete the edges v_tv_{t+1} and $v_{k-1}v_k$ and add edges v_tv_{k-1} and $v_{t+1}v_k$ from T obtaining the tree T'. With the assumption $d(v_t) < d(v_k)$ and the relation $d(v_{k-1}) > d(v_0) = 1$, by Proposition 1.1, we obtain that $\sigma(T') - \sigma(T) > 0$. This is again a contradiction. It follows that $d(v_t) \ge d(v_k)$, for $2 \le k \le t$ and thereby the case $i = 1$ is proven.

Now assume that $i \ge 2$ is even. Then $i - 1$ is odd, and by the induction hypothesis, we may assume that $d(v_{i-1}) \ge d(v_{t-i+2}) \ge d(v_k)$, for $i - 1 < k < t + 2 - i$. We want to show that $d(v_i) \ge d(v_{t+1-i}) \ge d(v_k)$, for $i < k < t + 1 - i$. Assume that this is not true and that $d(v_i) > d(v_{t+1-i}) > d(v_k)$. Here, we delete the edges $v_{i-1}v_i$ and v_kv_{k+1} and add the edges $v_{i-1}v_k$ and v_iv_{k+1} to T obtaining a tree T' with same degree sequence as T. For $i < k < t + 1 - i$, it holds that $d(v_{i-1}) > d(v_{k+1})$ and $d(v_i) > d(v_k)$. Thus, $\sigma(T') - \sigma(T) > 0$, which is a contradiction to the maximum optimality of T, and we may conclude that $d(v_i) \ge d(v_k)$, for $i \le k \le t + 1 - i$. In the same way, we can prove $d(v_i) \ge d(v_{t+1-i})$.

When $i \ge 2$ is odd, we can similarly show $d(v_i) \ge d(v_{t+1-i}) \ge d(v_k)$ by the same argument as we use above for even i. □

Lemma 3.1 implies the following weaker statement:

Corollary 3.1. *In a tree with maximum σ-irregularity, let $v_i \in L_i$ for $i = 0, 1, \dots$, then we may assume, for $j > i \ge 1$, that*

- *$d(v_i) \ge d(v_j)$ if i is odd;*
- *$d(v_i) \le d(v_j)$ if i is even.*

For a tree T, let d^* be the minimum degree of vertices in the set $L_1(T)$. Let $V_p^*(T)$ represent the set of pendant vertices whose adjacent vertices have a degree of d^* in tree T. Let $\overline{V_p^*(T)}$ be the set of pendant vertices in tree T that are not part of $V_p^*(T)$.

Lemma 3.2. *Let v' and v'' be two vertices of a tree T such that $v' \in V_p^*(T)$ and $v'' \in \overline{V_p^*(T)}$. We obtain trees T_1^* and T_2^* by identifying the roots r_i of an arbitrary tree T_i with v' and v'', respectively. Then, $\sigma(T_1^*) > \sigma(T_2^*)$.*

Proof. Suppose v_1 and v_2 are adjacent to v' and v'', respectively. Obviously, $d(v_1) < d(v_2)$. It holds

$$\begin{aligned}\sigma(T_1^*) - \sigma(T_2^*) &= (d(v_1) - (d(r_i) + 1))^2 + (1 - d(v_2))^2 - (1 - d(v_1))^2 - (d(v_2) - (d(r_i) + 1))^2 \\ &= (d(v_1) - (d(r_i) + 1))^2 + (1 - d(v_2))^2 \\ &= -2d(r_i)(d(v_1) - d(v_2)) > 0.\end{aligned}$$

□

By Lemma 3.2, a tree with maximum σ-irregularity is obtained by attaching a tree T_i to a vertex in T. We now construct an extremal tree with a given degree sequence using the following *adopting algorithm* [17]. The obtained tree will be referred to as an *adopting tree*.

Let $\mathcal{D} = (d_1, d_2, \dots, d_m)$ represent the degree sequence corresponding to the non-pendant vertices $v_1, v_2, \dots, v_m$. Recall that, as per the definition of a degree sequence given in the introduction, it is assumed that $d_1 \geq d_2 \geq \cdots \geq d_m$

1. We create subtrees T_i as follows:
   - T_1 is rooted at r_1 and is assigned $d_m - 1$ children whose degrees are $d_1, d_2, \dots, d_{d_m-1}$. These $d_m - 1$ children are exclusively adjacent to pendant vertices, in addition to being adjacent to a non-pendant root vertex.
   - T_2 is rooted at r_2 and is assigned $d_{d_m-1} - 1$ children whose degrees are $d_{d_m}, d_{d_m+1}, \dots, d_{d_m+d_{d_m-1}-2}$. Similarly, except to the root r_2, these $d_{d_m-1} - 1$ children are solely adjacent to pendant vertices.
   - We continue similarly to create trees $T_3, T_4, \dots$ rooted at $r_3, r_4, \dots$, each of which is assigned $d_{m-2} - 1, d_{m-3} - 1, \dots$ children, respectively.
2. We terminate this process by the tree T_l rooted at r_l when one of the following conditions is met: either there are fewer than $d_{m-l+1} - 1$ children available for assignment, or there are no remaining degrees to choose from for T_{l+1}. As a result, we obtain subtrees $T_1, T_2, \dots, T_l$ with $d(r_l) = d_{m-l+1}$.
3. Let $r = r_l$ and $T = T_l$. We obtain $T^{(l-1)}$ from T and T_{l-1} rooted at r_{l-1} by identifying a pendant vertex from T with r_{l-1}. Next, we let $T = T^{(l-1)}$. We obtain $T^{(l-2)}$ from T and T_{l-2} rooted at r_{l-2}. This process continues similarly for $T_{l-3}, \dots, T_1$.
4. We terminate the construction by setting $T = T^1$.

The tree T obtained by the above algorithm is not necessarily unique, since a subtree T_l may be rooted in several ways, as shown below in Example 3.1.

Theorem 3.1. *Given a degree sequence, an adopting tree T obtained by the adopting algorithm has the maximum σ-irregularity.*

Proof. It is easy to check that T satisfies Lemma 3.1 and Corollary 3.1. Next, we show that T indeed has the maximum σ-irregularity among the trees with the same degree sequence. Assume that it is not true and that another tree T', with the same degree sequence as T, not obtained by the adopting algorithm, has the maximum σ-irregularity, larger than $\sigma(T)$.

Next, we show that T' must contain the subtree T_1, as described in the adopting algorithm, otherwise, it cannot have maximum σ-irregularity. Recall that the subtree T_1 has a radius 2, and it is rooted at $r_1(v_m)$. Its children are vertices with degrees $d_1, d_2, \dots, d_{d_m-1}$ and its grandchildren all have a degree 1. A vertex of degree d_1 in T is adjacent to a vertex of degree d_m and $d_1 - 1$ vertices of degree 1. Assume that in T' this is not true and a vertex of degree d_1 is adjacent to a non-pendent vertex of degree d_j different from d_m. Observe that $d_j > d_m$, since d_m is the smallest degree of the non-pendent vertices. Let d_k be a degree of a non-pendent vertex in T' adjacent to a vertex of degree d_m. It holds that $d_k \geq d_m$. Also $d_k \neq d_1$, due to the above assumption that a vertex of degree d_1 is not adjacent to a non-pendent vertex of degree d_m. Thus, $d_1 > d_k$, since d_1 is the largest degree. Denote by v_j, v_k, and v_m, the vertices with degrees d_j, d_k, and d_m, respectively. Let $T'' = T' - v_1v_j - v_mv_k + v_1v_m + v_jv_k$. Then,

$$\sigma(T'') - \sigma(T') = -(d_1 - d_j)^2 - (d_m - d_k)^2 + (d_1 - d_m)^2 + (d_j + d_k)^2 = 2(d_1 - d_k)(d_j - d_m) > 0.$$

This is a contradiction on the assumption that T' has maximum σ-irregularity. It follows that T' has a maximum σ-irregularity if in T' v_1 is adjacent to a vertex with degree d_m. With this conclusion, applying the same argument we can show that if T' has a maximum σ-irregularity then v_2 is adjacent to a vertex with degree d_m. We may continue with the same argumentation for $v_3, v_4, \ldots, v_{m-1}$. Thus, we may conclude that T' has a maximum σ-irregularity if it contains T_1 as subtree.

Similarly, we may conclude that the rest of the subtrees T_i, $i = 2, \ldots, l$, must be subtrees of T' if T' has maximum σ-irregularity.

Lemma 3.2 guarantees the way of connecting the subtrees T_i, $i = 1, \ldots l$ such that the resulting tree has maximum σ-irregularity. Observe that in this way, the subtrees T_i, $i = 1, \ldots l$ are connected by the adopting algorithm. Therefore, T' must be isomorphic to one of the trees, if there is more than one, obtained by the adopting algorithm. Thus, we may conclude that T' and T have the maximum σ-irregularity. □

Example 3.1. *In Figure 2 four trees, which have maximum σ-irregularity among all trees with degree sequence $\mathcal{D} = (5, 5, 5, 4, 3, 3, 3, 2, 2)$, are presented. All trees T, T', T'', and T''' can be obtained by the "adopting algorithm".*

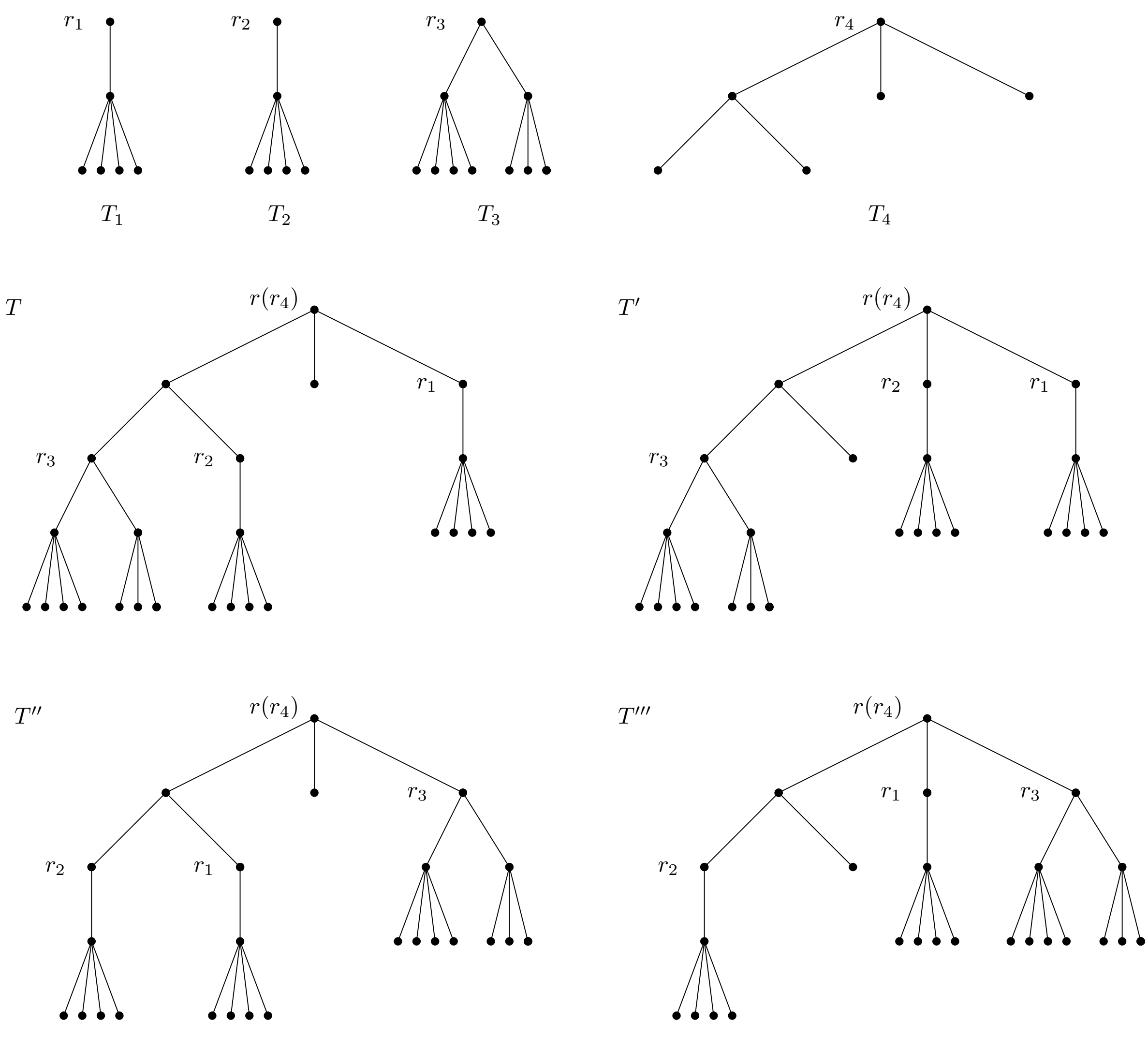

Figure 2: The subgraphs T_1, T_2, T_3, and T_4 generated from the degree sequence $\mathcal{D} = (5, 5, 5, 4, 3, 3, 3, 2, 2)$ as described in the first step of the "adopting algorithm". Based on these subtrees, the four trees T, T', T'', and T''' with maximum σ-irregularity are obtained.

Notice also here that for a given degree sequence, the adopting tree, which achieves the maximum σ-irregularity also achieved the maximum (general) Randić index for $\alpha > 0$ [24], and the minimum atom-bond connectivity index [17, 25].

4. Conclusion and further work

In this work, we characterize the trees with minimum and maximum σ-irregularity of given order and fixed degree sequence. The trees, which have minimal (respectively maximal) values for several graph topological invariants (including the (general) Randić index, the atom-bond connectivity index, the Albertson irregularity) also minimize/maximize the σ-irregularity.

It is easy to see that the greedy tree is always unique. The tree generated by the adopting algorithm is unique when all degrees larger than one occur only once in the degree sequence. However, in general, it is not unique.

Problem 4.1. *Find the sufficient and necessary conditions for the extremal tree with respect to σ-irregularity to be unique.*

When the tree with extremal σ-irregularity is not unique it will be of interest to find a way to enumerate all of them.

Problem 4.2. *Modify the greedy and adopting algorithms such that they generate all trees with minimal, respectively maximal, σ-irregularity.*

Acknowledgements

This work was partially supported by the Slovenian research agency ARRS (Program No. P1-0383 and Project No. J1-3002) and the National Natural Science Foundation of China (Grant No. 12271182 and Grant No. 11771362).

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