Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **12** (2023) 150–153
DOI: 10.47443/dml.2023.141

Research Article

Permutations avoiding the complement of a regular permutation group

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(Received: 28 August 2023. Accepted: 1 November 2023. Published online: 13 November 2023.)

Abstract

The class of permutations avoiding the complement of an arbitrary regular permutation group is described.

Keywords: permutation pattern; pattern avoidance; permutation group; regular group.

2020 Mathematics Subject Classification: 05A05, 05E16, 20B05.

1. Introduction

We approach permutations from two different points of view: the algebraic one of permutation groups and the combinatorial one of permutation patterns. These two concepts are well established and thoroughly studied, but they may not seem to have much in common at first sight. An interesting connection was, however, reported and investigated by Pöschel and the current author in [7]: for any subgroup G of the symmetric group S_n, the class $\mathrm{Av}(S_n \setminus G)$ of permutations avoiding the complement of G in S_n is comprised of levels that are permutation groups.

An interplay between permutation groups and permutation patterns was studied earlier by Atkinson and Beals in their papers [1, 2] on *group classes,* that is, permutation classes in which every level is a permutation group. More precisely, they showed that the level sequence of any group class eventually coincides with one of only a handful of possible "stable" families of groups. Moreover, they completely and explicitly described those group classes in which every level is a transitive group.

In [6], the current author refined and strengthened Atkinson and Beals's results on group glasses, on the one hand, by examining more carefully the local behaviour of the level sequence of $\mathrm{Av}(S_n \setminus G)$, for an arbitrary group $G \leq S_n$, and, on the other hand, by determining how fast this level sequence converges to one of the stable families of groups predicted by Atkinson and Beals. An exact description of the level sequence was discovered for nearly all permutation groups. However, for intransitive or imprimitive groups, only upper and lower bounds were found.

In this paper, we further sharpen the results of Atkinson and Beals [1, 2] and the current author [6] for a particular subfamily of transitive groups. Namely, we will find an exact description of the level sequence of $\mathrm{Av}(S_\ell \setminus G)$ when G is a regular group. Not only does this result improve the earlier work, but also the proof is quite simple and elegant.

2. Preliminaries

We assume that the reader is familiar with basic concepts and terminology related to permutations, permutation groups, and permutation patterns. For general background on these topics, we refer the reader, e.g., to the books by Bóna [3], Dixon and Mortimer [4], and Kitaev [5]. In this section, we will briefly introduce the necessary notions and notation, and we will quote a few lemmas that we need. Proofs and further details are provided in the papers [6, 7], upon which the current paper builds.

We denote the set of nonnegative integers and the set of positive integers by $\mathbb{N}$ and $\mathbb{N}_+$, respectively. For any $\ell, n \in \mathbb{N}_+$, let $[n]$ be the set $\{1, \dots, n\}$, and let $\binom{[n]}{\ell}$ be the set of all ℓ-element subsets of $[n]$.

For any $n \in \mathbb{N}_+$, the set S_n of all permutations of $[n]$, endowed with the operation of composition, constitutes the *symmetric group* (*of degree* n). Its subgroups are called *permutation groups* (*of degree* n). The subgroup of S_n generated by a subset $S \subseteq S_n$ is denoted by $\langle S \rangle$. We write $G \leq H$ to express that G is a subgroup of H. Permutations $\pi \in S_n$ will often be written as strings $\pi_1 \pi_2 \dots \pi_n$, where $\pi_i = \pi(i)$ for all $i \in [n]$. The usual cycle notation will also be used. We compose permutations, like all mappings, from right to left, i.e., for $\pi, \tau \in S_n$, $(\pi \circ \tau)(i) = \pi(\tau(i))$ for all $i \in [n]$.

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For any string $\mathbf{a} = a_1 a_2 \dots a_n$ and an index set $I = \{i_1, i_2, \dots, i_\ell\} \subseteq [n]$ with $i_1 < i_2 < \dots < i_\ell$, we denote the scattered substring $a_{i_1} a_{i_2} \dots a_{i_\ell}$ of $\mathbf{a}$ by $\mathbf{a}[I]$. For any string $\mathbf{u} = u_1 u_2 \dots u_\ell$ of distinct integers, the *reduced form* of $\mathbf{u}$, denoted by $\operatorname{red}(\mathbf{u})$, is the permutation of $[\ell]$ obtained from the string $\mathbf{u}$ by replacing its i-th smallest entry with i, for every $i \in [\ell]$. For $\pi = \pi_1 \pi_2 \dots \pi_n \in S_n$ and $I \subseteq [n]$, write $\pi_I := \operatorname{red}(\pi[I])$.

Example 2.1. *Let* $\pi = 642351$ *and* $I = \{1, 3, 4\}$. *Then* $\pi[I] = 623$ *and* $\pi_I = \operatorname{red}(\pi[i]) = 312$.

A permutation $\tau \in S_\ell$ is a *pattern* of a permutation $\pi \in S_n$, or π *involves* τ, denoted $\tau \leq \pi$, if $\tau = \pi_I$ for some $I \subseteq [n]$. The permutation π *avoids* τ if $\tau \nleq \pi$. The pattern involvement relation $\leq$ is a partial order on the set $\mathbb{P} := \bigcup_{n \geq 1} S_n$ of all finite permutations. Downward closed subsets of $(\mathbb{P}, \leq)$ are called *permutation classes*. Permutation classes can be specified with the help of forbidden patterns; for a set $S \subseteq \mathbb{P}$, the set of all permutations avoiding every member of S is denoted by $\operatorname{Av} S$. For $C \subseteq \mathbb{P}$ and $n \in \mathbb{N}_+$, the set $C^{(n)} := C \cap S_n$ is called the *n-th level* of C.

Fact 2.1 ([7, Lemma 2.8(iii)]). *Assume that* $\ell \leq m \leq n$. *If* $\sigma \in S_\ell$, $\tau \in S_n$ *and* $\sigma \leq \tau$, *then there exists* $\pi \in S_m$ *such that* $\sigma \leq \pi \leq \tau$.

Let $\ell, n \in \mathbb{N}_+$ with $\ell \leq n$. For $\tau \in S_n$, we denote by $\operatorname{Pat}^{(\ell)} \tau$ the set of all ℓ-patterns of τ, i.e., $\operatorname{Pat}^{(\ell)} \tau := \{\pi \in S_\ell : \pi \leq \tau\}$. We say that a permutation $\tau \in S_n$ is *compatible* with a set $S \subseteq S_\ell$ of ℓ-permutations if $\operatorname{Pat}^{(\ell)} \tau \subseteq S$, or, equivalently, if $\tau \in \operatorname{Av}(S_\ell \setminus S)$. For $S \subseteq S_\ell$, $T \subseteq S_n$, we write

$$\operatorname{Comp}^{(n)} S := \{\tau \in S_n \mid \operatorname{Pat}^{(\ell)} \tau \subseteq S\},$$
$$\operatorname{Pat}^{(\ell)} T := \bigcup_{\tau \in T} \operatorname{Pat}^{(\ell)} \tau.$$

Note that $\operatorname{Comp}^{(n)} S = \operatorname{Av}(S_\ell \setminus S) \cap S_n$. As observed in [7, Section 3], the operators $\operatorname{Comp}^{(n)}$ and $\operatorname{Pat}^{(\ell)}$ are precisely the upper and lower adjoints of the monotone Galois connection (residuation) between the power sets $\mathcal{P}(S_\ell)$ and $\mathcal{P}(S_n)$ induced by the pattern avoidance relation $\nleq$.

The operator $\operatorname{Comp}^{(n)}$ has the following remarkable property, which establishes the connection between permutation groups and permutation patterns that was mentioned in the introduction.

Proposition 2.2 ([7, Proposition 3.1]). *If G is a subgroup of S_ℓ, then $\operatorname{Comp}^{(n)} G$ is a subgroup of S_n.*

Using the standard terminology of the theory of permutation patterns, Proposition 2.2 can be rephrased as follows: for any permutation group G of rank ℓ, the class $\operatorname{Av}(S_\ell \setminus G)$ of all permutations avoiding $S_\ell \setminus G$, the complement of G in S_ℓ, is comprised of levels that are permutation groups.

The operators $\operatorname{Comp}^{(n)}$ and $\operatorname{Pat}^{(\ell)}$ satisfy the following "transitive property".

Lemma 2.1 ([6, Lemma 2.9]). *Assume that* $\ell \leq m \leq n$. *Then for all subsets* $S \subseteq S_\ell$, $T \subseteq S_n$,

$$\operatorname{Comp}^{(n)} \operatorname{Comp}^{(m)} S = \operatorname{Comp}^{(n)} S,$$
$$\operatorname{Pat}^{(\ell)} \operatorname{Pat}^{(m)} T = \operatorname{Pat}^{(\ell)} T.$$

The following permutations will be used many times in what follows:

- the *identity permutation* $\iota_n := 12 \dots n$,
- the *descending permutation* $\delta_n := n(n-1) \dots 1$,
- the *natural cycle* $\zeta_n := 23 \dots n1 = (1\ 2\ \cdots\ n)$.

The subgroup $\langle \zeta_n \rangle$ of S_n generated by the natural cycle ζ_n is called the *natural cyclic group* of degree n and is denoted by Z_n. The subgroup $\langle \zeta_n, \delta_n \rangle$ is called the *natural dihedral group* of degree n and is denoted by D_n. The *alternating group* of degree n is denoted by A_n.

Fact 2.3. *Recall that the* reverse *of a permutation* $\pi \in S_n$ *is* $\pi^{\mathrm{r}} = \pi \circ \delta_n$, *and the* complement *of* π *is* $\pi^{\mathrm{c}} = \delta_n \circ \pi$. *The combination of the two is the* reverse-complement *of* π, *i.e.*, $\pi^{\mathrm{rc}} = \delta_n \circ \pi \circ \delta_n$. *It is well known that pattern involvement is preserved under inverses, reverses, and complements, and hence also under reverse-complements, i.e.,*

$$\tau \leq \pi \iff \tau^{-1} \leq \pi^{-1}, \qquad \tau \leq \pi \iff \tau^{\mathrm{r}} \leq \pi^{\mathrm{r}}, \qquad \tau \leq \pi \iff \tau^{\mathrm{c}} \leq \pi^{\mathrm{c}}, \qquad \tau \leq \pi \iff \tau^{\mathrm{rc}} \leq \pi^{\mathrm{rc}}.$$

Lemma 2.2 ([6, Lemma 4.3]). *Let* $n, m \in \mathbb{N}_+$ *with* $n \leq m$. *Let* $G \leq S_n$. *Then* $\delta_m \in \operatorname{Comp}^{(m)} G$ *if and only if* $\delta_n \in G$.

Lemma 2.3 ([6, Lemma 4.5]). *Let $G \leq S_n$. Then the following statements hold.*

(i) *The following statements are equivalent.*

  (a) $Z_n \leq G$.

  (b) $Z_{n+1} \leq \operatorname{Comp}^{(n+1)} G$.

  (c) $\operatorname{Comp}^{(n+1)} G$ *contains a permutation* $\pi \in Z_{n+1} \setminus \{\iota_{n+1}\}$.

(ii) *The following statements are equivalent.*

  (a) $D_n \leq G$.

  (b) $D_{n+1} \leq \operatorname{Comp}^{(n+1)} G$.

  (c) $\operatorname{Comp}^{(n+1)} G$ *contains a permutation* $\pi \in D_{n+1} \setminus (Z_{n+1} \cup \{\delta_{n+1}\})$.

Lemma 2.4 ([6, Theorem 4.6]). *The following statements hold for all $n \in \mathbb{N}_+$.*

(i) $\operatorname{Comp}^{(n+1)} S_n = S_{n+1}$.

(ii) *If $n \geq 2$, then* $\operatorname{Comp}^{(n+1)} \{\iota_n\} = \{\iota_{n+1}\}$.

(iii) *If $n \geq 3$, then* $\operatorname{Comp}^{(n+1)} \langle \delta_n \rangle = \langle \delta_{n+1} \rangle$.

Lemma 2.5 ([6, Theorem 6.3]). *Let $G \leq S_n$, and assume that G contains the natural cycle ζ_n. Then the following statements hold.*

(i) *If $D_n \leq G$ and $G \notin \{S_n, A_n\}$, then* $\operatorname{Comp}^{(n+1)} G = D_{n+1}$.

(ii) *If $D_n \nleq G$, then* $\operatorname{Comp}^{(n+1)} G = Z_{n+1}$.

Finally, we would like to recall some terminology pertaining to the kinds of permutation groups that are central to our discussion. For a permutation group $G \leq S_n$ and an element $i \in [n]$, the *G-orbit* containing i is the set $\{\pi(i) \mid \pi \in G\}$. The G-orbits partition $[n]$. A permutation group G is *transitive* if it has only one G-orbit; otherwise it is *intransitive*. A permutation group $G \leq S_n$ *preserves* a partition Π of $[n]$ if for every $\pi \in G$ and for every block B of Π, the set $\pi(B) := \{\pi(b) \mid b \in B\}$ is again a block of Π. A transitive subgroup of S_n is *primitive* if it preserves no nontrivial partition of $[n]$; otherwise it is *imprimitive*. A permutation group $G \leq S_n$ is *regular* if it is transitive and for every pair $i, j \in [n]$, there exists exactly one $\pi \in G$ such that $\pi(i) = j$. It is immediate from the definition that a regular permutation group of degree n has exactly n elements.

Example 2.2.

(1) *For every $n \in \mathbb{N}_+$, the natural cyclic group Z_n is regular and primitive.*

(2) *The permutation group $\{1234, 2143, 3412, 4321\}$ is regular and imprimitive, as it preserves the partition $\{\{1,2\},\{3,4\}\}$.* (*This group is a representation of the Klein four-group.*)

3. Permutations compatible with a regular group

The main question studied in [6] is the following: given a permutation group G of rank n, describe the sequence

$$G, \ \operatorname{Comp}^{(n+1)} G, \ \operatorname{Comp}^{(n+2)} G, \ \dots, \tag{1}$$

i.e., the level sequence of the class of all permutations avoiding $S_n \setminus G$. By Proposition 2.2, the levels are permutation groups. An exact description of the sequence (1) was discovered for nearly all permutation groups. However, for intransitive or imprimitive groups, only upper and lower bounds were found. We are now going to sharpen the results of [6] for a particular subfamily of transitive groups (which includes both primitive and imprimitive groups), namely regular groups, and to find an exact description of the sequence (1) of compatible permutation groups in this case.

We start with an auxiliary result that provides insight on the pointwise distinctness of the patterns of a permutation – clearly a necessary condition for a permutation to be compatible with a regular group. In order to simplify notation, for any $\pi \in S_n$ and $i \in [n]$, we write $\pi{\restriction}_i := \pi_{[n] \setminus \{i\}}$.

Lemma 3.1. *Assume that $3 \leq \ell < n$. Let $\pi \in S_n$. Then π has distinct ℓ-patterns $\tau, \tau' \in \operatorname{Pat}^{(\ell)} \pi$ such that $\tau(i) = \tau'(i)$ for some $i \in [\ell]$ if and only if $\pi \notin D_n$.*

Proof. It is easy to verify that if $\pi \in D_n$, then any two distinct ℓ-patterns of π are also pointwise distinct. Assume now that $\pi \in S_n \setminus D_n$. We are going to show that there exist two distinct ℓ-patterns of π that coincide at some point. Since $D_n = \operatorname{Comp}^{(n)} D_4$, the permutation π has a 4-pattern τ that is not in D_4. By Fact 2.1, there exists a $\sigma \in S_{\ell+1}$ such that $\tau \leq \sigma \leq \pi$. Let $I \in \binom{[\ell]}{4}$ be such that $\tau = \sigma_I$, i.e., $I = \{i_1, i_2, i_3, i_4\}$, $i_1 < i_2 < i_3 < i_4$ and $\operatorname{red}(\sigma_{i_1}\sigma_{i_2}\sigma_{i_3}\sigma_{i_4}) = \tau$.

We need to consider different cases for τ. Since pattern involvement is preserved by taking inverses, reverses, and complements (see Fact 2.3) and since two permutations coincide at some point if and only if their inverses (reverses, complements) do, it is enough to prove the claim for just one representative of each set of the form

$$\{\mu, \mu^{\mathrm{r}}, \mu^{\mathrm{c}}, \mu^{\mathrm{rc}}, \mu^{-1}, (\mu^{-1})^{\mathrm{r}}, (\mu^{-1})^{\mathrm{c}}, (\mu^{-1})^{\mathrm{rc}}\},$$

where $\mu \in S_4 \setminus D_4$. These sets are

$$\{1243, 2134, 3421, 4312\}, \quad \{1324, 4231\}, \quad \{1342, 1423, 2314, 2431, 3124, 3241, 4132, 4213\}, \quad \{2413, 3142\}.$$

If $\tau \in \{1243, 1342\}$, then $\sigma{\restriction}_{i_2}(i_1) = \sigma(i_1) = \sigma{\restriction}_{i_4}(i_1)$ but $\sigma{\restriction}_{i_2}(i_3 - 1) = \sigma(i_3) - 1 = \sigma{\restriction}_{i_4}(i_3)$, so $\sigma{\restriction}_{i_2}$ and $\sigma{\restriction}_{i_4}$ are distinct ℓ-patterns of π that coincide at i_1.

If $\tau \in \{1324, 2413\}$, then $\sigma{\restriction}_{i_2}(i_1) = \sigma(i_1) = \sigma{\restriction}_{i_4}(i_1)$ but $\sigma{\restriction}_{i_2}(i_3 - 1) = \sigma(i_3) = \sigma{\restriction}_{i_4}(i_3)$, so $\sigma{\restriction}_{i_2}$ and $\sigma{\restriction}_{i_4}$ are distinct ℓ-patterns of π that coincide at i_1.

This completes the proof, because the cases considered above exhaust all possibilities. □

Lemma 3.2. *Assume that $G \leq S_n$ is regular and $G \neq Z_n$. Then $\operatorname{Comp}^{(n+1)} G \leq \langle \delta_{n+1} \rangle$.*

Proof. Let $\pi \in \operatorname{Comp}^{(n+1)} G$. If $\pi \notin D_{n+1}$, then, by Lemma 3.1, π has two distinct n-patterns that coincide at some point. But this is not possible since G is assumed to be regular.

If $\pi \in D_{n+1} \setminus (Z_{n+1} \cup \{\delta_{n+1}\})$, then $D_n \leq G$ by Lemma 2.3. But then $|G| \geq |D_n| = 2n$; hence G is not regular, so this case is not possible. If $\pi \in Z_{n+1} \setminus \{\iota_{n+1}\}$, then $Z_n \leq G$ by Lemma 2.3. But the only regular overgroup of Z_n is Z_n, and we are assuming that $G \neq Z_n$, so this case is not possible either.

The only remaining possibility is that $\pi \in \{\iota_{n+1}, \delta_{n+1}\} = \langle \delta_{n+1} \rangle$. □

We are now ready to state and prove our main result about the sequence (1) for an arbitrary regular group G.

Theorem 3.1. *Assume that $n \in \mathbb{N}_+$ and $G \leq S_n$ is regular.*

(i) *If $1 \leq n \leq 2$, then $\operatorname{Comp}^{(m)} G = S_m$ for every $m > n$.*

(ii) *If $n \geq 3$ and $G = Z_n$, then $\operatorname{Comp}^{(m)} G = Z_m$ for every $m > n$.*

(iii) *If $n \geq 3$ and $G \neq Z_n$ and $\delta_n \notin G$, then $\operatorname{Comp}^{(m)} G = \{\iota_m\}$ for every $m > n$.*

(iv) *If $n \geq 3$ and $\delta_n \in G$, then $\operatorname{Comp}^{(m)} G = \langle \delta_n \rangle$ for every $m > n$.*

Proof. (i) For $1 \leq n \leq 2$, the full symmetric group S_n is the only regular group of rank n, and it holds that $\operatorname{Comp}^{(m)} S_n = S_m$ for all $n, m \in \mathbb{N}_+$, $n \leq m$ by Lemma 2.4.

(ii) It follows immediately from Lemma 2.5 that $\operatorname{Comp}^{(n+1)} Z_n = Z_{n+1}$. Hence, by Lemma 2.1, $\operatorname{Comp}^{(m)} Z_n = Z_m$ for every $m > n$.

(iii) & (iv) By Lemma 3.2, it holds that $\operatorname{Comp}^{(n+1)} G \leq \langle \delta_{n+1} \rangle$. This, together with Lemma 2.2, implies that

$$\operatorname{Comp}^{(n+1)} G = \begin{cases} \{\iota_{n+1}\} & \text{if } \delta_n \notin G, \\ \langle \delta_{n+1} \rangle & \text{if } \delta_n \in G. \end{cases}$$

The statements now follow by Lemmas 2.1 and 2.4. □

The families $(S_n)_{n \in \mathbb{N}_+}$, $(Z_n)_{n \in \mathbb{N}_+}$, $(\{\iota_n\})_{n \in \mathbb{N}_+}$, and $(\langle \delta_n \rangle)_{n \in \mathbb{N}_+}$ of symmetric groups, natural cyclic groups, trivial groups, and groups generated by the descending permutation, respectively, are stable (see [1, 2]). Theorem 3.1 reveals that the level sequence of any regular group converges to a stable family of groups in at most one step.

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