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Research Article

Counterexamples to the total vertex irregularity strength's conjectures

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Abstract

The total vertex irregularity strength $\mathrm{tvs}(G)$ of a simple graph $G(V,E)$ is the smallest positive integer k so that there exists a function $\varphi : V \cup E \to [1,k]$ provided that all vertex-weights are distinct, where a vertex-weight is the sum of labels of a vertex and all of its incident edges. In the paper [Nurdin, E. T. Baskoro, A. N. M. Salman, N. N. Gaos, *Discrete Math.* **310** (2010) 3043–3048], two conjectures regarding the total vertex irregularity strength of trees and general graphs were posed as follows: (i) for every tree T, $\mathrm{tvs}(T) = \max\{\lceil (n_1+1)/2\rceil, \lceil (n_1+n_2+1)/3\rceil, \lceil (n_1+n_2+n_3+1)/4\rceil\}$, and (ii) for every graph G with minimum degree δ and maximum degree Δ, $\mathrm{tvs}(G) = \max\{\lceil (\delta + \sum_{j=1}^{i} n_j)/(i+1)\rceil : i \in [\delta, \Delta]\}$, where n_j denotes the number of vertices of degree j. In this paper, we disprove both of these conjectures by giving infinite families of counterexamples.

Keywords: vertex irregular total k-labeling; total vertex irregularity strength; trees; general graphs.

2020 Mathematics Subject Classification: 05C78.

1. Introduction

All (multi)graphs considered here are finite and undirected. We follow [5, 12] for terminologies and notations not defined in this paper. For a (multi)graph G, denote by δ the minimum degree of vertices of G, and by Δ the maximum degree of vertices of G. A vertex that has degree one is called a *pendant vertex*. Any non-pendant vertex of degree at least two is called an *exterior vertex* if it is adjacent to a pendant vertex, otherwise it is called an *interior vertex*. Any edge incident to a pendant vertex is called a *pendant edge*, otherwise it is called an *interior edge*. Given integers a, b with $a \leqslant b$, we write $[a,b] := \{a, a+1, \ldots, b\}$.

Let $G = (V,E)$ be a graph. Let k be a positive integer. We say that a function $\varphi : V \cup E \to [1,k]$ is a *vertex irregular total k-labeling* of G if $wt(u) \neq wt(v)$ for every pair of two vertices u, v of G, where $wt(v)$, the *vertex-weight* of v, is given by $wt(v) = \varphi(v) + \sum_{uv\in E} \varphi(uv)$. The smallest k for which G admits a vertex irregular total labeling is known as the *total vertex irregularity strength* of G and is denoted by $\mathrm{tvs}(G)$.

The notion of total vertex irregularity strength was introduced by Bača et al. [3] inspired by the work of Chartrand et al. [4] who defined the classical irregularity strength of graphs. Since then, this graph invariant has gained a significant interest from many graph theorists around the globe. Some recent results on this subject can be seen in [1, 7, 10, 13]. For a comprehensive survey on graph labelings, one may consult Gallian's survey [6].

In [9], a general lower bound for the total vertex irregularity strength of any tree T with maximum degree Δ was derived as follows:

$$\mathrm{tvs}(T) \geqslant \max\{t_i : i \in [1,\Delta]\}, \tag{1}$$

where

$$t_i = \lceil (\sum_{j=1}^{i} n_j + 1)/(i+1)\rceil$$

and n_j is the number of vertices of degree j. Susanto et al. [14] reduced the number of variables involved in (1) and proved that

$$\mathrm{tvs}(T) \geqslant \max\{t_1, t_2, t_3\}.$$

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For general graphs, it was also proved in [9] that

$$\operatorname{tvs}(G) \geqslant \max\left\{\left\lceil\frac{\delta+\sum_{j=1}^{i} n_j}{i+1}\right\rceil : i \in [\delta, \Delta]\right\}, \tag{2}$$

for every graph G with minimum degree δ and maximum degree Δ. In the same paper, they proposed the two conjectures given below; the first of which is a special case of the second.

Conjecture 1.1. [9] *For every tree T, $\operatorname{tvs}(T) = \max\{t_1, t_2, t_3\}$.*

Conjecture 1.2. [9] *For any graph G consisting of n_i vertices of degree i with minimum degree δ and maximum degree Δ,*

$$\operatorname{tvs}(G) = \max\left\{\left\lceil\frac{\delta+\sum_{j=1}^{i} n_j}{i+1}\right\rceil : i \in [\delta, \Delta]\right\}.$$

Results supporting Conjectures 1.1 and 1.2 have been provided for many tree classes [13, 16–18], and for general graph classes [8, 11]. However, in general, both of these conjectures are still open.

Recently, we characterized all trees having total vertex irregularity strength t_1 [15]. In the quest of characterizing trees with total vertex irregularity strength t_2 and t_3, unexpectedly, we find counterexamples to Conjecture 1.1. In particular, we construct an infinite family of trees with maximum degree at least three having total vertex irregularity strength one more than that of Conjecture 1.1. By utilizing some graph operations, we also obtain an infinite family of general graphs with total vertex irregularity strength one more than that of Conjecture 1.2. Our main purpose in this paper is to prove the following theorem.

Theorem 1.1. *There are infinitely many connected graphs G consisting of n_i vertices of degree i with minimum degree one and maximum degree $\Delta \geqslant 3$ that satisfy*

$$\operatorname{tvs}(G) = \max\left\{\left\lceil\frac{\sum_{j=1}^{i} n_j + 1}{i+1}\right\rceil : i \in [1, \Delta]\right\} + 1.$$

Consequently, there are infinitely many trees T with maximum degree $\Delta \geqslant 3$ that satisfy

$$\operatorname{tvs}(T) = \max\{t_1, t_2, t_3\} + 1.$$

The proof of Theorem 1.1 is given in Section 3. Beforehand, we present in Section 2 some definitions that we need to construct our counterexamples. In Section 4, we give some consequences of Theorem 1.1 and provide some questions for further study.

2. Some definitions

We begin by defining two graph operations that will be needed in constructing our counterexamples.

Definition 2.1. *Let G be a (multi)graph and $uv \in E(G)$. The **D-substitution** of uv is defined by removing uv and adding a digon (two parallel edges) e_1, e_2 with $e_1 = e_2 = xy$, and connecting u to x and v to y.*

Definition 2.2. *Let G be a (multi)graph and $uv, u'v' \in E(G)$, $uv \neq u'v'$. The **E-subdivision** of uv and $u'v'$ is defined by subdividing uv and $u'v'$ once each, and joining the two subdivided vertices.*

Our counterexamples are constructed in three steps, where the second step is defined recursively. The next definition provides the basis of the recursive step.

Definition 2.3. *Let $p \geqslant 2$, $q \in \{0, 1\}$, and $\partial \geqslant 4$. Let T_0 be a tree consisting only of $p + 2 + (\partial - 2)q$ vertices of degree one, p vertices of degree three and q vertices of degree ∂. Define $\mathcal{T}_{p,q}$ as the class of all trees constructed from T_0 by attaching exactly two pendant vertices to every vertex of degree one in T_0.*

Obviously, every $T \in \mathcal{T}_{p,q}$ has order $4p + (3\partial - 5)q + 6$ with maximum degree $\Delta = 3$ if $q = 0$, and $\Delta = \partial$ if $q = 1$. Also, every exterior vertex in T is of degree three. Now we are ready to define the recursive step of our construction.

Definition 2.4. *Let $p \geqslant 2$ and $q \in \{0, 1\}$. Let $\mathcal{G}_{p,q}$ be the class of all (multi)graphs defined recursively as follows.*
Basis step: *Every tree $T \in \mathcal{T}_{p,q}$ is a member of $\mathcal{G}_{p,q}$.*
Recursive step: *If a (multi)graph G is a member of $\mathcal{G}_{p,q}$ then a (multi)graph G' obtained from G by D-substituting r ($\geqslant 0$) edges of G and E-subdividing s ($\geqslant 0$) pairs of two edges of G, where $r + s > 0$, is also a member of $\mathcal{G}_{p,q}$.*

From Definition 2.4, it follows that for every (multi)graph $G \in \mathcal{G}_{p,q}$ there exists a non-negative integer g, a tree $T \in \mathcal{T}_{p,q}$, and a sequence of (multi)graphs $T = G_0, G_1, \ldots, G_g = G$ so that G_{i+1} is obtained from G_i by D-substituting r_i edges of G_i and E-subdividing s_i pairs of two edges of G_i, where $r_i + s_i > 0$ for each i. Moreover, the integer g and the subsequence $G_1, G_2, \ldots, G_{g-1}$ may not be unique. If there are two integers g and g' with corresponding subsequences $G_1, G_2, \ldots, G_{g-1}$ and $G'_1, G'_2 \ldots, G'_{g'-1}$, respectively, then $\sum_{i=0}^{g-1}(r_i + s_i) = \sum_{i=0}^{g'-1}(r'_i + s'_i)$. Figure 1 illustrates two different subsequences G_1 and G'_1, G'_2 in the construction of a graph $G \in \mathcal{G}_{2,0}$ from a basis tree $T \in \mathcal{T}_{2,0}$.

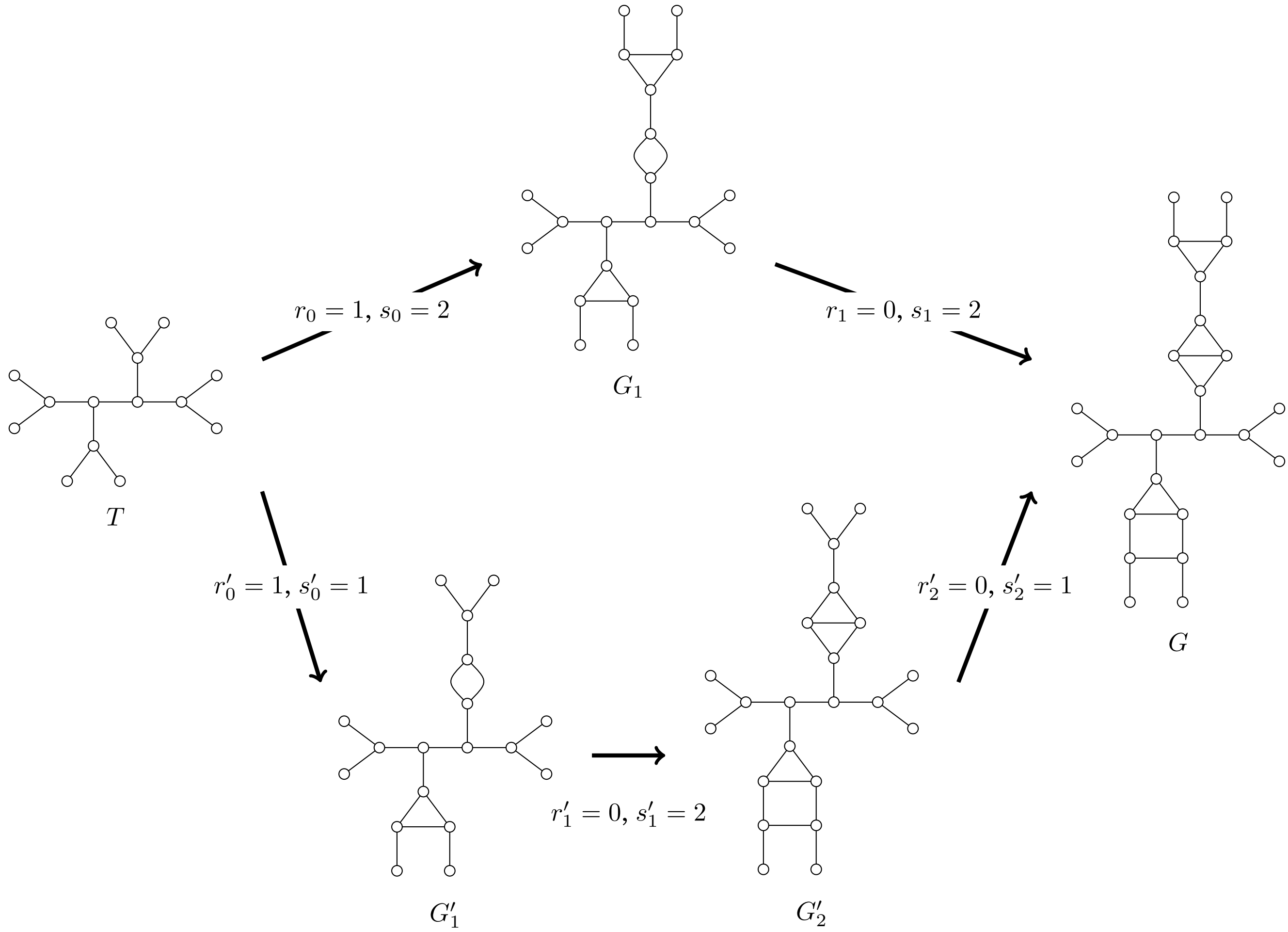

Figure 1: Two different constructions of a (multi)graph $G \in \mathcal{G}_{2,0}$ from a basis tree $T \in \mathcal{T}_{2,0}$.

The last step of the construction is given in the next definition.

Definition 2.5. *Let $p \geqslant 2$ and $q \in \{0, 1\}$. Let $G \in \mathcal{G}_{p,q}$ with sequence $T = G_0, G_1, \ldots, G_g = G$, for some $T \in \mathcal{T}_{p,q}$ and non-negative integer g. Suppose $E_{Int} = E^1_{Int} \cup E^2_{Int}$ be a partition of interior edges of G, where E^1_{Int} is the set of all interior edges joining vertices of degree three, and E^2_{Int} is the set of all other interior edges in G.*

Define G^* as a graph obtained from G by subdividing every edge in E^1_{Int} either one or two times, and subdividing every edge in E^2_{Int} either zero or one time so that G^* contains exactly $4p + (\partial - 2)q + 6\sum_{i=0}^{g-1}(r_i + s_i) + 1$ vertices of degree two but no multiple edges. Denote by $\mathcal{G}^*_{p,q}$ the class of all possible G^* constructed from all $G \in \mathcal{G}_{p,q}$.

We note that the edge subdivisions in Definition 2.5 are always feasible. To see this, let us consider the argument as follows. By simple calculation, the number of interior edges in $T \in \mathcal{T}_{p,q}$ is $2p + (\partial - 1)q + 1$. Furthermore, every D-substitution of an edge in G_i contributes three new interior edges to G_{i+1}, and every E-subdivision of a pair of two edges in G_i contributes three new interior edges to G_{i+1}. So, the number of interior edges in $G \in \mathcal{G}_{p,q}$ is

$$|E^1_{Int}| + |E^2_{Int}| = |E_{Int}| = 2p + (\partial - 1)q + 3\sum_{i=0}^{g-1}(r_i + s_i) + 1.$$

As $|E^2_{Int}| = \partial q$, we have

$$2|E^1_{Int}| + |E^2_{Int}| = 4p + (\partial - 2)q + 6\sum_{i=0}^{g-1}(r_i + s_i) + 2.$$

Thus, if we subdivide all edges in E^1_{Int} twice and $|E^2_{Int}| - 1$ in E^2_{Int} once, or we subdivide $|E^1_{Int}| - 1$ in E^1_{Int} twice, one edge in E^1_{Int} once, and all edges in E^2_{Int} once, then we obtain the required number of degree two in G^*.

Since any interior edge in $G \in \mathcal{G}_{p,q}$ is subdivided at most two times, any vertex of degree two in $G^* \in \mathcal{G}^*_{p,q}$ is always adjacent to a vertex of degree three. Moreover, since any pendant edge in G is not subdivided, any pendant vertex in G^* is always adjacent to a vertex of degree three.

By Definition 2.3, it is clear that any tree $T \in \mathcal{T}_{p,q}$ has $2p + (\partial - 2)q + 2$ vertices of degree three and $p + (\partial - 2)q + 2$ exterior vertices. So, the difference between these two numbers is $p \geqslant 2$. Furthermore, the D-substitution of an edge in G_i increases the number of exterior vertices in G_{i+1} by at most one, and increases the number of vertices of degree three in G_{i+1} by two. Also, the E-subdivision of a pair of two edges in G_i increases the number of exterior vertices in G_{i+1} by at most one, and increases the number of vertices of degree three in G_{i+1} by two.

Let ℓ denotes the number of exterior vertices of $G^* \in \mathcal{G}^*_{p,q}$. Then, by the above argument, we have that $\ell + 2$ is at most $2p + (\partial - 2)q + 2\sum_{i=0}^{g-1}(r_i + s_i) + 2$, the number of vertices of degree three in G^*.

3. Proof of Theorem 1.1

First, we prove the first part of the theorem. Let $G^* \in \mathcal{G}^*_{p,q}$ for $p \geqslant 2$ and $q \in \{0, 1\}$. Our aim is to show that

$$\operatorname{tvs}(G^*) = \max\left\{\left\lceil \frac{\sum_{j=1}^{i} n_j + 1}{i+1} \right\rceil : i \in [1, \Delta]\right\} + 1.$$

We know that G^* has $2p + 2(\partial - 2)q + 4$ vertices of degree one, $4p + (\partial - 2)q + 6\sum_{i=0}^{g-1}(r_i + s_i) + 1$ vertices of degree two, $2p + (\partial - 2)q + 2\sum_{i=0}^{g-1}(r_i + s_i) + 2$ vertices of degree three, and q vertex of degree ∂. With respect to (2),

$$\operatorname{tvs}(G^*) \geqslant \max\left\{\left\lceil \frac{\sum_{j=1}^{i} n_j + 1}{i+1} \right\rceil : i \in [1, \Delta]\right\} = 2p + (\partial - 2)q + 2\sum_{i=0}^{g-1}(r_i + s_i) + 2.$$

Let

$$k = 2p + (\partial - 2)q + 2\sum_{i=0}^{g-1}(r_i + s_i) + 2.$$

For the sake of contradiction, assume that there exists a vertex irregular total k-labeling of G^*. Then the integers $2, 3, \ldots, 4k$ must be realizable as the vertex weights of degrees one, two, and three. In particular, the weights $3k + 1, 3k + 2, \ldots, 4k$ must be assigned to the vertices of degree three since otherwise, a vertex of degree j, $j \in \{1, 2\}$ or one of its incident edges would receive label greater than k, a contradiction.

Let $v_1, v_2, \ldots, v_k$ be the vertices of degree three with $wt(v_i) = 3k + i$ for $i \in [1, k]$. Clearly, the weight 2 must be assigned to a pendant vertex, say u_1. The vertex u_1 must be adjacent to v_1 since otherwise, the vertex v_j for some $j > 1$, or one of its incident edges would receive label greater than k, a contradiction. So v_1 is labeled k, u_1v_1 is labeled 1 and other incident edges of v_1 are labeled k.

Assume that the weight 3 is assigned to a vertex of degree two, say x. Then x is adjacent to v_j for some $j > 1$. As xv_j is labeled 1 and $wt(v_j) > 3k + 1$, v_j or one of its incident edges would receive label greater than k, a contradiction. Thus the weight 3 must be assigned to a pendant vertex, say u_2. By similar argument with the previous one, we get that u_2 must be adjacent to v_2, and so v_2 is labeled k, u_2v_2 is labeled 2 and other incident edges of v_2 are labeled k.

By repeating the above process for the weights $4, 5, \ldots, \ell + 1$, we find that for $i \in [3, \ell]$, the weight $i + 1$ must be assigned to a pendant vertex u_i that is adjacent to v_i, which implies that v_i is labeled k, u_iv_i is labeled i and other incident edges of v_i are labeled k.

As $\ell + 2 \leqslant k$, the weight $\ell + 2$ must be assigned to a vertex of degree two, say y. Then y is adjacent to v_j for some $j > \ell$. Since the label of yv_j is at most ℓ and $wt(v_j) > 3k + \ell$, v_j or one of its incident edges would receive label greater than k, a contradiction. Therefore

$$\operatorname{tvs}(G^*) \geqslant k + 1.$$

Let us construct a total labeling φ on vertices and edges of G^* as follows. Let $v_1, v_2, \ldots, v_\ell$ be the exterior vertices of G^* so that v_i is adjacent to two pendant vertices for $i \in [1, 2p + 2(\partial - 2)q - \ell + 4]$, and v_i is adjacent to one pendant vertex for $i \in [2p + 2(\partial - 2)q - \ell + 5, \ell]$. By the symbol v_{ij} we mean the jth pendant vertex adjacent to v_i. Then, we define

$$\begin{aligned}
\varphi(v_{i1}) &= 1 \quad \text{for } i \in [1, \ell],\\
\varphi(v_{i2}) &= i \quad \text{for } i \in [1, 2p + 2(\partial - 2)q - \ell + 4],\\
\varphi(v_iv_{i1}) &= i \quad \text{for } i \in [1, \ell],\\
\varphi(v_iv_{i2}) &= k \quad \text{for } i \in [1, 2p + 2(\partial - 2)q - \ell + 4],
\end{aligned}$$

$$\varphi(v_i) = k \quad \text{for } i \in [1, \ell].$$

Let $x_1, x_2, \ldots, x_{k-\ell}$ be the interior vertices of degree three in G^*. For $i \in [1, k-\ell]$, let y_i be a vertex of degree two so that y_i is adjacent to x_i and another vertex y'_i of degree two, and $y_i \neq y'_j$ for $i \neq j$. Then, we define

$$\begin{aligned}
\varphi(x_i) &= k+1 \quad \text{for } i \in [1, k-\ell],\\
\varphi(y_i) &= 1 \quad \text{for } i \in [1, k-\ell],\\
\varphi(y'_i) &= 2p + 2(\partial-2)q - \ell + i + 4 \quad \text{for } i \in [1, k-\ell-1],\\
\varphi(y'_{k-\ell}) &= k-\ell,\\
\varphi(x_iy_i) &= \ell + i - 1 \quad \text{for } i \in [1, k-\ell],\\
\varphi(y_iy'_i) &= 1 \quad \text{for } i \in [1, k-\ell-1],\\
\varphi(y_{k-\ell}y'_{k-\ell}) &= 2p + 2(\partial-2)q - \ell + 5.
\end{aligned}$$

Suppose that $z_1, z_2, \ldots, z_{\ell-p-(\partial-2)q-2}, z'_1, z'_2, \ldots, z'_{\ell-p-(\partial-2)q-2}$ are the vertices of degree two such that $z_iz'_i \in E(G^*)$ for $i \in [1, \ell-p-(\partial-2)q-2]$. Then, we define

$$\begin{aligned}
\varphi(z_i) &= 2 \quad \text{for } i \in [1, \ell-p-(\partial-2)q-2],\\
\varphi(z'_i) &= 3 \quad \text{for } i \in [1, \ell-p-(\partial-2)q-2],\\
\varphi(z_iz'_i) &= k + 2p + 2(\partial-2)q - 2\ell + 2i + 2 \quad \text{for } i \in [1, \ell-p-(\partial-2)q-2].
\end{aligned}$$

Finally, we assign k to all the remaining edges, $2, 3, \ldots, k$ to the $(k-1)$ remaining vertices of degree two, and 1 to a vertex of degree ∂ (if exists).

From the above construction, the weights of pendant vertices form the set

$$\{2, 3, \ldots, \ell+1\} \cup \{k+1, k+2, \ldots, k+2p+2(\partial-2)q-\ell+4\}.$$

For the vertices of degree two, three, and ∂ (if exists), their corresponding weights are of the forms $\{\ell+2, \ell+3, \ldots, k\} \cup \{k+2p+2(\partial-2)q-\ell+5, k+2p+2(\partial-2)q-\ell+6, \ldots, 3k\}$, $\{3k+1, 3k+2, \ldots, 4k\}$, and $\{\partial k+1\}$, respectively. Moreover, it is obvious that the labels used in the construction are at most $k+1$. Thus φ is a vertex irregular total $(k+1)$-labeling of G^*, and so

$$\operatorname{tvs}(G^*) = k+1. \tag{3}$$

Next, we prove the second part of the theorem. Let $T \in \mathcal{T}_{p,q}$ for $p \geqslant 2$ and $q \in \{0, 1\}$. Then, T^* is a tree and a member of $\mathcal{G}^*_{p,q}$. Therefore,

$$\operatorname{tvs}(T^*) = \max\left\{\left\lceil \frac{\sum_{j=1}^{i} n_j + 1}{i+1} \right\rceil : i \in [1, \Delta]\right\} + 1 = \max\{t_1, t_2, t_3\} + 1 = 2p + (\partial-2)q + 3.$$

This finishes the proof of the theorem.

An example of a graph $G^* \in \mathcal{G}^*_{2,1}$ with total vertex irregularity strength 15 can be found in Figure 2.

4. Some consequences and questions

Ali et al. [2] defined a modification of a vertex irregular total labeling called a modular vertex irregular total labeling. A total labeling $\varphi : V \cup E \to [1, k]$ of a graph G on n vertices is called a *modular vertex irregular total k-labeling* of G if the *modular total vertex-weights*, $wt(v) = \varphi(v) + \sum_{uv \in E} \varphi(uv) \pmod n$, are all distinct. The least integer k for which there exists a modular vertex irregular total labeling is called the *modular total vertex irregularity strength* of G, denoted by $\operatorname{mtvs}(G)$. Obviously, for a graph G,

$$\operatorname{mtvs}(G) \geqslant \operatorname{tvs}(G). \tag{4}$$

Recall that the set of vertex-weights under the vertex irregular total labeling φ of G^* described in the previous section is $\{2, 3, \ldots, 4k\}$ if $q = 0$, and $\{2, 3, \ldots, 4k, \partial k + 1\}$ if $q = 1$, where $k = 2p + (\partial-2)q + 2\sum_{i=0}^{g-1}(r_i + s_i) + 2$. Evidently, when $q = 0$, the modular total vertex-weights in the set $\{2, 3, \ldots, 4k\}$ are all distinct.

Now let $q = 1$, and $\partial = 4a + b$ for some integers $a \geqslant 1$ and $b \in [0, 3]$. Then $\partial k + 1 = (4a+b)k + 1 \equiv bk + 1 \pmod{4k}$. If $b = 0$ then the modular total vertex-weights in the set $\{2, 3, \ldots, 4k, \partial k + 1\}$ are all distinct. If $b \in [1, 3]$ then we modify the labeling φ in the following way. Let $u_1, u_2, \ldots, u_{4k}$ denote the vertices of G^* so that under the labeling φ,

$$wt(u_i) = i + 1 \quad \text{for } i \in [1, 4k-1],$$

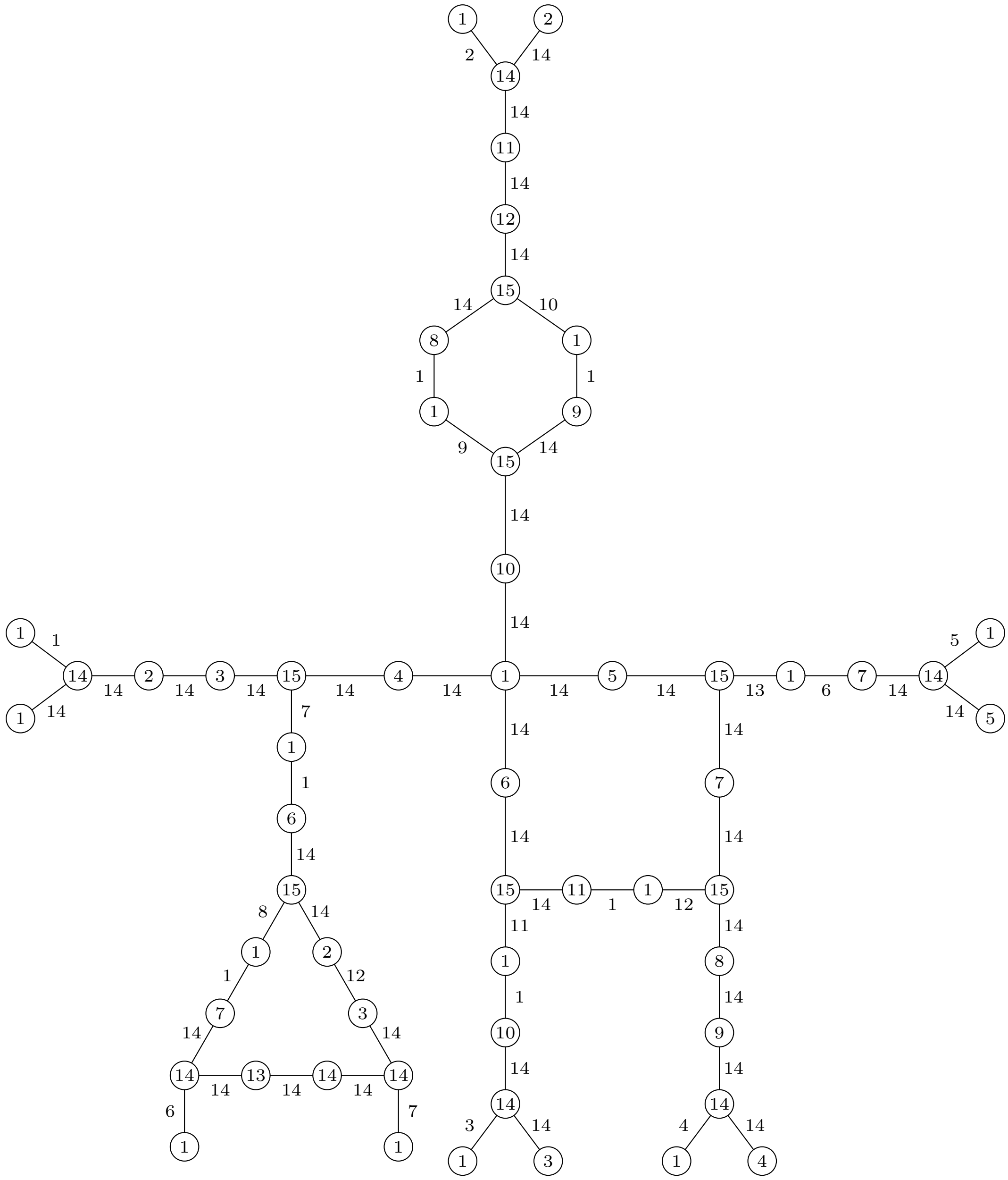

Figure 2: A vertex irregular total 15-labeling of a graph $G^* \in \mathcal{G}^*_{2,1}$.

$$wt(u_{4k}) = \partial k + 1.$$

For $i \in [bk, 3k+\ell-1]$, increase by 1 the label of the vertex u_i. Next, for $i \in [3k+\ell, 4k-1]$ and some $j \in [k+2p+2\partial-\ell+1, 3k-1]$, increase by 1 the label of the edge u_iu_j, and decrease by 1 the label of the vertex u_j.

Then, the weights of the vertices $u_{bk}, u_{bk+1}, \ldots, u_{4k-1}$ increase by 1 and the others remain unchanged, that is

$$\{wt(u_i) : i \in [1, 4k]\} = \{2, 3, \ldots, 4k+1, \partial k + 1\} \backslash \{bk+1\},$$

which means that the modular total vertex-weights in this set are all distinct. Furthermore, the labels used in the modified labeling belong to the set $\{1, 2, \ldots, k+1\}$. Therefore, $\mathrm{mtvs}(G^*) \leqslant k+1$. Combining this with (3) and (4), $\mathrm{mtvs}(G^*) = k+1$. Thus, the next corollary holds.

Corollary 4.1. *Let $p \geqslant 2$ and $q \in \{0, 1\}$. Let $G \in \mathcal{G}_{p,q}$ with sequence $T = G_0, G_1, \ldots, G_g = G$, for some $T \in \mathcal{T}_{p,q}$ and some non-negative integer g. Then, for every graph $G^* \in \mathcal{G}^*_{p,q}$,*

$$\mathrm{mtvs}(G^*) = 2p + (\partial - 2)q + 2\sum_{i=0}^{g-1}(r_i + s_i) + 3.$$

Let $T \in \mathcal{T}_{p,1}$ for $p \geqslant 2$. Clearly, the number of vertices of degree one, two, three, and ∂ in T is $2p + 2\partial$, $4p + \partial - 1$, $2p + \partial$, and 1, respectively. If $\partial = 4$ then

$$n_3 = 2p + 4 = 1 + \frac{4p+3}{2} + \frac{3}{2} = n_4 + \frac{n_2}{2} + \frac{3}{2} \quad \Leftrightarrow \quad n_2 = 2n_3 - 2n_4 - 3.$$

Similarly, if $\partial = 5$ then

$$n_4 = 0 = 2p + 5 - \frac{4p+4}{2} - \frac{3}{2} - \frac{3}{2} = n_3 - \frac{n_2}{2} - \frac{3n_5}{2} - \frac{3}{2} \quad \Leftrightarrow \quad n_2 = 2n_3 - 2n_4 - 3n_5 - 3.$$

Consequently, the results of Susilawati et al. [16] (Theorems 3 and 4) and Susilawati et al. [17] (Theorems 2.3 and 2.4) should be revised. In particular, for a tree T with $t_2 = t_3$, $\mathrm{tvs}(T)$ may not be equal to $\max\{t_1, t_2, t_3\}$. This leads to the following question.

Question 4.1. *What are the necessary and sufficient conditions for a tree T with $\mathrm{tvs}(T) > \max\{t_1, t_2, t_3\}$?*

In Theorem 1.1, we have presented infinite families of trees and general graphs containing pendant vertices, whose the total vertex irregularity strength is one more than that of Conjectures 1.1 and 1.2, respectively. Until now, we could not find graphs without pendant vertices which are counterexamples to Conjecture 1.2, and so we propose the following.

Question 4.2. *Does there exist a graph without pendant vertices which is a counterexample to Conjecture 1.2?*

Finally, we also ask a natural question regarding the upper bound on the total vertex irregularity strength of arbitrary graphs.

Question 4.3. *Does there exist a constant c so that for every graph G consisting of n_i vertices of degree i with minimum degree δ and maximum degree Δ,*

$$\mathrm{tvs}(G) \leqslant \max\left\{\left\lceil \frac{\delta + \sum_{j=1}^{i} n_j}{i+1} \right\rceil : i \in [\delta, \Delta]\right\} + c?$$

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References

[1] A. Ahmad, Computing the total vertex irregularity strength associated with zero divisor graph of commutative ring, *Kragujevac J. Math.* **46** (2022) 711–719.
[2] G. Ali, M. Bača, M. Lascsáková, A. Semaničová-Feňovčíková, A. Aloqaily, N. Mlaiki, Modular total vertex irregularity strength of graphs, *AIMS Math.* **8** (2023) 7662–7671.
[3] M. Bača, S. Jendrol', M. Miller, J. Ryan, On irregular total labellings, *Discrete Math.* **307** (2007) 1378–1388.
[4] G. Chartrand, M. S. Jacobson, J. Lehel, O. R. Oellermann, S. Ruiz, F. Saba, Irregular networks, *Congr. Numer.* **64** (1988) 197–210.
[5] G. Chartrand, L. Lesniak, P. Zhang, *Graphs and Digraphs*, Sixth Edition, CRC Press, New York, 2016.
[6] J. A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.* (2022) #DS6.
[7] M. Imran, A. Ahmad, M. K. Siddiqui, T. Mehmood, Total vertex irregularity strength of generalized prism graphs, *J. Discrete Math. Sci. Cryptogr.* **25** (2022) 1855–1865.
[8] P. Jeyanthi, A. Sudha, Total vertex irregularity strength of some graphs, *Palest. J. Math.* **7** (2018) 725–733.
[9] Nurdin, E. T. Baskoro, A. N. M. Salman, N. N. Gaos, On the total vertex irregularity strength of trees, *Discrete Math.* **310** (2010) 3043–3048.
[10] Rikayanti, E. T. Baskoro, Algorithm for computing the total vertex irregularity strength of some cubic graphs, *Int. J. Math. Comput. Sci.* **16** (2021) 897–905.
[11] Rikayanti, G. B. Tanidi, S. W. Saputro, E. T. Baskoro, The total vertex irregularity strength of symmetric cubic graphs of the Foster's Census, *Indonesian J. Combin.* **6** (2022) 105–119.
[12] K. H. Rosen, *Discrete Mathematics and its Applications*, Seventh Edition, McGraw-Hill, New York, 2012.
[13] R. Simanjuntak, Susilawati, E. T. Baskoro, Total vertex irregularity strength for trees with many vertices of degree two, *Electron. J. Graph Theory Appl.* **8** (2020) 415–421.
[14] F. Susanto, R. Simanjuntak, E. T. Baskoro, A note on vertex irregular total labeling of trees, *Indonesian J. Combin.* **7** (2023) 1–7.
[15] F. Susanto, R. Simanjuntak, E. T. Baskoro, Characterization of trees with total vertex irregularity strength t_1, Submitted.
[16] Susilawati, E. T. Baskoro, R. Simanjuntak, Total vertex irregularity strength of trees with maximum degree four, *AIP Conf. Proc.* **1707** (2016) #020022.
[17] Susilawati, E. T. Baskoro, R. Simanjuntak, Total vertex irregularity strength of trees with maximum degree five, *Electron. J. Graph Theory Appl.* **6** (2018) 250–257.
[18] Susilawati, E. T. Baskoro, R. Simanjuntak, J. Ryan, On the vertex irregular total labeling for subdivision of trees, *Australas. J. Combin.* **71** (2018) 293–302.