Permutations avoiding the complement of a regular permutation group

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Abstract

The class of permutations avoiding the complement of an arbitrary regular permutation group is described.

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1. Introduction

We approach permutations from two different points of view: the algebraic one of permutation groups and the combinatorial one of permutation patterns. These two concepts are well established and thoroughly studied, but they may not seem to have much in common at first sight. An interesting connection was, however, reported and investigated by Pöschel and the current author in [7]: for any subgroup $G$ of the symmetric group $S_n$, the class $Av(S_n \setminus G)$ of permutations avoiding the complement of $G$ in $S_n$ is comprised of levels that are permutation groups.

An interplay between permutation groups and permutation patterns was studied earlier by Atkinson and Beals in their papers [1, 2] on group classes, that is, permutation classes in which every level is a permutation group. More precisely, they showed that the level sequence of any group class eventually coincides with one of only a handful of possible “stable” families of groups. Moreover, they completely and explicitly described those group classes in which every level is a transitive group.

In [6], the current author refined and strengthened Atkinson and Beals’s results on group classes, on the one hand, by examining more carefully the local behaviour of the level sequence of $Av(S_n \setminus G)$, for an arbitrary group $G \leq S_n$, and, on the other hand, by determining how fast this level sequence converges to one of the stable families of groups predicted by Atkinson and Beals. An exact description of the level sequence was discovered for nearly all permutation groups. However, for intransitive or imprimitive groups, only upper and lower bounds were found.

In this paper, we further sharpen the results of Atkinson and Beals [1, 2] and the current author [6] for a particular subfamily of transitive groups. Namely, we will find an exact description of the level sequence of $Av(S_n \setminus G)$ when $G$ is a regular group. Not only does this result improve the earlier work, but also the proof is quite simple and elegant.

2. Preliminaries

We assume that the reader is familiar with basic concepts and terminology related to permutations, permutation groups, and permutation patterns. For general background on these topics, we refer the reader, e.g., to the books by Bóna [3], Dixon and Mortimer [4], and Kitaev [5]. In this section, we will briefly introduce the necessary notions and notation, and we will quote a few lemmas that we need. Proofs and further details are provided in the papers [6, 7], upon which the current paper builds.

We denote the set of nonnegative integers and the set of positive integers by $\mathbb{N}$ and $\mathbb{N}_+$, respectively. For any $\ell, n \in \mathbb{N}_+$, let $[n]$ be the set $\{1, \ldots, n\}$, and let $\binom{n}{\ell}$ be the set of all $\ell$-element subsets of $[n]$.

For any $n \in \mathbb{N}_+$, the set $S_n$ of all permutations of $[n]$, endowed with the operation of composition, constitutes the symmetric group (of degree $n$). Its subgroups are called permutation groups (of degree $n$). The subgroup of $S_n$ generated by a subset $S \subseteq S_n$ is denoted by $\langle S \rangle$. We write $G \leq H$ to express that $G$ is a subgroup of $H$. Permutations $\pi \in S_n$ will often be written as strings $\pi_1 \pi_2 \ldots \pi_n$, where $\pi_i = \pi(i)$ for all $i \in [n]$. The usual cycle notation will also be used. We compose permutations, like all mappings, from right to left, i.e., for $\pi, \tau \in S_n$, $(\pi \circ \tau)(i) = \pi(\tau(i))$ for all $i \in [n]$.

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For any string $a = a_1 a_2 \ldots a_n$ and an index set $I = \{i_1, i_2, \ldots, i_\ell\} \subseteq [n]$ with $i_1 < i_2 < \cdots < i_\ell$, we denote the scattered substring $a_{i_1} a_{i_2} \ldots a_{i_\ell}$ of $a$ by $a[I]$. For any string $u = u_1 u_2 \ldots u_\ell$ of distinct integers, the reduced form of $u$, denoted by $\text{red}(u)$, is the permutation of $[\ell]$ obtained from the string $u$ by replacing its $i$-th smallest entry with $i$, for every $i \in [\ell]$. For $\pi = \pi_1 \pi_2 \ldots \pi_n \in \mathbb{S}_n$ and $I \subseteq [n]$, write $\pi_I := \text{red}(\pi[I])$.

**Example 2.1.** Let $\pi = 642351$ and $I = \{1, 3, 4\}$. Then $\pi[I] = 623$ and $\pi_I = \text{red}(\pi[i]) = 312$.

A permutation $\tau \in S_\ell$ is a pattern of a permutation $\pi \in S_n$, or $\pi$ involves $\tau$, denoted $\tau \preceq \pi$, if $\tau = \pi[I]$ for some $I \subseteq [n]$. The permutation $\pi$ avoids $\tau$ if $\tau \not\preceq \pi$. The pattern involvement relation $\preceq$ is a partial order on the set $\mathbb{P} := \bigcup_{n \geq 1} S_n$ of all finite permutations. Downward closed subsets of $(\mathbb{P}, \preceq)$ are called permutation classes. Permutation classes can be specified with the help of forbidden patterns; for a set $S \subseteq \mathbb{P}$, the set of all permutations avoiding every member of $S$ is denoted by $\text{Av}(S)$. For $C \subseteq \mathbb{P}$ and $n \in \mathbb{N}_+$, the set $\text{Comp}(n) := C \cap S_n$ is called the $n$-th level of $C$.

**Fact 2.1** ([7, Lemma 2.8(iii)]). Assume that $\ell \leq m \leq n$. If $\sigma \in S_{\ell}$, $\tau \in S_n$ and $\sigma \leq \tau$, then there exists $\pi \in S_m$ such that $\sigma \leq \pi \leq \tau$.

Let $\ell, n \in \mathbb{N}_+$ with $\ell \leq n$. For $\tau \in S_n$, we denote by $\text{Pat}(\ell) \tau$ the set of all $\ell$-patterns of $\tau$, i.e., $\text{Pat}(\ell) \tau := \{\pi \in S_\ell : \pi \preceq \tau\}$. We say that a permutation $\tau \in S_n$ is compatible with a set $S \subseteq S_\ell$ of $\ell$-permutations if $\text{Pat}(\ell) \tau \subseteq S$, or, equivalently, if $\tau \in \text{Av}(S_\ell \setminus S)$. For $S \subseteq S_\ell, T \subseteq S_\ell$, we write

\[
\text{Comp}(n) S := \{\tau \in S_n \mid \text{Pat}(\ell) \tau \subseteq S\}, \\
\text{Pat}(\ell) T := \bigcup_{\tau \in T} \text{Pat}(\ell) \tau.
\]

Note that $\text{Comp}(n) S = \text{Av}(S_\ell \setminus S) \cap S_n$. As observed in [7, Section 3], the operators $\text{Comp}(n)$ and $\text{Pat}(\ell)$ are precisely the upper and lower adjoints of the monotone Galois connection (residuation) between the power sets $\mathcal{P}(S_\ell)$ and $\mathcal{P}(S_n)$ induced by the pattern avoidance relation $\preceq$.

The operator $\text{Comp}(n)$ has the following remarkable property, which establishes the connection between permutation groups and permutation patterns that was mentioned in the introduction.

**Proposition 2.2** ([7, Proposition 3.1]). If $G$ is a subgroup of $S_\ell$, then $\text{Comp}(n) G$ is a subgroup of $S_n$.

Using the standard terminology of the theory of permutation patterns, Proposition 2.2 can be rephrased as follows: for any permutation group $G$ of rank $\ell$, the class $\text{Av}(S_\ell \setminus G)$ of all permutations avoiding $S_\ell \setminus G$, the complement of $G$ in $S_\ell$, is comprised of levels that are permutation groups.

The operators $\text{Comp}(n)$ and $\text{Pat}(\ell)$ satisfy the following “transitive property”.

**Lemma 2.1** ([6, Lemma 2.9]). Assume that $\ell \leq m \leq n$. Then for all subsets $S \subseteq S_\ell, T \subseteq S_n$,

\[
\text{Comp}(n) \text{Comp}(m) S = \text{Comp}(n) S, \\
\text{Pat}(\ell) \text{Pat}(m) T = \text{Pat}(\ell) T.
\]

The following permutations will be used many times in what follows:

- the identity permutation $\iota_n := 12 \ldots n$,
- the descending permutation $\delta_n := n(n-1) \ldots 1$,
- the natural cycle $\zeta_n := 23 \ldots n1 = (1 2 \ldots n)$.

The subgroup $\langle \zeta_n \rangle$ of $S_n$ generated by the natural cycle $\zeta_n$ is called the natural cyclic group of degree $n$ and is denoted by $Z_n$. The subgroup $\langle \zeta_n, \delta_n \rangle$ is called the natural dihedral group of degree $n$ and is denoted by $D_n$. The alternating group of degree $n$ is denoted by $A_n$.

**Fact 2.3.** Recall that the reverse of a permutation $\pi \in S_n$ is $\pi^r = \pi \circ \delta_n$, and the complement of $\pi$ is $\pi^c = \delta_n \circ \pi \circ \delta_n$. It is well known that pattern involvement is preserved under inverses, reverses, and complements, and hence also under reverse-complements, i.e.,

\[
\tau \leq \pi \iff \tau^{-1} \leq \pi^{-1}, \quad \tau \preceq \pi \iff \tau^r \preceq \pi^r, \quad \tau \leq \pi \iff \tau^c \leq \pi^c, \quad \tau \preceq \pi \iff \tau^c \preceq \pi^c.
\]

**Lemma 2.2** ([6, Lemma 4.3]). Let $n, m \in \mathbb{N}_+$ with $n \leq m$. Let $G \leq S_n$. Then $\delta_n \in \text{Comp}(m) G$ if and only if $\delta_n \in G$. 
Lemma 2.3 ([6, Lemma 4.5]). Let $G \leq S_n$. Then the following statements hold.

(i) The following statements are equivalent.
(a) $Z_n \leq G$.
(b) $Z_{n+1} \leq \text{Comp}^{(n+1)} G$.
(c) $\text{Comp}^{(n+1)} G$ contains a permutation $\pi \in Z_{n+1} \setminus \{\iota_{n+1}\}$.

(ii) The following statements are equivalent.
(a) $D_n \leq G$.
(b) $D_{n+1} \leq \text{Comp}^{(n+1)} G$.
(c) $\text{Comp}^{(n+1)} G$ contains a permutation $\pi \in D_{n+1} \setminus (Z_{n+1} \cup \{\delta_{n+1}\})$.

Lemma 2.4 ([6, Theorem 4.6]). The following statements hold for all $n \in \mathbb{N}_+$.

(i) $\text{Comp}^{(n+1)} S_n = S_{n+1}$.
(ii) If $n \geq 2$, then $\text{Comp}^{(n+1)} \{\iota_n\} = \{\iota_{n+1}\}$.
(iii) If $n \geq 3$, then $\text{Comp}^{(n+1)} \{\delta_n\} = \{\delta_{n+1}\}$.

Lemma 2.5 ([6, Theorem 6.3]). Let $G \leq S_n$, and assume that $G$ contains the natural cycle $\zeta_n$. Then the following statements hold.

(i) If $D_n \leq G$ and $G \notin \{S_n, A_n\}$, then $\text{Comp}^{(n+1)} G = D_{n+1}$.
(ii) If $D_n \notin G$, then $\text{Comp}^{(n+1)} G = Z_{n+1}$.

Finally, we would like to recall some terminology pertaining to the kinds of permutation groups that are central to our discussion. For a permutation group $G \leq S_n$, and an element $i \in [n]$, the $G$-orbit containing $i$ is the set $\{\pi(i) \mid \pi \in G\}$. The $G$-orbits partition $[n]$. A permutation group $G$ is transitive if it has only one $G$-orbit; otherwise it is intransitive. A permutation group $G \leq S_n$ preserves a partition $\Pi$ of $[n]$ if for every $\pi \in G$ and for every block $B$ of $\Pi$, the set $\pi(B) := \{\pi(b) \mid b \in B\}$ is again a block of $\Pi$. A transitive subgroup of $S_n$ is primitive if it preserves no nontrivial partition of $[n]$; otherwise it is imprimitive. A permutation group $G \leq S_n$ is regular if it is transitive and for every pair $i, j \in [n]$, there exists exactly one $\pi \in G$ such that $\pi(i) = j$. It is immediate from the definition that a regular permutation group of degree $n$ has exactly $n!$ elements.

Example 2.2.

(1) For every $n \in \mathbb{N}_+$, the natural cyclic group $Z_n$ is regular and primitive.

(2) The permutation group $\{1234, 2143, 3412, 4321\}$ is regular and imprimitive, as it preserves the partition $\{\{1, 2\}, \{3, 4\}\}$. (This group is a representation of the Klein four-group.)

3. Permutations compatible with a regular group

The main question studied in [6] is the following: given a permutation group $G$ of rank $n$, describe the sequence

$$G, \text{Comp}^{(n+1)} G, \text{Comp}^{(n+2)} G, \ldots,$$

i.e., the level sequence of the class of all permutations avoiding $S_n \setminus G$. By Proposition 2.2, the levels are permutation groups. An exact description of the sequence (1) was discovered for nearly all permutation groups. However, for intransitive or imprimitive groups, only upper and lower bounds were found. We are now going to sharpen the results of [6] for a particular subfamily of transitive groups (which includes both primitive and imprimitive groups), namely regular groups, and to find an exact description of the sequence (1) of compatible permutation groups in this case.

We start with an auxiliary result that provides insight on the pointwise distinctness of the patterns of a permutation – clearly a necessary condition for a permutation to be compatible with a regular group. In order to simplify notation, for any $\pi \in S_n$ and $i \in [n]$, we write $\pi|_i := \pi_{[n] \setminus \{i\}}$.

Lemma 3.1. Assume that $3 \leq \ell < n$. Let $\pi \in S_n$. Then $\pi$ has distinct $\ell$-patterns $\tau, \tau' \in \text{Pat}(\ell, \pi)$ such that $\tau(i) = \tau'(i)$ for some $i \in [\ell]$ if and only if $\pi \notin D_n$. 

Proof. It is easy to verify that if \( \pi \in D_n \), then any two distinct \( \ell \)-patterns of \( \pi \) are also pointwise distinct. Assume now that \( \pi \in S_n \setminus D_n \). We are going to show that there exist two distinct \( \ell \)-patterns of \( \pi \) that coincide at some point. Since \( D_n = \text{Comp}^{(n)} D_4 \), the permutation \( \pi \) has a 4-pattern \( \tau \) that is not in \( D_4 \). By Fact 2.1, there exists a \( \sigma \in S_{\ell+1} \) such that \( \tau \leq \sigma \leq \pi \). Let \( I \in \left( \mathcal{P} \right) \) be such that \( \tau = \sigma_1 \), i.e., \( I = \{i_1, i_2, i_3, i_4\}, i_1 < i_2 < i_3 < i_4 \) and \( \text{red}(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \tau \).

We need to consider different cases for \( \tau \). Since pattern involvement is preserved by taking inverses, reverses, and complements (see Fact 2.3) and since two permutations coincide at some point if and only if their inverses (reverses, complements) do, it is enough to prove the claim for just one representative of each set of the form

\[
\{\mu, \mu', \mu'', \mu''', \mu^{-1}, (\mu^{-1})', (\mu^{-1}''), (\mu^{-1}''')\},
\]

where \( \mu \in S_4 \setminus D_4 \). These sets are

\[
\{1243, 2134, 3421, 4312\}, \quad \{1324, 4231\}, \quad \{1342, 1423, 2314, 2431, 3124, 3421, 4132, 4213\}, \quad \{2413, 3142\}.
\]

If \( \tau \in \{1243, 2134\} \), then \( \sigma_{i_2}(i_1) = \sigma(i_1) = \sigma_{i_4}(i_1) \) but \( \sigma_{i_2}(i_3 - 1) = \sigma(i_3) - 1 = \sigma_{i_4}(i_3) \), so \( \sigma_{i_2} \) and \( \sigma_{i_4} \) are distinct \( \ell \)-patterns of \( \pi \) that coincide at \( i_1 \).

If \( \tau \in \{1324, 2413\} \), then \( \sigma_{i_2}(i_1) = \sigma(i_1) = \sigma_{i_4}(i_1) \) but \( \sigma_{i_2}(i_3 - 1) = \sigma(i_3) = \sigma_{i_4}(i_3) \), so \( \sigma_{i_2} \) and \( \sigma_{i_4} \) are distinct \( \ell \)-patterns of \( \pi \) that coincide at \( i_1 \).

This completes the proof, because the cases considered above exhaust all possibilities.

\[\square\]

**Lemma 3.2.** Assume that \( G \leq S_n \) is regular and \( G \neq Z_n \). Then \( \text{Comp}^{(n+1)} G \leq \langle \delta_{n+1} \rangle \).

**Proof.** Let \( \pi \in \text{Comp}^{(n+1)} G \). If \( \pi \not\in D_{n+1} \), then, by Lemma 3.1, \( \pi \) has two distinct \( n \)-patterns that coincide at some point. But this is not possible since \( G \) is assumed to be regular.

If \( \pi \in D_{n+1} \setminus \langle Z_{n+1} \cup \langle \delta_{n+1} \rangle \rangle \), then \( D_n \leq G \) by Lemma 2.3. But then \( |G| \geq |D_n| = 2n \); hence \( G \) is not regular, so this case is not possible. If \( \pi \in Z_{n+1} \setminus \{t_{n+1}\} \), then \( Z_n \leq G \) by Lemma 2.3. But the only regular overgroup of \( Z_n \) is \( Z_n \), and we are assuming that \( G \neq Z_n \), so this case is not possible either.

The only remaining possibility is that \( \pi \in \{t_{n+1}, \delta_{n+1}\} = \langle \delta_{n+1} \rangle \).

\[\square\]

We are now ready to state and prove our main result about the sequence (1) for an arbitrary regular group \( G \).

**Theorem 3.1.** Assume that \( n \in \mathbb{N}_+ \) and \( G \leq S_n \) is regular.

(i) If \( 1 \leq n \leq 2 \), then \( \text{Comp}^{(m)} G = S_m \) for every \( m > n \).

(ii) If \( n \geq 3 \) and \( G = Z_n \), then \( \text{Comp}^{(m)} G = Z_m \) for every \( m > n \).

(iii) If \( n \geq 3 \) and \( G \neq Z_n \) and \( \delta_n \not\in G \), then \( \text{Comp}^{(m)} G = \{t_m\} \) for every \( m > n \).

(iv) If \( n \geq 3 \) and \( \delta_n \in G \), then \( \text{Comp}^{(m)} G = \langle \delta_n \rangle \) for every \( m > n \).

**Proof.** (i) For \( 1 \leq n \leq 2 \), the full symmetric group \( S_n \) is the only regular group of rank \( n \), and it holds that \( \text{Comp}^{(m)} S_n = S_m \) for all \( m, n \in \mathbb{N}_+ \), \( n \leq m \) by Lemma 2.4.

(ii) It follows immediately from Lemma 2.5 that \( \text{Comp}^{(n+1)} Z_n = Z_{n+1} \). Hence, by Lemma 2.1, \( \text{Comp}^{(m)} Z_n = Z_m \) for every \( m > n \).

(iii) & (iv) By Lemma 3.2, it holds that \( \text{Comp}^{(n+1)} G \leq \langle \delta_{n+1} \rangle \). This, together with Lemma 2.2, implies that

\[
\text{Comp}^{(n+1)} G = \begin{cases} \{t_{n+1}\} & \text{if } \delta_n \not\in G, \\ \langle \delta_{n+1} \rangle & \text{if } \delta_n \in G. \end{cases}
\]

The statements now follow by Lemmas 2.1 and 2.4.

\[\square\]

The families \( (S_n)_{n \in \mathbb{N}_+} \), \( (Z_n)_{n \in \mathbb{N}_+} \), \( \{t_n\}_{n \in \mathbb{N}_+} \), and \( \langle \delta_n \rangle_{n \in \mathbb{N}_+} \) of symmetric groups, natural cyclic groups, trivial groups, and groups generated by the descending permutation, respectively, are stable (see [1, 2]). Theorem 3.1 reveals that the level sequence of any regular group converges to a stable family of groups in at most one step.

**References**


