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Research Article

Variant alternating Euler sums of higher order

Anthony Sofo[1,2,*]

[1] *College of Engineering and Science, Victoria University, Melbourne, Australia*
[2] *School of Science, Royal Melbourne Institute of Technology University, Melbourne, Australia*

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Abstract

A family of alternating variant Euler sums of higher order is investigated. A number of different examples concerning the main theorem are given. A Log-PolyLog integral in terms of special functions is also evaluated.

Keywords: variant Euler sums; alternating harmonic sums; polygamma functions; Riemann zeta functions.

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1. Introduction, preliminaries, and notations

There are many particular cases describing the representation of alternating variant Euler sums in closed form; for instance, see [3, 4, 10, 11]. The aim of this paper is to collect all these individual results and present them in a unifying general theorem describing the general nature in terms of parameter values. From this unifying theorem all the particular published examples follow directly. In this regard we study alternating variant Euler sums of the form

$$\frac{1}{2^{1+p}}\mathrm{S}^{+-}_{t,p+1}\left(\frac{1}{2}\right)=\sum_{n=1}^{\infty}(-1)^n\frac{H_n^{(t)}}{(n+\frac{1}{2})^{p+1}}\qquad(\,n,p,\,t\in\mathbb{N}).\tag{1}$$

In this investigation we let $\mathbb{N},\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$ denote the sets of positive integers, complex numbers, real numbers, rational numbers, and integers, respectively. The notation introduced by Flajolet and Salvy [5]

$$\begin{aligned}\mathrm{S}^{++}_{p,q}&=\sum_{n=1}^{\infty}\frac{H_n^{(p)}}{n^q},\ \mathrm{S}^{+-}_{p,q}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n^{(p)}}{n^q},\\ \frac{1}{2^q}\mathrm{S}^{++}_{p,q}\left(\frac{1}{2}\right)&=\sum_{n=1}^{\infty}\frac{H_n^{(p)}}{(n+\frac{1}{2})^q},\ \frac{1}{2^q}\mathrm{S}^{++}_{p,q}\left(\frac{1}{2}\right)=\sum_{n=1}^{\infty}\frac{H_n^{(p)}}{(n+\frac{1}{2})^q},\end{aligned}$$

will be employed in this study. The representation of these Euler sums in terms of special functions has its beginnings with Euler in 1742 in his communications with Goldbach. Nielsen [7] continued this area of study and it is now known that $\mathrm{S}^{++}_{p,q}$ can be evaluated in the cases $p=1,p=q,p+q$ odd, $p+q$ even with the pairs $(2,4)$ and $(4,2)$. There also exists the reciprocity identity, see [1] or [15].

$$\mathrm{S}^{++}_{p,q}+\mathrm{S}^{++}_{q,p}=\zeta(p)\zeta(q)+\zeta(p+q).$$

The alternating Euler sum $\mathrm{S}^{+-}_{p,q}$ can also be expressed in terms of special functions for odd weight $p+q$, the pairs $(1,3)\,,(2,2)$ and for $q=1,p\in\mathbb{N}$. The variant Euler sum

$$\mathrm{S}^{+-}_{p,q}\left(\frac{1}{2}\right)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n^{(p)}}{(2n+1)^q}$$

may also be expressed in terms of special functions. In this investigation we explicitly give a closed form representation of the alternating variant Euler sum (1) in terms of special functions in the case of even weight $p+q$. The two case $(p,q)=(1,2)\,,(2,1)$ have been published in the various papers [3, 4]. The evaluation of $\mathrm{S}^{+-}_{p,q}\left(\frac{1}{2}\right)$ would then form the set $\left\{\mathrm{S}^{++}_{p,q},\mathrm{S}^{+-}_{p,q},\mathrm{S}^{++}_{p,q}\left(\frac{1}{2}\right)\text{ and }\mathrm{S}^{+-}_{p,q}\left(\frac{1}{2}\right)\right\}$ of Euler sums which admit a representation in terms of special functions.

The harmonic numbers H_n are given by

$$H_n=\sum_{j=1}^{n}\frac{1}{j}=\gamma+\psi\left(n+1\right)\quad(n\in\mathbb{Z}_{\geqslant0})\quad\text{and}\quad H_0:=0.\tag{2}$$

*E-mail address: anthony.sofo@vu.edu.au

Here γ is the familiar Euler-Mascheroni constant (see, e.g., [21, Section 1.2]) and $\psi(z)$ denotes the digamma (or psi) function defined by

$$\psi(z) := \frac{d}{dz}(\log\Gamma(z)) = \frac{\Gamma'(z)}{\Gamma(z)} \quad (\mathbb{C}\backslash\mathbb{Z}_{<0}),$$

where $\Gamma(z)$ is the Gamma function (see, e.g., [21, Section 1.1]).

The generalized harmonic numbers $H_n^{(t)}(b)$ of order t are defined by

$$H_n^{(t)}(b) := \sum_{j=1}^{n} \frac{1}{(j+b)^t} \quad (t \in \mathbb{C},\ b \in \mathbb{C}\setminus\{-1,-2,-3,\cdots\},\ n \in \mathbb{N}), \tag{3}$$

and $H_n^{(t)} := H_n^{(t)}(0)$ are the harmonic numbers of order t. The Riemann Zeta function $\zeta(z)$ is defined by

$$\zeta(z) := \lim_{n\to\infty} H_n^{(z)} = \sum_{j=1}^{\infty} \frac{1}{j^z} \quad (\Re(z) > 1) \tag{4}$$

and the Dirichlet eta function $\eta(z)$ is given by $\eta(z) = \left(1 - 2^{1-z}\right)\zeta(z)$, where, for $z = 0, 1$ we have

$$\eta(1) = \log 2 \quad \text{and} \quad \eta(0) = \frac{1}{2}. \tag{5}$$

The Dirichlet lambda function $\lambda(s)$ is defined as the term-wise arithmetic mean of the Dirichlet eta function and the Riemann zeta function:

$$\lambda(s) = \frac{\eta(s)+\zeta(s)}{2} = \lim_{n\to\infty} O_n^{(s)} = \sum_{j=1}^{\infty} \frac{1}{(2j-1)^s} \quad (\Re(s) > 1). \tag{6}$$

The Bernoulli numbers B_n and the Euler numbers E_n may be defined via generating functions:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \quad (|z| < 2\pi) \tag{7}$$

and

$$\frac{2e^z}{e^{2z}+1} = \operatorname{sech} z = \sum_{n=0}^{\infty} E_n \frac{z^n}{n!} \quad \left(|z| < \frac{\pi}{2}\right). \tag{8}$$

It is noted that

$$B_{2n+1} = 0 \ \ (n \in \mathbb{N}) \quad \text{and} \quad E_{2n+1} = 0,\ (n \in \mathbb{Z}_{\geqslant 0}). \tag{9}$$

The first few of these Bernoulli and Euler numbers are:

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},\ B_8 = -\frac{1}{30},\ \ldots$$

and

$$E_0 = 1,\ E_2 = -1,\ E_4 = 5,\ E_6 = -61,\ E_8 = 1385,\ \ldots.$$

The polylogarithm function $\mathrm{Li}_p(z)$ of order p is defined by

$$\begin{aligned} \mathrm{Li}_p(z) &= \sum_{j=1}^{\infty} \frac{z^j}{j^p} \quad (|z| \leqslant 1;\ p \geq 2) \\ &= \int_0^z \frac{\mathrm{Li}_{p-1}(x)}{x}\,\mathrm{d}x \quad (p \geq 3). \end{aligned} \tag{10}$$

The dilogarithm function $\mathrm{Li}_2(z)$ is given by

$$\begin{aligned} \mathrm{Li}_2(z) &= \sum_{j=1}^{\infty} \frac{z^j}{j^2} \quad (|z| \leqslant 1) \\ &= -\int_0^z \frac{\log(1-x)}{x}\,\mathrm{d}x. \end{aligned} \tag{11}$$

The polylogarithm function $\mathrm{Li}_p(z)$ of order p in (10) can be extended as follows (see, e.g., [21, p. 198], or [6]):

$$\mathrm{Li}_s(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^s},\ (s \in \mathbb{C} \text{ and } |z| < 1;\ \Re(s) > 1 \text{ and } |z| = 1). \tag{12}$$

The polygamma function $\psi^{(k)}(z)$ defined by

$$\psi^{(k)}(z) := \frac{d^k}{dz^k}\{\psi(z)\} = (-1)^{k+1}\, k! \sum_{r=0}^{\infty} \frac{1}{(r+z)^{k+1}} = (-1)^{k+1}\, k!\, \zeta(k+1, z) \tag{13}$$

$$(k \in \mathbb{N};\;\; z \in \mathbb{C} \setminus \{0, -1, -2, -3, \cdots\})$$

has the recurrence

$$\psi^{(k)}(z+1) = \psi^{(k)}(z) + \frac{(-1)^k\, k!}{z^{k+1}}. \tag{14}$$

The generalized (or Hurwitz) zeta function, $\zeta(s,z)$ is defined by

$$\zeta(s,z) = \sum_{m=0}^{\infty} \frac{1}{(m+z)^s} \quad (\Re(s) > 1,\, z \in \mathbb{C} \setminus \{0, -1, -2, -3, \cdots\})\,. \tag{15}$$

An important property of the generalized (or Hurwitz) zeta function is:

$$\zeta(s,1) = \zeta(s) \quad \text{and} \quad \zeta(s,z) = \zeta(s, n+z) + \sum_{j=0}^{n-1} \frac{1}{(j+z)^s} \quad (n \in \mathbb{N})\,. \tag{16}$$

The Dirichlet beta function $\beta(z)$ is defined by

$$\beta(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^z} \quad (\Re(z) > 0). \tag{17}$$

which admits other representations such as::

$$\beta(z) = 4^{-z}\left\{\zeta\left(z, \frac{1}{4}\right) - \zeta\left(z, \frac{3}{4}\right)\right\} = \frac{i}{2}\left\{\mathrm{Li}_z(-i) - \mathrm{Li}_z(i)\right\}, \tag{18}$$

and $\beta(2)$ is known as Catalan's constant.

Euler sum representation of the form (1), in terms of special functions such as the Riemann zeta function, the Dirichlet beta functions and others are important in various applications of mathematics and to the authors knowledge no representation for the general case (1) exists in the literature.. Other relevant articles on Euler sums include, for example, [2,8,19], and the excellent monographs [20–22]. The papers [9, 14, 16–18, 23] also explored various other Euler sums. The Euler sum (1) cannot be evaluated directly using a present CAS software package.

2. The main theorem

The following main theorem is established.

Theorem 2.1. *Let $p \in \mathbb{Z}_{\geqslant 0}$, $t \in \mathbb{N} \geq 2$ with $p+t$ an odd integer. Then the following formula holds:*

$$\frac{1}{2^p}\mathrm{S}^{+-}_{t,p+1}\left(\frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n\, H_n^{(t)}}{(2n+1)^{p+1}} = \frac{(-1)^p}{2p!}\left(\frac{\pi}{2}\right)^{p+1} |E_p|\, \zeta\,(t)$$

$$+\frac{(-1)^{p+1}}{2p!}\sum_{j=0}^{p} 2^t j!\left(\frac{\pi}{2}\right)^{p+1-j}\binom{p}{j}\binom{t+j-1}{j}|E_{p-j}|\,\lambda\,(t+j) \tag{19}$$
$$+(-1)^p\sum_{j=0}^{\lfloor \frac{t}{2}\rfloor} 2^{t-2j} j!\,\eta\,(2j)\binom{p+t-2j}{t-2j}\beta\,(p+t-2j+1)\,,$$

where $H_n^{(t)}$ are harmonic numbers of order t, E_p are the Euler numbers, $\beta\,(\cdot)$ are the Dirichlet Beta functions, $\lambda\,(\cdot)$ are the Dirichlet lambda functions and $\eta\,(\cdot)$ are the Dirichlet eta functions.

Proof. Let $|a| < 1$ and consider

$$X\,(a,t) = \int_0^{\infty} \frac{x^a\, \mathrm{Li}_t(-x^2)}{1+x^2}\, \mathrm{d}x = \int_0^{\pi/2} \tan^a(\theta)\;\mathrm{Li}_t(-\tan^2(\theta))\, \mathrm{d}\theta, \tag{20}$$

using the properties of the polylog function, and can be easily confirmed on "Mathematica", we find

$$=\frac{\pi}{2}\sec\left(\frac{a\pi}{2}\right)\left(\zeta(t)-\zeta\left(t,\frac{1-a}{2}\right)\right),$$

where $\zeta\left(t,\frac{1-a}{2}\right)$ is the Hurwitz zeta function. We now differentiate p times, with respect to a both sides of the resultant identity, which is permissible since the integrand is uniformly convergent on $|a|<1$. Finally take the limit as a approaches zero, so that we obtain

$$\begin{aligned}\lim_{a->0}\frac{d^p}{da^p}X(a,t) &= \int_0^\infty \log^p(x)\,\frac{\mathrm{Li}_t(-x^2)}{1+x^2}\,\mathrm{d}x\\ &= \frac{\pi}{2}\lim_{a->0}\frac{d^p}{da^p}\left(\sec\left(\frac{a\pi}{2}\right)\left(\zeta(t)-\zeta\left(t,\frac{1-a}{2}\right)\right)\right).\end{aligned} \tag{21}$$

Putting

$$f(x;p,t)=\log^p(x)\,\frac{\mathrm{Li}_t(-x^2)}{1+x^2}$$

we have, from the publication [10], in the case where $p+t$ is of odd order

$$\int_0^\infty f(x;p,t)\,\mathrm{d}x=2\int_0^1 f(x;p,t)\,\mathrm{d}x+2\sum_{j=0}^{\lfloor\frac{t}{2}\rfloor}\frac{2^{t-2j}}{(t-2j)!}\eta(2j)\binom{p+t-2j}{t-2j}\int_0^1\frac{\log^{p+t-2j}(x)}{1+x^2}\,\mathrm{d}x,$$

It is known that (see [10])

$$\int_0^1 f(x;p,t)\,\mathrm{d}x=(-1)^p\,p!\sum_{n=0}^\infty\frac{(-1)^n H_n^{(t)}}{(2n+1)^{p+1}}.$$

and that

$$\int_0^1\frac{\log^{p+t-2j}(x)}{1+x^2}\,\mathrm{d}x=(p+t-2j)!\,\beta(p+t+1-2j).$$

From (21), we may evaluate, using the well known relation

$$\sec(z)=\sum_{j=0}^\infty\frac{(-1)^j E_{2j}z^{2j}}{(2j)!},$$

$$\lim_{a->0}\frac{d^p}{da^p}\left(\sec\left(\frac{a\pi}{2}\right)\,\zeta(t)\right)=\left(\frac{\pi}{2}\right)^p|E_p|\,\zeta(t).$$

Utilizing the definition (15), we have

$$\lim_{a->0}\frac{d^p}{da^p}\left(\zeta\left(t,\frac{1-a}{2}\right)\right)=2^t p!\binom{p+t-1}{p}\lambda(p+t)$$

and so using the binomial expansion on the product of derivatives we have

$$\lim_{a->0}\frac{d^p}{da^p}\left(\sec\left(\frac{a\pi}{2}\right)\left(\zeta(t)-\zeta\left(t,\frac{1-a}{2}\right)\right)\right)=\left(\frac{\pi}{2}\right)^p|E_p|\,\zeta(t)-\sum_{j=0}^p 2^t j!\left(\frac{\pi}{2}\right)^{p+1-j}\binom{p}{j}\binom{t+j-1}{j}|E_{p-j}|\,\lambda(t+j).$$

Combining these results together delivers the result (19) and the proof is finished. The integral (20) is obtained by the substitution $x=\tan(\theta)$ .We note that the special case of $t=1$ is listed in the following corollaries. □

A number of corollaries follow from Theorem 2.1 and we express them in the following results.

Corollary 2.1. *Let $p+1=t$, $t\in\mathbb{N}\geq 2$. Then the following formula holds:*

$$\begin{aligned}\frac{1}{2^{t-1}}\mathrm{S}_{t,t}^{+-}\left(\frac{1}{2}\right)=\sum_{n=1}^\infty\frac{(-1)^n H_n^{(t)}}{(2n+1)^t}&=\frac{(-1)^{t-1}}{2(t-1)!}\left(\frac{\pi}{2}\right)^t|E_{t-1}|\,\zeta(t)\\ &+\frac{(-1)^t}{2(t-1)!}\sum_{j=0}^{t-1}2^t j!\left(\frac{\pi}{2}\right)^{t-j}\binom{t-1}{j}\binom{t+j-1}{j}|E_{t-1-j}|\,\lambda(t+j)\\ &+(-1)^{t-1}\sum_{j=0}^{\lfloor\frac{t}{2}\rfloor}2^{t-2j}j!\,\eta(2j)\binom{2t-2j-1}{t-2j}\beta(2t-2j)\ ,\end{aligned}$$

Proof. Follows directly from Theorem 2.1. □

Corollary 2.2. *Let $p = 0$, and put $t = 2t - 1$, $t \in \mathbb{N}$. Then the following formula holds:*

$$\mathrm{S}^{+-}_{2t-1,1}\left(\frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(2t-1)}}{2n+1} = -2^{2t-3}\pi\eta(2t-1) + \sum_{j=0}^{\lfloor t-\frac{1}{2}\rfloor} 2^{2t-2j-1}\,\eta(2j)\,\beta(2t-2j)\,,$$

Proof. Follows directly from Theorem 2.1, and for the case $t = 1$

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{2n+1} = G - \frac{\pi}{2}\ln 2,$$

where G is the Catalan constant. □

Corollary 2.3. *Let $t = 1$ and replace p by $2p$, $p \in \mathbb{N}$. Then the following formula holds:*

$$\frac{1}{2^{2p+1}}\mathrm{S}^{+-}_{1,2p+1}\left(\frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{(2n+1)^{2p+1}} = \frac{1}{(2p)!}\left(\frac{\pi}{2}\right)^{2p+1} |E_{2p}|\ln 2 - (2p+1)\,\beta(2p+2)$$
$$+\frac{1}{2\,(2p)!}\left(\frac{\pi}{2}\right)^{2p+1}\sum_{j=1}^{p}(-1)^{j+1}\binom{2p}{2j}\frac{2^{4j}}{\zeta(2j)}|E_{2p-2j}|\;B_{2j}\;\lambda(2j+1)\,, \tag{22}$$

here B_j are the Bernoulli numbers, E_j are the Euler numbers and $\lambda(j)$ are the Dirichlet Lambda functions and are defined in the introduction. (This result first appeared in [11].)

Proof. Follows directly from Theorem 2.1 and we have used the famous Euler identity

$$\zeta(2n) = \frac{(-1)^{n+1} B_{2n}(2\pi)^{2n}}{2\,(2n)!}.$$

□

Some particular instances of the above identities are demonstrated in the following example.

Example 2.1. *From Theorem 2.1, with $t = 3, p = 4$ we have that*

$$\frac{1}{2^4}\mathrm{S}^{+-}_{3,5}\left(\frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n H_n^{(3)}}{(2n+1)^5} = \frac{1}{48}\left(\frac{\pi}{2}\right)^5 |E_4|\,\zeta(3)$$
$$-\frac{1}{48}\sum_{j=0}^{4} 2^3 j!\left(\frac{\pi}{2}\right)^{5-j}\binom{4}{j}\binom{2+j}{j}|E_{4-j}|\,\lambda(3+j)$$
$$+24\sum_{j=0}^{1} 2^{3-2j} j!\,\eta(2j)\binom{7-2j}{3-2j}\beta(8-2j)\,,$$
$$= \quad 140\beta(8) + 10\eta(2)\,\beta(6) - \frac{\pi}{2}\left(\frac{5}{128}\pi^4\zeta(3) + \frac{93}{32}\pi^2\zeta(5) + \frac{1905}{32}\zeta(7)\right).$$

The following identity was given in [12]

$$\sum_{n=0}^{\infty} \frac{(-1)^n H_n^{(2)}}{2n+1} = 2G\ln 2 - \frac{11\pi^3}{96} - \frac{\pi\ln^2 2}{8} + 4\mathcal{G}$$

where

$$\mathcal{G} = \frac{1}{2}\int_0^1 \frac{\log^2(1-x)}{1+x^2}\,dx = \operatorname{Im}\left(Li_3\left(\frac{1+i}{2}\right)\right) \approx .570077$$

and in [18] the equivalent expression

$$\mathcal{G} = \sum_{n\geq 1}\frac{\sin\left(\frac{n\pi}{4}\right)}{2^{n/2}n^3}$$

was given. The related symmetrical result

$$\sum_{n=0}^{\infty} \frac{(-1)^n H_n}{(2n+1)^2} = 2\mathcal{G} - \frac{\pi^3}{64} - \frac{\pi\ln^2 2}{16} - G\ln 2$$

is valid.

The following lemma was given in [13].

Lemma 2.1. *Let* $a \in \mathbb{C}\backslash\{0,-1,-2,-3,\cdots\}$, $p,t \in \mathbb{N}, t \geq 2$ *with* $p+t$ *of odd weight, then*

$$\sum_{n\geqslant 1}\frac{(-1)^n\ H_{an}^{(t)}}{(2n+1)^{p+1}} = \zeta(t)\beta(p+1) + \frac{(-1)^{p+t}}{p!\,(t-1)!}\int_0^1\int_0^1 \frac{\ln^{t-1}(x)\ln^p(y)}{(1-x)(1+x^a y^2)}\,\mathrm{d}x\mathrm{d}y,$$

which, in terms of a first integral can be written as

$$\sum_{n\geqslant 1}\frac{(-1)^n\ H_{an}^{(t)}}{(2n+1)^{p+1}} = \zeta(t)\beta(p+1) + \frac{(-1)^p\, i}{2\,(t-1)!}\int_0^1 \frac{\ln^{t-1}(x)\left(\mathrm{Li}_{p+1}(-ix^{a/2})-\mathrm{Li}_{p+1}(ix^{a/2})\right)}{(1-x)\,x^{a/2}}\,\mathrm{d}x.$$

Remark 2.1. *If we now consider Theorem 2.1 and apply Lemma 2.1, for* $a=1$ *we may evaluate that*

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(t)}}{(2n+1)^{p+1}} = \zeta(t)\beta(p+1) + \frac{(-1)^{p+t}}{2p!\,(t-1)!}\int_0^1\frac{\ln^{t-1}(x)}{(1-x)}\,\mathrm{d}x\left(\int_0^1\left(\frac{\ln^p(y)}{1-iy\sqrt{x}}+\frac{\ln^p(y)}{1+iy\sqrt{x}}\right)\mathrm{d}y\right)$$

$$= \zeta(t)\beta(p+1) + \frac{(-1)^p\, i}{2\,(t-1)!}\int_0^1\frac{\ln^{t-1}(x)}{(1-x)\sqrt{x}}\left[\sum_{j=1}^{p+1}\frac{(-1)^{j+1}\ln^{p+1-j}(y)\left(\mathrm{Li}_j(-iy\sqrt{x})-\mathrm{Li}_j(iy\sqrt{x})\right)}{(p+1-j)!}\right]_0^1\mathrm{d}x$$

$$= \zeta(t)\beta(p+1) + \frac{(-1)^p\, i}{2\,(t-1)!}\int_0^1\frac{\ln^{t-1}(x)\left(\mathrm{Li}_{p+1}(-i\sqrt{x})-\mathrm{Li}_{p+1}(i\sqrt{x})\right)}{(1-x)\sqrt{x}}\,\mathrm{d}x. \tag{23}$$

Now matching (23) with (19) we find the remarkable integral identity

$$\frac{(-1)^p\, i}{2\,(t-1)!}\int_0^1\frac{\ln^{t-1}(x)\left(\mathrm{Li}_{p+1}(-i\sqrt{x})-\mathrm{Li}_{p+1}(i\sqrt{x})\right)}{(1-x)\sqrt{x}}\,\mathrm{d}x = -\zeta(t)\beta(p+1)$$

$$+\frac{(-1)^{p+1}}{2p!}\sum_{j=0}^{p}2^t j!\left(\frac{\pi}{2}\right)^{p+1-j}\binom{p}{j}\binom{t+j-1}{j}|E_{p-j}|\,\lambda(t+j)$$

$$+(-1)^p\sum_{j=0}^{\lfloor \frac{t}{2}\rfloor}2^{t-2j}j!\binom{p+t-2j}{t-2j}\eta(2j)\,\beta(p+t-2j+1) + \frac{(-1)^p}{2p!}\left(\frac{\pi}{2}\right)^{p+1}|E_p|\,\zeta(t).$$

The case $(p,t)=(1,1)$ *has been given in [4].*

3. Concluding remarks

We have analyzed the Euler sum (1), in the case of even weight, and have expressed it in terms of special functions adding to the set of known Euler sum representation. It may be possible to represent the Euler sum (1) in closed form for odd weight other than weight three, but to the authors knowledge no result of this type exits in the published literature. A remarkable identity for a Log-Polylog integral has also been highlighted.

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