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Research Article

Two series which generalize Dirichlet's lambda and Riemann's zeta functions at positive integer arguments

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Abstract

The series $\sum_{k=0}^{\infty} \frac{G_N(k)}{(2k+1)^r}$ and $\sum_{k=1}^{\infty} \frac{H_N(k)}{k^r}$ are considered, where $G_N(k)$ and $H_N(k)$ are the Borwein-Chamberland sums appeared in the expansions of integer powers of the arcsine reported in the paper [D. Borwein, M. Chamberland, *Int. J. Math. Math. Sci.* **2007** (2007) #1981]. For $3 \le r \in \mathbb{N}$, representations for these series in terms of zeta values are derived, extending a theorem proved in the paper [J. Ewell, *Canad. Math. Bull.* **34** (1991) 60–66]. Several corollaries (especially for the case $r = 3$) are obtained, extending some known representations, including Euler's famous rapidly converging series for $\zeta(3)$. The technique can be applied to the case $r = 2$ and it yields generalizations of the formulas $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

Keywords: Borwein-Chamberland sums; Borwein-Chamberland expansions; Ewell's theorem; Euler's series for $\zeta(3)$; Euler sums.

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1. Introduction

Let $\lambda(s)$ and $\zeta(s)$ be the Dirichlet lambda and the Riemann zeta functions, respectively; i.e., the functions initially defined for $\Re(s) > 1$ as $\lambda(s) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^s}$, $\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$, and then extended to all complex $s \neq 1$ by analytic continuation. An active area of recent research has been the study of zeta and associated functions that are themselves expressed in terms of zeta values. A substantial collection of such results may be found in the book by Srivastava and Choi [12], or in the research-expository article [11]. Historically, the first example of a series in terms of values of the Riemann zeta function was given by Euler [4]. In contemporary notation that series is

$$\zeta(3) \;=\; -\frac{4\pi^2}{7} \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+1)(2k+2)2^{2k}},$$

which was rediscovered by Ewell [5]. The above sum converges more rapidly than $\sum_{k=1}^{\infty} \frac{1}{k^3}$ itself, and exemplifies the interest in such representations.

The aim of the present work is to study two interesting sums arising from the Maclaurin expansions of integer powers of $\arcsin(z)$, discovered in 2007 by Borwein and Chamberland [3]. These expansions are as follows: for odd powers, one has

$$\arcsin^{2N+1}(z) = (2N+1)! \sum_{k=0}^{\infty} \frac{G_N(k)\binom{2k}{k}}{(2k+1)4^k} z^{2k+1}, \tag{1}$$

where

$$G_0(k) = 1, \;\; G_N(k) = \sum_{m_1=0}^{k-1} \frac{1}{(2m_1+1)^2} \sum_{m_2=0}^{m_1-1} \frac{1}{(2m_2+1)^2} \cdots \sum_{m_N=0}^{m_{N-1}-1} \frac{1}{(2m_N+1)^2}; \tag{2}$$

for even powers there holds

$$\arcsin^{2N}(z) = (2N)! \sum_{k=1}^{\infty} \frac{H_N(k)\,4^k}{k^2\binom{2k}{k}} z^{2k}, \tag{3}$$

where

$$H_1(k) = \frac{1}{4}, \;\; H_{N+1}(k) \;=\; \frac{1}{4} \sum_{m_1=1}^{k-1} \frac{1}{(2m_1)^2} \sum_{m_2=1}^{m_1-1} \frac{1}{(2m_2)^2} \cdots \sum_{m_N=1}^{m_{N-1}-1} \frac{1}{(2m_N)^2}. \tag{4}$$

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Let $3 \le r \in \mathbb{N}$. We consider the series $\sum_{k=0}^{\infty} \frac{G_N(k)}{(2k+1)^r}$ and $\sum_{k=1}^{\infty} \frac{H_N(k)}{k^r}$, which generalize $\lambda(r)$ and $\zeta(r)$. We shall derive representations in terms of zeta values for these series, generalizing and extending several known results, including Euler's representation of $\zeta(3)$ mentioned above. Additionally, we investigate a set of sums also motivated by Euler's work, the simplest example of which is the following representation:

$$\sum_{k=1}^{\infty} \left\{ \sum_{m=1}^{k-1} \frac{1}{m^2} \right\} \frac{1}{k^3} = -\frac{11}{2}\zeta(5) + 3\zeta(2)\zeta(3).$$

2. Main results

The reader is referred to [1, 12] for all preliminaries. In what follows, $[\xi]_M$ shall denote the Pochhammer symbol (rising factorial): for $\xi \in \mathbb{C}$, define $[\xi]_0 = 1;\ \ [\xi]_M = \xi(\xi+1)\cdots(\xi+M-1),\ \ \forall M \in \mathbb{N}$. We also set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Throughout the paper, B_n $(n \in \mathbb{N}_0)$ shall denote the Bernoulli numbers. Our first result is a generalization of the main theorem from Ewell's 1991 paper [6].

Theorem 2.1. *For each $m \in \mathbb{N}_0,\ r \in \mathbb{N}$, and for every $N = 0, 1, 2, \ldots$, define the coefficients $A_{2m}(r, N)$ as follows:*

$$A_{2m}(1, N) = [2m+1]_{2N} \cdot B_{2m}, \quad A_{2m}(r, N) = [2m+1]_{2N} \cdot \sum \frac{\binom{2m}{2i_1, \ldots, 2i_r} B_{2i_1} \cdots B_{2i_r}}{(2i_1 + 2N + 1) \cdots (2i_1 + \cdots + 2i_{r-1} + 2N + 1)},$$

where the sum is over all $(i_1, i_2, \ldots, i_r) \in \mathbb{N}_0^r$ such that $i_1 + i_2 + \cdots + i_r = m$. Then, for every integer $r > 2$, it holds that

$$\sum_{k=0}^{\infty} \frac{G_N(k)}{(2k+1)^r} = \frac{1}{(2N+1)!} \left(\frac{\pi}{2}\right)^{2N+2} \sum_{m=0}^{\infty} (-1)^m A_{2m}(r-2, N) \frac{\pi^{2m}}{(2m+2N+2)!}. \tag{5}$$

When $N = 0$, Theorem 2.1 reduces to [6, Theorem 1.2]. For the proof of Theorem 2.1, one needs the following lemma (cf. [6, Lemma 2.1]), which is stated here in a slightly improved form:

Lemma 2.1. *For every $n \in \mathbb{N}$, it holds that*

$$\int_0^t \frac{\left(\arcsin \tau\right)^n}{\tau} d\tau = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} B_{2k}}{(2k)!} \frac{\left(\arcsin t\right)^{2k+n}}{2k+n}.$$

Proof. The result is easily obtained upon integration from 0 to $\arcsin t$ of the identity

$$u^n \cot u = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} B_{2k}}{(2k)!} u^{2k+n-1}$$

and changing the variables in the integral on the left-hand side. □

Proof of Theorem 2.1. We prove by induction over $r \ge 3$ that

$$(2N+1)! \sum_{k=0}^{\infty} \frac{G_N(k)\binom{2k}{k}}{4^k} \frac{x^{2k+1}}{(2k+1)^{r-1}} = \sum_{m=0}^{\infty} (-1)^m\, 2^{2m} A_{2m}(r-2, N) \frac{\left(\arcsin x\right)^{2m+2N+1}}{(2m+2N+1)!}. \tag{6}$$

Indeed, take the expansion of $\arcsin^{2N+1}(t)$, divide both sides by t, integrate both sides from 0 to x, and apply Lemma 2.1:

$$\begin{aligned}
(2N+1)! \sum_{k=0}^{\infty} \frac{G_N(k)\binom{2k}{k}}{4^k} \frac{x^{2k+1}}{(2k+1)^2} &= \int_0^x \frac{\left(\arcsin t\right)^{2N+1}}{t} dt \\
&= \sum_{i=0}^{\infty} (-1)^i \frac{2^{2i} B_{2i}}{(2i)!} \frac{\left(\arcsin x\right)^{2i+2N+1}}{2i+2N+1} \\
&= \sum_{i=0}^{\infty} (-1)^i \frac{2^{2i} B_{2i} \cdot [2i+1]_{2N}}{(2i+2N+1)!} \left(\arcsin x\right)^{2i+2N+1},
\end{aligned}$$

establishing (6) for $r = 3$.

Now, take the last relation, divide both sides by x, integrate both sides from 0 to x, and apply Lemma 2.1 again:

$$\begin{aligned}
(2N+1)!\sum_{k=0}^{\infty}\frac{G_N(k)\binom{2k}{k}}{4^k}\frac{x^{2k+1}}{(2k+1)^3} &= \sum_{i=0}^{\infty}(-1)^i\frac{2^{2i}B_{2i}}{(2i)!}\frac{1}{2i+2N+1}\int_0^x\frac{\Big(\arcsin t\Big)^{2i+2N+1}}{t}dt \\
&= \sum_{i=0}^{\infty}(-1)^i\frac{2^{2i}B_{2i}}{(2i)!}\frac{1}{2i+2N+1}\sum_{j=0}^{\infty}(-1)^j\frac{2^{2j}B_{2j}}{(2j)!}\frac{\Big(\arcsin x\Big)^{2i+2j+2N+1}}{2i+2j+2N+1} \\
&= \sum_{i,j=0}^{\infty}(-1)^{i+j}\frac{2^{2i+2j}B_{2i}B_{2j}}{(2i)!(2j)!}\frac{1}{2i+2N+1}\frac{\Big(\arcsin x\Big)^{2i+2j+2N+1}}{2i+2j+2N+1} \\
&= \sum_{m=0}^{\infty}(-1)^m\frac{2^{2m}}{2m+2N+1}\Big(\arcsin x\Big)^{2m+2N+1}\sum_{i+j=m}\frac{B_{2i}B_{2j}}{(2i)!(2j)!}\frac{1}{2i+2N+1} \\
&= \sum_{m=0}^{\infty}(-1)^m\frac{2^{2m}}{(2m+2N+1)!}\Big(\arcsin x\Big)^{2m+2N+1}\cdot[2m+1]_{2N}\sum_{i+j=m}\frac{\binom{2m}{2i,2j}B_{2i}B_{2j}}{2i+2N+1} \\
&= \sum_{m=0}^{\infty}(-1)^m\,2^{2m}\,A_{2m}(2,N)\,\frac{\Big(\arcsin x\Big)^{2m+2N+1}}{(2m+2N+1)!}.
\end{aligned}$$

Thus (6) is true also for $r=4$. Suppose now that there holds

$$(2N+1)!\sum_{k=0}^{\infty}\frac{G_N(k)\binom{2k}{k}}{4^k}\frac{t^{2k+1}}{(2k+1)^{r-2}} = \sum_{m=0}^{\infty}(-1)^m\,2^{2m}A_{2m}(r-3,N)\frac{\Big(\arcsin t\Big)^{2m+2N+1}}{(2m+2N+1)!}.$$

Repeating the above steps of dividing both sides by t, integrating both sides from 0 to x, and applying Lemma 2.1 yields

$$\begin{aligned}
(2N+1)!\sum_{k=0}^{\infty}\frac{G_N(k)\binom{2k}{k}}{4^k}\frac{x^{2k+1}}{(2k+1)^{r-1}} &= \sum_{l=0}^{\infty}(-1)^l\frac{2^{2l}A_{2l}(r-3,N)}{(2l+2N+1)!}\int_0^x\frac{\Big(\arcsin t\Big)^{2l+2N+1}}{t}dt \\
&= \sum_{l=0}^{\infty}(-1)^l\frac{2^{2l}A_{2l}(r-3,N)}{(2l+2N+1)!}\sum_{j=0}^{\infty}(-1)^j\frac{2^{2j}B_{2j}}{(2j)!}\frac{\Big(\arcsin x\Big)^{2l+2j+2N+1}}{2l+2j+2N+1} \\
&= \sum_{m=0}^{\infty}(-1)^m\frac{2^{2m}\Big(\arcsin x\Big)^{2m+2N+1}}{2m+2N+1}\sum_{l+j=m}\frac{A_{2l}(r-3,N)B_{2j}}{(2l+2N+1)!(2j)!}\frac{1}{2l+2j+2N+1} \\
&= \sum_{m=0}^{\infty}(-1)^m\frac{2^{2m}\Big(\arcsin x\Big)^{2m+2N+1}}{2m+2N+1}\frac{1}{(2m)!}\sum_{l+j=m}\frac{\binom{2m}{2l+2N,2j}}{2l+2N+1}\frac{A_{2l}(r-3,N)B_{2j}}{2l+2j+2N+1} \\
&= \sum_{m=0}^{\infty}(-1)^m\frac{2^{2m}\Big(\arcsin x\Big)^{2m+2N+1}}{(2m+2N+1)!}[2m+1]_{2N}\cdot\sum_{l+j=m}\frac{\binom{2m}{2l+2N,2j}}{2l+2N+1}\frac{A_{2l}(r-3,N)B_{2j}}{2l+2j+2N+1} \\
&= \sum_{m=0}^{\infty}(-1)^m\frac{2^{2m}\Big(\arcsin x\Big)^{2m+2N+1}}{(2m+2N+1)!}A_{2l}(r-2,N),
\end{aligned}$$

and thus (6) is established. Now, put $x=\sin\theta$ in (6) and integrate both sides from 0 to $\frac{\pi}{2}$:

$$\begin{aligned}
(2N+1)!\sum_{k=0}^{\infty}\frac{G_N(k)\binom{2k}{k}}{4^k}\frac{1}{(2k+1)^{r-1}}\int_0^{\frac{\pi}{2}}(\sin\theta)^{2k+1}d\theta &= \sum_{m=0}^{\infty}(-1)^m\,2^{2m}A_{2m}(r-2,N)\,\frac{1}{(2m+2N+1)!}\int_0^{\frac{\pi}{2}}\theta^{2m+2N+1}\,d\theta \\
&= \Big(\frac{\pi}{2}\Big)^{2N+2}\sum_{m=0}^{\infty}(-1)^m A_{2m}(r-2,N)\frac{\pi^{2m}}{(2m+2N+2)!}.
\end{aligned}$$

In the last equation, the Wallis integral $\int_0^{\frac{\pi}{2}}(\sin\theta)^{2k+1}d\theta$ is equal to $\dfrac{4^k}{\binom{2k}{k}}\dfrac{1}{2k+1}$, and hence the required result follows. □

The following companion result to Theorem 2.1 is obtained if the even-power expansions (3) are considered:

Theorem 2.2. *For each $m \in \mathbb{N}_0,\ r \in \mathbb{N}$, and for every $N = 1, 2, 3, \ldots$, define the coefficients $D_{2m}(r, N)$ as follows:*

$$D_{2m}(1,N) \;=\; [2m+1]_{2N-1}\cdot B_{2m}\,, \quad D_{2m}(r,N) \;=\; [2m+1]_{2N-1}\cdot \sum \frac{\binom{2m}{2i_1,\ldots,2i_r} B_{2i_1}\cdots B_{2i_r}}{(2i_1+2N)\cdots(2i_1+\cdots+2i_{r-1}+2N)}\,,$$

where the sum is over all $(i_1, i_2, \ldots, i_r) \in \mathbb{N}_0^r$ such that $i_1 + i_2 + \cdots + i_r = m$. Then, for every integer $r > 2$, it holds that

$$\sum_{k=1}^{\infty} \frac{H_N(k)}{k^r} \;=\; \frac{2^{r-2}}{(2N)!}\left(\frac{\pi}{2}\right)^{2N} \sum_{m=0}^{\infty} (-1)^m D_{2m}(r-2,N)\frac{\pi^{2m}}{(2m+2N+1)!}. \tag{7}$$

Proof. The proof is very similar to that of Theorem 2.1, so we only sketch it. Beginning with the expansion (3) and applying the same steps as in the previous proof, one shows by induction that

$$\frac{(2N)!}{2^{r-2}} \sum_{k=1}^{\infty} \frac{H_N(k)4^k}{\binom{2k}{k}} \frac{x^{2k}}{k^r} \;=\; \sum_{m=0}^{\infty} (-1)^m\, 2^{2m} D_{2m}(r-2,N) \frac{\Big(\arcsin x\Big)^{2m+2N}}{(2m+2N)!}. \tag{8}$$

Setting $x = \sin\theta$ in (8) and integrating both sides from 0 to $\frac{\pi}{2}$ yields

$$\begin{aligned}\frac{(2N)!}{2^{r-2}} \sum_{k=1}^{\infty} \frac{H_N(k)4^k}{\binom{2k}{k}} \frac{1}{k^r} \int_0^{\frac{\pi}{2}} (\sin\theta)^{2k} d\theta \;&=\; \sum_{m=0}^{\infty} (-1)^m\, 2^{2m} D_{2m}(r-2,N)\, \frac{1}{(2m+2N)!} \int_0^{\frac{\pi}{2}} \theta^{2m+2N}\, d\theta \\ &=\; \left(\frac{\pi}{2}\right)^{2N+2} \sum_{m=0}^{\infty} (-1)^m D_{2m}(r-2,N) \frac{\pi^{2m}}{(2m+2N+1)!}.\end{aligned}$$

In the last equation, the Wallis integral $\int_0^{\frac{\pi}{2}} (\sin\theta)^{2k} d\theta$ is equal to $\frac{\pi}{2}\frac{\binom{2k}{k}}{4^k}$, and therefore (7) follows. □

3. Corollaries of the main results

Upon setting $r = 3$ in (5) and (7), respectively, and making use of Euler's formula

$$\zeta(2m) = \frac{(-1)^{m+1} B_{2m}}{2(2m)!}(2\pi)^{2m}, \tag{9}$$

the following corollary is obtained:

Corollary 3.1. *The following fast-converging-series formulas:*

(A). $$\sum_{k=0}^{\infty} \frac{G_N(k)}{(2k+1)^3} \;=\; \frac{-2}{(2N+1)!}\left(\frac{\pi}{2}\right)^{2N+2} \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+2N+1)(2k+2N+2)2^{2k}},$$

(B). $$\sum_{k=1}^{\infty} \frac{H_N(k)}{k^3} \;=\; \frac{-4}{(2N)!}\left(\frac{\pi}{2}\right)^{2N} \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+2N)(2k+2N+1)2^{2k}},$$

hold for $N = 0, 1, 2\cdots$, in (A), and $N = 1, 2, 3\cdots$, in (B).

We note that the series given in Corollary 3.1 are themselves generalizations of two well-known results: setting $N = 0$ in Corollary 3.1(A) gives

$$\zeta(3) \;=\; \frac{-4\pi^2}{7} \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+1)(2k+2)2^{2k}}$$

(first example of such series, given in 1772 by Euler [4]; rediscovered in 1990 by Ewell [5]); setting $N = 1$ in Corollary 3.1(B) gives

$$\zeta(3) \;=\; -2\pi^2 \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+2)(2k+3)2^{2k}}$$

(Zhang and Williams [13], 1993).

Regarding Theorems 2.1 and 2.2, it may be mentioned that with an application of (9), the coefficients $A_{2m}(r,N)$, and also $D_{2m}(r,N)$, can be rewritten in terms of $\zeta(2i_1)\cdots\zeta(2i_r)$ instead of $B_{2i_1}\cdots B_{2i_r}$. After some manipulations, an extension of Corollary 2.2 from Ewell [6] will be obtained, with a similar result arising from Theorem 2.2.

Next, we turn our attention to two other interesting representations of the series

$$\sum_{k=0}^{\infty} \frac{G_N(k)}{(2k+1)^3}, \quad \sum_{k=1}^{\infty} \frac{H_N(k)}{k^3}, \tag{10}$$

respectively – representations which involve finite sums of zeta values. They are obtained by eliminating the rapidly convergent sums which appear in Corollary 3.1, with the help of two formulas given by Srivastava in [10].

Recall that Euler considered sums of the form

$$\sigma_h(s,t) = \sum_{k=1}^{\infty} \left[\frac{1}{1^s} + \frac{1}{2^s} + \cdots + \frac{1}{(k-1)^s} \right] \frac{1}{k^t},$$

which are now called Euler sums, and discovered (without proof) that $\sigma_h(s,t)$ can be evaluated in terms of the ζ-function when $s+t$ is odd (see [2] and the references therein). The series (10) are interesting Euler-type sums that generalize $\sigma_h(2,3)$, with

$$\sum_{k=1}^{\infty} \frac{H_2(k)}{k^3} = \sum_{k=1}^{\infty} \left\{ \sum_{m=1}^{k-1} \frac{1}{m^2} \right\} \frac{1}{k^3}$$

being exactly $\sigma_h(2,3)$.

Corollary 3.2. *We have*

(A). $$\sum_{k=0}^{\infty} \frac{G_N(k)}{(2k+1)^3} = \frac{(-1)^N}{2^{4N+3}}(N+1)\left[2^{2N+3}-1\right]\zeta(2N+3) + \frac{(-1)^{N+1}}{2^{4N+2}} \sum_{j=1}^{N} \frac{j\left(2^{2j}-1\right)}{B_{2N-2j+2}} \zeta(2N-2j+2)\zeta(2j+1),$$

(B). $$\sum_{k=1}^{\infty} \frac{H_N(k)}{k^3} = \frac{(-1)^{N+1}}{2^{4N-1}}\left[(2N-1)2^{2N}-2N\right]\zeta(2N+1) + \frac{(-1)^N}{2^{4N-3}} \sum_{j=1}^{N-1} \frac{j(2j-1)\left(2^{2j}-1\right)}{(2N-2j+1)B_{2N-2j}} \zeta(2N-2j)\zeta(2j+1),$$

for $N = 0,1,2\cdots$, in (A) and $N = 1,2,3\cdots$, in (B).

Proof. The following equations, valid for $n \in \mathbb{N}$, are known: (see [10, Equations (3.31) and (3.32)]):

$$\zeta(2n+1) = (-1)^{n-1} \frac{(2\pi)^{2n}}{(2n)!\left(2^{2n+1}-1\right)} \cdot \left[\sum_{j=1}^{n-1} (-1)^j \binom{2n-1}{2j-1} \frac{(2j)!\left(2^{2j}-1\right)}{(2\pi)^{2j}} \zeta(2j+1) - 2\sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+2n-1)(2k+2n)2^{2k}} \right], \tag{11}$$

$$\zeta(2n+1) = (-1)^{n-1} \frac{(2\pi)^{2n}}{(2n)!\left[(2n-1)2^{2n}-2n\right]} \cdot \left[2n\sum_{j=1}^{n-1} (-1)^j \binom{2n-1}{2j-2} \frac{(2j)!\left(2^{2j}-1\right)}{(2\pi)^{2j}} \zeta(2j+1) - 2\sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+2n)(2k+2n+1)2^{2k}} \right]. \tag{12}$$

Setting $n = N+1$ in (11) gives

$$\zeta(2N+3) = \frac{(-1)^N (2\pi)^{2N+2}}{(2N+2)!\left(2^{2N+3}-1\right)} \cdot \left[\sum_{j=1}^{N} (-1)^j \binom{2N+1}{2j-1} \frac{(2j)!\left(2^{2j}-1\right)}{(2\pi)^{2j}} \zeta(2j+1) - 2\sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+2N+1)(2k+2N+2)2^{2k}} \right]. \tag{13}$$

Solve (13) for $\sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+2N+1)(2k+2N+2)2^{2k}}$ and substitute it in the formula from Corollary 3.1(A); also use Equation (9) to eliminate the even powers of π. Part (A) then follows after some simple manipulations. The proof of Part (B) is similar, with a utilization of the second Srivastava formula (12). □

Setting $N = 0, 1, 2,$ in Corollary 3.2(A) gives:

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} = \frac{7}{8}\zeta(3)\,,$$

$$\sum_{k=0}^{\infty} \left\{ \sum_{m=0}^{k-1} \frac{1}{(2m+1)^2} \right\} \frac{1}{(2k+1)^3} = -\frac{31}{64}\zeta(5) + \frac{9}{32}\zeta(2)\zeta(3)\,,$$

$$\sum_{k=0}^{\infty} \left\{ \sum_{m=0}^{k-1} \frac{1}{(2m+1)^2} \sum_{p=0}^{m-1} \frac{1}{(2p+1)^2} \right\} \frac{1}{(2k+1)^3} = \frac{381}{2048}\zeta(7) - \frac{45}{256}\zeta(2)\zeta(5) + \frac{45}{512}\zeta(4)\zeta(3).$$

Setting $N = 1, 2,$ in Corollary 3.2(B) gives:

$$\sum_{k=1}^{\infty} \frac{1}{k^3} = \zeta(3) \quad \text{and} \quad \sum_{k=1}^{\infty} \left\{ \sum_{m=1}^{k-1} \frac{1}{m^2} \right\} \frac{1}{k^3} = -\frac{11}{2}\zeta(5) + 3\zeta(2)\zeta(3).$$

Note that the last relation (in which the left-hand side is $\sigma_h(2,3)$) is in agreement with the formula

$$\sigma_h(2, 2n-1) = -\frac{2n^2+n+1}{2}\zeta(2n+1) + \zeta(2)\zeta(2n-1) + \sum_{k=1}^{n-1} 2k\zeta(2k+1)\zeta(2n-2k)$$

given by Borwein, Borwein and Girgensohn [2].

4. Two extensions of the Basel series

The simplest application of the Borwein-Chamberland expansions is obtained upon setting $z = \sin\theta$ in (1) and (3), respectively, integrating both sides of the resulting equations from 0 to $\frac{\pi}{2}$, and applying the two Wallis formulas already mentioned; this will result in the two series:

$$\sum_{k=0}^{\infty} \frac{G_N(k)}{(2k+1)^2} = \frac{1}{(2N+2)!}\left(\frac{\pi}{2}\right)^{2N+2} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{H_N(k)}{k^2} = \frac{1}{(2N+1)!}\left(\frac{\pi}{2}\right)^{2N},$$

which are extensions of the classical formulas

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6},$$

respectively.

5. Some notes regarding two related papers

In this section, we give credit to the paper by Muzaffar [8] and to the one by Nimbran, Levrie and Sofo [9], which are closely related to our investigation, with some minimal intersection of results (see below). Both of them utilize similar methods to the ones presented herein, most notably the expansions of powers of arcsine combined with the two Wallis formulas. In [8] the expansions of $(\arcsin x)^q$, with $1 \le q \le 4$, are used to obtain explicit formulae for the sums of several interesting series, and also exact values are calculated for several integrals of type $\int_0^{\pi/2} \theta^m \csc^n\theta \, d\theta$. In [9] various sums are obtained which involve harmonic/odd harmonic numbers and central binomial coefficients, again by utilizing the expansions of powers of the arcsine. Certain definite integrals are solved, for example ones having the form $\int_0^1 \frac{(\arcsin x)^p}{x^q}\, dx$ for various $p, q \le 6$. A host of series and integrals are evaluated in terms of zeta, beta, polylogarithm, polygamma and Clausen functions, and basic constants.

Several special cases of our results have appeared in [8] and [9]. They are as follows: (i) Lemma 2.1 (originally given by Ewell [6]) with $t = 1$ is the first half of Theorem 1 in [9]. (Unfortunately, the paper [9] has no mention of Ewell's important contributions.) (ii) Corollary 3.2(A) with $N = 1$, i.e., the formula $\sum_{k=0}^{\infty} \left\{ \sum_{m=0}^{k-1} \frac{1}{(2m+1)^2} \right\} \frac{1}{(2k+1)^3} = -\frac{31}{64}\zeta(5) + \frac{9}{32}\zeta(2)\zeta(3)$ appears in [9, p. 11]. (iii) An equivalent of the formula $\sum_{k=1}^{\infty} \frac{H_2(k)}{k^2} = \frac{1}{5!}\left(\frac{\pi}{2}\right)^4$ is given in [8, Eqs. (4.4)] and also in [9, eq. (5.16)]. (iv) The formula $\sum_{k=0}^{\infty} \frac{G_1(k)}{(2k+1)^2} = \frac{1}{4!}\left(\frac{\pi}{2}\right)^4 = \frac{\pi^4}{384}$ is given in [8, Eqs. (4.3)] and in [9, p. 11].

6. Concluding remarks

Recently, we gave (see [7]) two interesting representations which express the series $\sum_{k=0}^{\infty} \frac{G_N(k)}{(2k+1)^3}$ and $\sum_{k=1}^{\infty} \frac{H_N(k)}{k^3}$ in terms of Euler-type integrals (log-sine integrals). As particular cases, there follow very simple proofs of Euler's equation

$$\zeta(3) = \frac{2\pi^2}{7} \log 2 + \frac{16}{7} \int_0^{\frac{\pi}{2}} x \log(\sin x)\, \mathrm{d}x$$

and of the similar formula

$$\zeta(3) = \frac{2\pi^2}{9} \log 2 + \frac{16}{3\pi} \int_0^{\frac{\pi}{2}} x^2 \log(\sin x)\, \mathrm{d}x.$$

A fitting conclusion to the present work is the following question which we pose as an open problem: is it possible to find a closed-form evaluation of the sums $\sum_{k=0}^{\infty} \frac{G_N(k)}{(2k+1)^{2m}}$ and $\sum_{k=1}^{\infty} \frac{H_N(k)}{k^{2m}}$ for every $m \in \mathbb{N}$? It is clear that the two series given in Section 4 are the first step in that quest. If such evaluations should be found, they would be generalizations of the famous formula (9) given by Euler.

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