Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **12** (2023) 130–137
DOI: 10.47443/dml.2023.083

Research Article

On zonal and inner zonal labelings of plane graphs of maximum degree 3

Andrew Bowling[1], Ping Zhang[2,*]

[1]*Chemical Engineering Department, University of Minnesota Duluth, Duluth, Minnesota 55812, USA*
[2]*Department of Mathematics, Western Michigan University, Kalamazoo, Michigan 49008-5248, USA*

(Received: 18 April 2023. Received in revised form: 9 August 2023. Accepted: 11 August 2023. Published online: 25 August 2023.)

Abstract

A zonal labeling of a plane graph G is an assignment of the two nonzero elements of the ring $\mathbb{Z}_3$ of integers modulo 3 to the vertices of G such that the sum of the labels of the vertices on the boundary of each region of G is the zero element of $\mathbb{Z}_3$. A plane graph possessing such a labeling is a zonal graph. If there is at most one exception, then the labeling is inner zonal and the graph is inner zonal. In 2019, Chartrand, Egan, and Zhang proved that showing the existence of zonal labelings in all cubic maps is equivalent to giving a proof of the Four Color Theorem. It is shown that every inner zonal cubic map is zonal, thereby establishing an improvement of the 2019 result. It is also shown that (i) while certain 2-connected plane graphs of maximum degree 3 may not be zonal, they must be inner zonal and (ii) no connected cubic plane graph with bridges can be inner zonal.

Keywords: zonal and inner zonal labeling; zonal and inner zonal graph; Four Color Problem; cubic map.

2020 Mathematics Subject Classification: 05C10, 05C15, 05C78.

1. Introduction

Let G be a connected plane graph each of whose vertices is labeled with one of the two nonzero elements 1 and 2 of the ring $\mathbb{Z}_3$ of integers modulo 3. The *value* of a region (zone) R of G is the sum in $\mathbb{Z}_3$ of the labels of the vertices on the boundary of R. Such a labeling of G is said to be a *zonal labeling* if the value of each zone in G is the zero element of $\mathbb{Z}_3$. Let $\mathbb{Z}_3^* = \mathbb{Z}_3 - \{0\} = \{1, 2\}$. Hence, a labeling $\ell : V(G) \to \mathbb{Z}_3^*$ of a plane graph G is *zonal* if the value $\ell(R)$ of each zone R with boundary B, defined by $\ell(R) = \sum_{x \in V(B)} \ell(x)$, is 0 in $\mathbb{Z}_3$. If G admits a zonal labeling, then G is *zonal*. For example, Figure 1 shows the plane graphs $K_4 - e$ and K_4 together with a labeling for each using labels from the set $\mathbb{Z}_3^*$. We can obtain a value of 0 for all zones in K_4 by assigning each vertex of K_4 the label 1 (or by assigning each vertex of K_4 the label 2). For $K_4 - e$, however, we can obtain a value of 0 for the interior zones but not for the exterior zone. In fact, there is no way to obtain the label 0 for all three zones of $K_4 - e$. Thus, K_4 is zonal, while $K_4 - e$ is not.

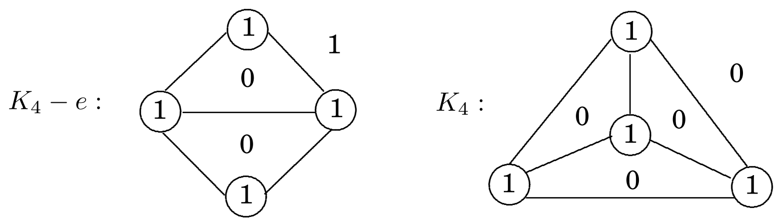

Figure 1: The planar graphs $K_4 - e$ and K_4.

This concept was introduced by Cooroo Egan in 2014 (see [5]) and studied in [2–4, 6]. We refer to the book [7] for graph theory notation and terminology not described in this paper.

There is a close connection between zonal labelings of planar graphs and the famous Four Color Problem:

Can the countries of every map be colored with four or fewer colors so that every two countries with a common boundary line are colored differently?

This problem was introduced in 1852 by the British mathematician Francis Guthrie (see [10], for example). A computer-aided solution to this problem was obtained in 1976 by Appel and Haken [1], resulting in the Four Color Theorem stated on the top of the next page.

*Corresponding author (ping.zhang@wmich.edu).

The regions of every plane graph can be colored with at most four colors so that every two regions with a common boundary line are colored differently.

A connected bridgeless cubic plane graph (or multigraph) is referred to as a *cubic map*. Thus, every cubic map is a 3-regular 2-connected plane graph. For example, the three graphs K_4, $K_3 \,\square\, K_2$ (the Cartesian product of K_3 and K_2) and the 3-cube Q_3 in Figure 2 are cubic maps.

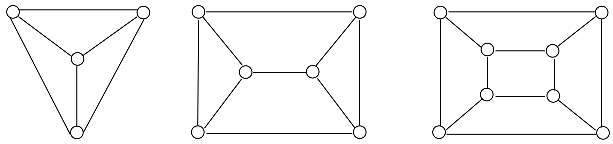

Figure 2: Cubic maps.

Long before the Four Color Problem was solved, the following was known (see [7], for example).

Theorem 1.1. *The Four Color Conjecture is true if and only if the regions of every cubic map can be colored with four or fewer colors so that every two regions with a common boundary line are colored differently.*

Theorem 1.1 then says that to solve the Four Color Theorem, we need only consider cubic maps. The mathematician Peter Tait brought edge colorings to the forefront of the Four Color Problem rather than only coloring regions or vertices. A *proper vertex coloring* of a graph G is a vertex coloring in which every two adjacent vertices are colored differently. The *chromatic number* of G is the minimum number of colors in a proper vertex coloring of G. A *proper edge coloring* of a graph G is one in which every two adjacent edges are colored differently. The *chromatic index* of G is the minimum number of colors in a proper edge coloring of G. In 1880 Tait [9] proved the following.

Theorem 1.2. (Tait's Theorem) *The regions of a cubic map G can be colored with four or fewer colors so that every two adjacent regions are colored differently if and only if the chromatic index of G is 3.*

Tait thought that his theorem would lead to a verification of the Four Color Conjecture as he thought it would be easy to prove that there is a proper coloring of the edges of every cubic map with three colors. But, as it turned out, this problem is equivalent to, and therefore just as difficult as, the Four Color Problem. However, we now know the following edge version of the Four Color Theorem: *The chromatic index of every cubic map is 3.* In 2019, Chartrand, Egan, and Zhang [5] proved that showing the existence of zonal labelings in all cubic maps is equivalent to giving a proof of the Four Color Theorem. More precisely, they proved the following result.

Theorem 1.3. *A cubic map M has chromatic index 3 if and only if M is zonal.*

By Theorems 1.1, 1.2, and 1.3, the Four Color Problem deals with

(1) coloring the regions of a map or the vertices of a planar graph with 4 colors,

(2) properly coloring the edges of a cubic map with 3 colors, and

(3) assigning a zonal labeling to the vertices of a cubic map with the 2 elements in $\mathbb{Z}_3^*$.

Hence, because of the Four Color Theorem, we know that every cubic map is zonal. We also know that if every cubic map has a zonal labeling, then the Four Color Theorem is true. While we know that every cubic map has a zonal labeling (because of the Four Color Theorem), the question is whether one could give an independent proof that every cubic map has a zonal labeling without using the Four Color Theorem. Therefore, Theorem 1.3 provides a new perspective of the Four Color Theorem as well as a new approach to a potentially different solution of the Four Color Problem.

2. Inner zonal labelings of cubic maps

For a plane graph G, a labeling $\ell : V(G) \to \mathbb{Z}_3^*$ is an *inner zonal labeling* of G if the value $\ell(R)$ of every region R of G is the zero element in $\mathbb{Z}_3$ with at most one exception. A plane graph G is *inner zonal* if G has an inner zonal labeling. We may always assume that an inner zonal plane graph G is embedded in the plane so that the value of every interior region of G is 0 in $\mathbb{Z}_3$. Consequently, every zonal plane graph is also inner zonal. We show that every inner zonal cubic map is, in fact, zonal thereby establishing a result that is an improvement of Theorem 1.3. Since showing that every cubic map is inner zonal would imply that every planar graph is 4-colorable, this would indicate that an independent proof of this fact would likely be nontrivial. In order to provide an initial step with this goal in mind, rather than considering cubic maps in general, we consider certain plane graphs that are related to cubic maps, namely connected plane graphs having maximum degree 3. We show that (1) while certain 2-connected plane graphs of maximum degree 3 may not be zonal, they must be inner zonal and (2) no connected cubic plane graph with bridges can be inner zonal.

There are inner zonal plane graphs that are not zonal. For example, the labeling of the nonzonal plane graph $K_4 - e$ given in Figure 1 is an inner zonal labeling and so $K_4 - e$ is inner zonal. On the other hand, every inner zonal cubic map is zonal, as we show next.

Theorem 2.1. *Every inner zonal cubic map is zonal.*

Proof. Let G be an inner cubic zonal map where R_0 is the exterior region and $R_1, R_2, \ldots, R_t$ are the interior regions of G. Let ℓ be an inner zonal labeling of G where $\ell(R_i) = 0$ in $\mathbb{Z}_3$ for $1 \le i \le t$. Therefore,

$$\sum_{i=1}^{t} \ell(R_i) = 0 \text{ in } \mathbb{Z}_3. \tag{1}$$

Since G is cubic and 2-connected, every vertex of G lies on the boundaries of exactly three regions of G. Hence, the sum of the values of all regions $R_0, R_1, R_2, \ldots, R_t$ of G is

$$\sum_{i=0}^{t} \ell(R_i) = \sum_{v \in V(G)} 3\ell(v) = 0 \text{ in } \mathbb{Z}_3. \tag{2}$$

It follows by (1) and (2) that $\ell(R_0) = \sum_{i=0}^{t} \ell(R_i) - \sum_{i=1}^{t} \ell(R_i) = 0$ in $\mathbb{Z}_3$. Thus, ℓ is a zonal labeling of G and so G is zonal. □

Since every zonal plane graph is also inner zonal, the following corollary is a consequence of Theorem 2.1.

Corollary 2.1. *A cubic map is zonal if and only if it is inner zonal.*

Since an inner zonal labeling is less restrictive than a zonal labeling, it may be less challenging to determine the existence of an inner zonal labeling in a cubic map rather than a zonal labeling. Consequently, Corollary 2.1 provides an improvement of the 2019 result (Theorem 1.3) by Chartrand, Egan, and Zhang.

Combining well-known results on graph colorings, results on zonal graphs, and Corollary 2.1, we have the following.

Theorem 2.2. *The following six statements are equivalent.*

1. *The Four Color Theorem: The regions of every plane graph can be colored with four or fewer colors so that every two regions with a common boundary are colored differently.*
2. *The regions of every cubic map can be colored with four or fewer colors so that every two regions with a common boundary are colored differently.*
3. *The chromatic number of every planar graph is at most* 4.
4. *The chromatic index of every cubic map is* 3.
5. *Every cubic map is zonal.*
6. *Every cubic map is inner zonal.*

Proof. By Theorem 1.1, Statements 1, 2 and 3 are equivalent. By Theorem 1.2 (Tait's Theorem), Statements 2 and 4 are equivalent. By Theorem 1.3, Statements 4 and 5 are equivalent. By Corollary 2.1, Statements 5 and 6 are equivalent. Therefore, these six statements are equivalent. □

Because of the Four Color Theorem and Corollary 2.1, we know that every cubic map is, in fact, inner zonal. We also know (without using the Four Color Theorem) that if every cubic map is inner zonal, then the Four Color Theorem is true. Consequently, the question arises as to whether there is an independent proof that shows every cubic map is inner zonal without using the Four Color Theorem. Therefore, as with zonal labelings, inner zonal labelings provide a possible new perspective and approach to the Four Color Problem.

3. Inner zonal labelings of plane graphs of maximum degree 3

We now investigate inner zonality in connected plane graphs having maximum degree 3. First, we introduce some additional definitions and notation as well as some preliminary results. For a labeling ℓ of the vertices of a graph G with the labels 1 and 2 of $\mathbb{Z}_3^*$, the vertex labeling $\bar{\ell}$ of G defined by $\bar{\ell}(v) = 3 - \ell(v) = 2\ell(v)$ for each vertex v of G is called the *complementary labeling* of ℓ. Thus, for every region R of a plane graph G,

$$\bar{\ell}(R) = 2\ell(R). \tag{3}$$

It is immediate that a labeling ℓ of a connected plane graph is zonal if and only if its complementary labeling $\bar{\ell}$ is zonal. This is also the case for inner zonal labelings.

Observation 3.1. *Let $\ell : V(G) \to \mathbb{Z}_3^*$ be a labeling of of a plane graph G. Then ℓ is an inner zonal labeling if and only if its complementary labeling $\bar{\ell}$ is an inner zonal labeling.*

It was observed in [5] that every cycle is zonal and therefore inner zonal. By Corollary 2.1, every cubic map is inner zonal. Both cycles and cubic maps are 2-connected graphs, where the first is 2-regular and the second is 3-regular. This gives rise to the following question: *What happens for 2-connected plane graphs that lie between cycles and cubic maps*? These are 2-connected plane graphs in which every vertex has degree 2 or 3, where there are some vertices of each degree. To investigate this problem, we first investigate those 2-connected plane graphs obtained by (i) adding two vertices of degree 3 (a chord) to a cycle or (ii) adding a vertex of degree 2 to a cubic map. We begin with (i). For an integer $n \geq 4$, let $C_n + e$ be the graph obtained by adding a chord e to the n-cycle C_n. Thus, $C_n + e$ is a 2-connected subgraph of a cubic map. As we saw in Figure 1, the graph $C_4 + e$ is not zonal but it is inner zonal. It was shown in [4] that $C_n + e$ is zonal if and only if $C_n + e$ is triangle-free. Therefore, if the chord e joins two vertices at distance 2 in C_n, then $C_n + e$ is not zonal. On the other hand, for $n \geq 4$, every graph $C_n + e$ is inner zonal. In fact, these graphs belong to a class of inner zonal subgraphs of cubic maps. The following theorem appeared in [5] whose proof is independent of the Four Color Theorem.

Theorem 3.1. *Every plane graph G with maximum degree $\Delta(G) \leq 3$ where the boundary cycle of the exterior zone is a Hamiltonian cycle of G is inner zonal.*

There are nonzonal plane graphs that satisfy the conditions described in Theorem 3.1. For example, consider the plane graph G of Figure 3 where the boundary cycle of the exterior zone is a Hamiltonian cycle of G and $\Delta(G) = 3$. By Theorem 3.1, G is inner zonal. We claim that G is not zonal. Assume, to the contrary, that G has a zonal labeling ℓ. Then $\ell(R_i) = 0$ for $i = 1, 2, 3$ and so $\sum_{i=1}^{9} \ell(v_i) = 0$. Thus, $\ell(R_5) = \sum_{i=1}^{10} \ell(v_i) = \ell(v_{10}) \neq 0$, a contradiction. Thus, G is not zonal.

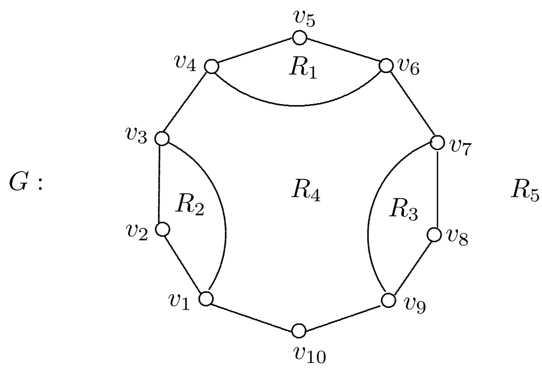

Figure 3: A nonzonal graph G.

We now investigate those 2-connected plane graphs obtained by adding a vertex of degree 2 to a cubic map or to a connected cubic plane graph. First, we introduce an additional concept. A connected plane graph G is a *nearly cubic plane graph* if all vertices of G have degree 3 except for one vertex having degree 2. Thus, a nearly cubic plane graph is a graph obtained by subdividing one edge of a connected cubic plane graph exactly once. Four nearly cubic plane graphs are shown in Figure 4, three of which are 2-connected and the fourth one contains bridges and so is not 2-connected.

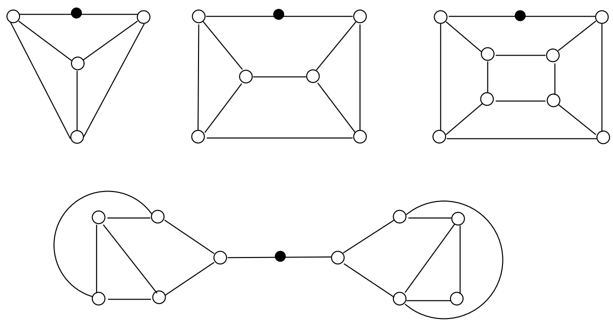

Figure 4: Nearly cubic plane graphs.

Proposition 3.1. *No nearly cubic plane graph is zonal.*

Proof. Assume, to the contrary, that there is a nearly cubic plane graph G that is zonal. Then G has a zonal labeling ℓ. Let $\mathcal{R}$ be the set of all regions of G. Since $\ell(R) = 0$ for each $R \in \mathcal{R}$, it follows that

$$\sum_{R\in\mathcal{R}} \ell(R) = 0 \text{ in } \mathbb{Z}_3. \tag{4}$$

Next, let w be the vertex of degree 2 in G. Then w lies on the boundaries of two regions R_1 and R_2 of G. Every other vertex of G lies on the boundaries of three regions of G. Hence,

$$\sum_{R\in\mathcal{R}} \ell(R) = \sum_{v\in V(G)} \ell(v) = 2\ell(w) + \sum_{v\in V(G)-\{w\}} 3\ell(v) = 2\ell(w) \neq 0 \text{ in } \mathbb{Z}_3,$$

which contradicts (4). □

There are many nearly cubic plane graphs that are not inner zonal. For example, let G be the nearly cubic plane graph of Figure 4 whose vertices are labeled as shown in Figure 5. We claim that G is not inner zonal. Assume, to the contrary, that G has an inner zonal labeling ℓ. Then ℓ must assign the same label to the vertices lying on each triangle of G. Thus, $u, v, , w, x$ must be labeled the same. By Observation 3.1, we may assume that $u, v, , w, x$ are labeled 1. However then, the value of R is $1 + 1 + 1 + \ell(y) \in \{1, 2\}$ in $\mathbb{Z}_3$. Similarly, the value of R' cannot be 0 in $\mathbb{Z}_3$, a contradiction. Therefore, G is not inner zonal. Since this graph G has bridges, it is not 2-connected. On the other hand, every 2-connected nearly cubic plane graph is inner zonal, as we show next. For a region R, we write $V(R)$ for the vertex set of the boundary of R.

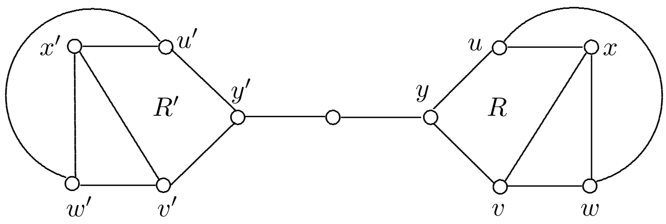

Figure 5: A nearly cubic plane graph that is not inner zonal.

Theorem 3.2. *Every 2-connected nearly cubic plane graph is inner zonal.*

Proof. Let G be a 2-connected nearly cubic plane graph and let w be the vertex of degree 2 in G. Then w lies on the boundary of two regions R_1 and R_2. Let u and v be the two neighbors of w and so (u, w, v) is a 3-path in G and v lies on the boundaries of R_1 and R_2 as well as a third region R_3 in G. Let $R_4, R_5, \ldots, R_t$ be the remaining regions of G. Next, let G' be the graph (or multigraph) constructed from G by replacing the 3-path (u, w, v) with the edge uv.

Denote the t corresponding regions in G' by $R'_1, R'_2 \ldots, R'_t$ such that $V(R'_i) = V(R_i)$ for $3 \le i \le t$ and $V(R'_i) \cup \{w\} = V(R_i)$ for $i = 1, 2$. This is shown in Figure 6. Then G' is a cubic map.

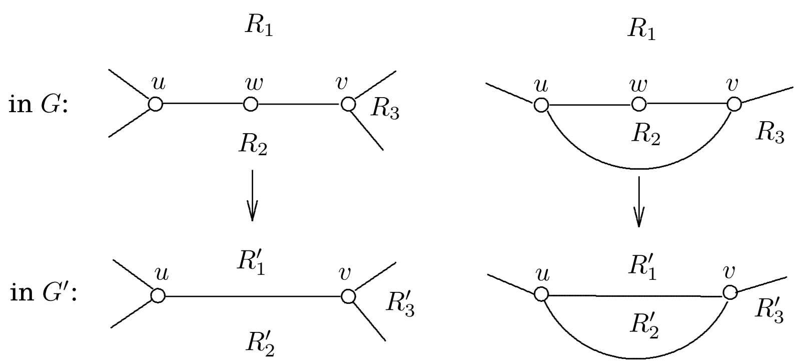

Figure 6: Constructing the graph (multigraph) G.

By Theorem 1.3, the cubic map G' is zonal and therefore has a zonal labeling ℓ'. By Observation 3.1, we may assume that $\ell'(v) = 2$ in $\mathbb{Z}_3$. We now define a labeling of $\ell : V(G) \to \mathbb{Z}_3^*$ by

$$\ell(x) = \begin{cases} 1 & \text{if } x \in \{v, w\} \\ \ell'(x) & \text{otherwise.} \end{cases}$$

We claim that ℓ is an inner zonal labeling of G.

- ⋆ Since $\ell'(v) = \ell(v) + \ell(w)$, it follows for $i = 1, 2$ that $\ell(R_i) = \ell'(R_i) - \ell'(v) + \ell(v) + \ell(w) = \ell'(R_i) = 0$ in $\mathbb{Z}_3$.
- ⋆ If $i = 3$, then $\ell(R_i) = \ell'(R_i) - \ell'(v) + \ell(v) = \ell'(R_i) - 1 = 2 \neq 0$ in $\mathbb{Z}_3$.
- ⋆ If $4 \leq i \leq t$, then $\ell(R_i) = \ell'(R_i) = 0$ in $\mathbb{Z}_3$.

Thus, the only region of G with a nonzero value is R_3, and so ℓ is an inner zonal labeling of G. □

There are infinitely many zonal (inner zonal) graphs obtained from a given zonal (inner zonal) graph G by inserting vertices of degree 2 to G (or subdividing the edges of G). In order to present this fact, we first present a useful observation [4].

Observation 3.2. *If X is a set of vertices of a graph with $|X| \geq 2$, then there is a labeling $\ell : X \to \mathbb{Z}_3^*$ of X such that $\sum_{x \in X} \ell(x) = 0$ in $\mathbb{Z}_3$.*

Theorem 3.3. *Let H be a plane graph, let Z be a nonempty set of edges of H and let G be the graph obtained by subdividing each edge in Z at least twice. Then* (1) *G is zonal if H is zonal and* (2) *G is inner zonal if H is inner zonal. Consequently, if H is a cubic map, then G is zonal; while if H is a nearly cubic plane graph, then G is inner zonal.*

Proof. Let $Z = \{e_1, e_2, \ldots, e_t\}$ be a nonempty set of edges of H and let G be the graph obtained by subdividing each edge in Z at least twice. For $1 \leq i \leq t$, let $e_i = u_i v_i$ and let Q_i be the $u_i - v_i$ path of order 4 or more that replaces e_i in G and let $X_i = V(Q_i) - \{u_i, v_i\}$. Since $|X_i| \geq 2$ for $1 \leq i \leq t$, it follows by Observation 3.2 that there is a labeling $\ell_i : X_i \to \mathbb{Z}_3^*$ of X_i such that $\sum_{x \in X_i} \ell_i(x) = 0$ in $\mathbb{Z}_3$. Let ℓ_H be a zonal or inner zonal labeling of the graph H, depending on whether H is zonal or inner zonal. We define a labeling $\ell : V(G) \to \mathbb{Z}_3^*$ by

$$\ell(v) = \begin{cases} \ell_H(v) & \text{if } v \in V(H) \\ \ell_i(v) & \text{if } v \in X_i \text{ where } 1 \leq i \leq t. \end{cases}$$

There is a one-to-one correspondence between the set of all regions of G and the set of all regions of H. For each region R' of G, let R be the corresponding region of H. Let B' be the boundary of R' in G and B the boundary of R in H. If $B' = B$, then $\ell(R') = \ell_H(R)$. If $B' \neq B$, then B contains at least one edge in Z. We may assume that B contains $e_1, e_2, \ldots, e_r$. Then $V(B') = V(B) \cup (\cup_{i=1}^r X_i)$. Thus, $\ell(R') = \ell_H(R) + \sum_{i=1}^r \left(\sum_{x \in X_i} \ell_i(x)\right) = \ell_H(R)$. Therefore, $\ell(R') = \ell_H(R)$ for every region R' of G. Consequently, (1) and (2) hold. □

By Corollary 2.1, every cubic map (namely 2-connected cubic plane graph) is zonal. The following theorem appeared in [5] whose proof is independent of the Four Color Theorem.

Theorem 3.4. *If G is a connected cubic plane graph with bridges, then G is not zonal.*

By Theorem 3.4, no connected cubic plane graph with bridges is zonal. We also saw that the nearly cubic plane graph of Figure 5 contains bridges and is not inner zonal. In fact, this is the case for all connected cubic plane graphs with bridges.

Theorem 3.5. *If G is a connected cubic plane graph with bridges, then G is not inner zonal.*

Proof. Assume, to the contrary, that there exists a connected cubic plane graph G with bridges such that G is inner zonal. Let $e = w_1 w_2$ be a bridge of G. We may assume that e lies on the boundary of the exterior region of G. Let G_1 and G_2 be the two components of $G - e$ where $w_i \in V(G_i)$ for $i = 1, 2$. Let $\ell : V(G) \to \mathbb{Z}_3^*$ be an inner zonal labeling of G. Then either the value of each interior region of G_1 is 0, or the value of each interior region of G_2 is 0, or both. Assume, without loss of generality, that the value of each interior region of G_1 is 0. Let B_1 be the boundary of the exterior region R_1 of G_1 where $V(B_1)=\{w_1, v_1, v_2, \ldots, v_k\}$. Let ℓ_1 be the restriction of ℓ to G_1. Since G_1 is a nearly cubic plane graph and $\ell_1(R) = \ell(R) = 0$ for each interior region R of G_1, it follows by Proposition 3.1 that $\ell_1(R_1) = \ell(w_1) + \sum_{i=1}^k \ell(v_i) = a \neq 0$ in $\mathbb{Z}_3$. Hence, ℓ_1 is an inner zonal labeling of G_1, where $\ell_1(w_1) = \ell(w_1) \in \mathbb{Z}_3^*$. By Observation 3.1, the complementary labeling $\overline{\ell}_1$ of ℓ_1 is also an inner zonal labeling of G_1. Furthermore, $\overline{\ell}_1(R) = 0$ for every interior region R of G_1 and $\overline{\ell}_1(R_1) = 2a \neq 0$ in $\mathbb{Z}_3$ for the exterior region R_1 of G_1.

Let G_1' be another copy of G_1, where the vertex w_1 in G_1 is denoted by w_1' in G_1' and each vertex v of G_1 is denoted by v' in G_1'. We now construct a new graph G' from G_1 and G_1' by adding the edge $w_1 w_1'$. Thus, G' is a connected cubic plane graph with the bridge $w_1 w_1'$. Let R' be the exterior region of G' whose boundary B' contains the bridge $w_1 w_1'$. Hence, $V(B') = V(B_1) \cup \{w_1', v_1', v_2', \ldots, v_k'\}$. Define a labeling $\ell' : V(G') \to \mathbb{Z}_3^*$ of G' as follows: If $x \in V(G_1)$, then define

$\ell'(x) = \ell_1(x)$; while if $x' \in V(G'_1)$ where its corresponding vertex x belongs to G_1, then define $\ell'(x') = \overline{\ell}_1(x)$. Hence, the value of the exterior region R' of G' is

$$\ell'(R') = \left[\ell_1(w_1) + \sum_{i=1}^{k} \ell_1(v_i)\right] + \left[\overline{\ell}_1(w_1) + \sum_{i=1}^{k} \overline{\ell}_1(v_i)\right] = a + 2a = 0.$$

If R is an interior region of G', then R is either an interior region of G_1 or an interior region of G'_1. Thus, either $\ell'(R) = \ell_1(R) = 0$ or $\ell'(R) = \overline{\ell}_1(R) = 0$. Therefore, ℓ' is a zonal labeling of G' and so G' is zonal, which contradicts Theorem 3.4. □

By the proof of Theorem 3.2, if G is a 2-connected nearly cubic planar graph, then there is an inner zonal labeling ℓ of G such that every vertex on the boundary of the exterior region R of G has degree 3 and $\ell(R) \neq 0$. In fact, this is true for every inner zonal labeling of every nearly cubic plane graph, as we show next.

Proposition 3.2. *Let G be a nearly cubic plane graph. If ℓ is an inner zonal labeling of G, then there exists a region R all of whose boundary vertices have degree 3 such that $\ell(R) \neq 0$.*

Proof. Let G be a nearly cubic plane graph and let w be the vertex of degree 2 in G. Suppose that ℓ is an inner zonal labeling of G. Assume, to the contrary, that $\ell(R) = 0$ for every region R all of whose boundary vertices have degree 3. Since G is not zonal by Proposition 3.1, there is a region R_0 in G whose boundary contains the vertex w of degree 2 in G such that $\ell(R_0) \neq 0$. Furthermore, $\ell(R) = 0$ for every region $R \neq R_0$ of G. We may assume that G is embedded in the plane such that R_0 is the exterior region of G. By Observation 3.1, the complementary labeling $\overline{\ell}$ is also an inner zonal labeling of G. Thus, $\overline{\ell}(R) = 0$ for every region $R \neq R_0$ in G. Next, let G_1 and G_2 be two vertex-disjoint copies of G, where w is labeled w_i in G_i and the region R_0 in G_i is denoted by $R_{0,i}$ for $i = 1, 2$. We now construct a new graph G' from G_1 and G_2 by adding the edge w_1w_2. Thus, G' is a connected cubic plane graph with the bridge w_1w_2. The argument in the proof of Theorem 3.5 shows that G' is zonal, which is impossible by Theorem 3.4. □

4. Intermediate value result and problem

Every 2-connected plane graph G of order $n \geq 4$ with $\Delta(G) \leq 3$ has size m where $n \leq m \leq 3n/2$. If $m = n$, then G is an n-cycle and if $m = 3n/2$, then G is a cubic map. In these two instances, more can be said.

> *Every 2-connected plane graph G of order $n \geq 4$ with $\Delta(G) \leq 3$ having size n or $3n/2$ is zonal.*

This brings up the following question.

> *Is every 2-connected plane graph G of order n and size m with $\Delta(G) = 3$ zonal?*

We saw in Figure 1 that the answer to this question is no for $n = 4$ as the graph $K_4 - e$ has order $n = 4$ and size $m = 5$ (where then $n < m < 3n/2$) is not zonal. In fact, the answer to this question is no for every integer $n \geq 4$ and integer m with $n < m < 3n/2$. In order to verify this fact, we first present a lemma.

Lemma 4.1. *Let G be a plane graph. If there exist distinct regions $R_0, R_1, \ldots, R_k$ of G where the boundary R_i is B_i for $0 \leq i \leq k$ such that $\{V(B_1), V(B_2), \ldots, V(B_k)\}$ form a partition of $V(B_0) - \{v\}$ for some $v \in V(B_0)$, then G is not zonal.*

Proof. Assume, to the contrary, that G has a zonal labeling ℓ. Then $\ell(R_i) = 0$ in $\mathbb{Z}_3$ for $1 \leq i \leq k$. However then,

$$\ell(R_0) \;=\; \ell(v) + \sum_{x \in V(B_0)-\{v\}} \ell(x) = \ell(v) + \sum_{i=1}^{k} \sum_{x \in V(B_i)} \ell(x) = \ell(v) + \sum_{i=1}^{k} \ell(R_i) = \ell(v) \neq 0$$

in $\mathbb{Z}_3$, which is a contradiction. □

Theorem 4.1. *For each pair n, m of integers with $n \geq 4$ and $n < m < 3n/2$, there is a 2-connected plane graph G of order n and size m with $\Delta(G) = 3$ such that G is not zonal.*

Proof. We saw that if the chord e joins two vertices at distance 2 in the cycle C_n of order $n \geq 4$, then $C_n + e$ is not zonal. Thus, there is a 2-connected plane graph G of order $n \geq 4$ and size $m = n + 1$ with $\Delta(G) = 3$ such that G is not zonal. Thus, we may assume that $m \geq n + 2$. By Proposition 3.1, we may assume that G has at least two vertices of degree 2 and at least four vertices of degree 3. Thus, $n \geq 6$ and $m = n + p$ where $2 \leq p \leq \lfloor \frac{n-2}{2} \rfloor$. We consider two cases, according to whether n is odd or n is even.

Case 1. *$n \geq 7$ is odd*. Let $n = 2k + 1$ and let $C = (w, u_1, u_2, \ldots, u_k, v_k, v_{k-1}, \ldots, v_1, w)$ be a $(2k + 1)$-cycle. There are two subcases, depending on the parity of p.

Subcase 1.1. *$m = n + p$ where $p \geq 3$ is odd.* Since G has at least two vertices of degree 2, it follows that $n \geq 9$. Let G be the graph obtained from C by adding the p chords u_iv_i for $1 \leq i \leq p$. Let G be embedded in the plane so that the boundary of the exterior region R is C. For $1 \leq j \leq (p-1)/2$, let R_j be the region of G whose boundary is the 4-cycle $(u_{2j-1}, u_{2j}, v_{2j}, v_{2j-1}, u_{2j-1})$ and for $j = (p+1)/2$, let R_j be the region of G whose boundary is the cycle $(u_{2j-1}, u_{2j}, \ldots, u_k, v_k, v_{k-1}, \ldots, v_{2j}, v_{2j-1}, u_{2j-1})$. Let B_j be the boundary of R_j for $1 \leq j \leq (p+1)/2$. Since $B_1, B_2, \ldots, B_{\frac{p+1}{2}}$ are pairwise disjoint and $\bigcup_{i=1}^{\frac{p+1}{2}} V(B_i) = V(C) - \{w\}$, where C is the boundary of the exterior region R of G, it follows by Lemma 4.1 that G is not zonal.

Subcase 1.2. *$m = n+p$ where $p \geq 2$ is even*. In this case, $n \geq 7$. Let G be the graph obtained from C by adding the chord u_1v_1 and the $p-1$ chords u_iv_{i+1} for $2 \leq i \leq p$. Let G be embedded in the plane so that the boundary of the exterior region R is C. Let R_1 be the region of G whose boundary is the 3-cycle (w, u_1, v_1, w). For $2 \leq j \leq p/2$ where $p \geq 4$, let R_j be the region of G whose boundary is the 4-cycle $(u_{2j-2}, u_{2j-1}, v_{2j}, v_{2j-1}, u_{2j-1})$ and for $j = (p+2)/2$, let R_j be the region of G whose boundary is the cycle $(u_{2j-2}, u_{2j-1}, \ldots, u_k, v_k, v_{k-1}, \ldots, v_{2j}, v_{2j-1}, u_{2j-2})$. Let B_j be the boundary of R_j for $1 \leq j \leq (p+2)/2$. Since these boundaries are pairwise disjoint and $\bigcup_{i=1}^{\frac{p+2}{2}} V(B_i) = V(C) - \{v_2\}$, it follows by Lemma 4.1 that G is not zonal.

Case 2. *n is even*. Let $n = 2k+2$. There are two subcases, depending on the parity of p.

Subcase 2.1. *$m = n + p$ where $p \geq 3$ is odd.* Then $n = 2k + 2 \geq 8$. Let $C = (x, u_1, u_2, \ldots, u_k, y, v_k, v_{k-1}, \ldots, v_1, x)$ be a $(2k+2)$-cycle. Let G be the graph obtained from C by adding the p chords u_iv_i for $1 \leq i \leq p$. Let G be embedded in the plane so that the boundary of the exterior region R is C. For $1 \leq j \leq (p-1)/2$, let R_j be the region of G whose boundary is the 4-cycle $(u_{2j-1}, u_{2j}, v_{2j}, v_{2j-1}, u_{2j-1})$ and for $j = (p+1)/2$, let R_j be the region of G whose boundary is the cycle $(u_{2j-1}, u_{2j}, \cdots, u_k, y, v_k, v_{k-1}, \cdots, v_{2j}, v_{2j-1}, u_{2j-1})$. Let B_j be the boundary of R_j for $1 \leq j \leq (p+1)/2$. Since these boundaries are pairwise disjoint and $\bigcup_{i=1}^{\frac{p+1}{2}} V(B_i) = V(C) - \{x\}$, it follows by Lemma 4.1 that G is not zonal.

Subcase 2.2. *$m = n + p$ where $p \geq 2$ is even*. Thus, $n = 2k + 2 \geq 6$. Let $C = (x, u_1, u_2, \ldots, u_k, v_k, v_{k-1}, \ldots, v_1, x)$ be a $(2k+1)$-cycle. Let G be the graph obtained from C by (1) adding the vertex y and joining y to v_1, x, u_1 and (2) if $n \geq 10$ and $p \geq 4$, then adding the $p-2$ chords $u_{i-1}v_{i+1}$ for $1 \leq i \leq p-2$. Thus, G has order n and size $m = n + p$. Let G be embedded in the plane so that the boundary of the exterior region is C. Let R_1 be the region of G whose boundary is the 5-cycle $(y, u_1, u_2, v_2, v_1, y)$. If $n = 6$ or $n = 8$, then $p = 2$ only. If $n \geq 10$ and $p \geq 4$, then let $R_{p/2}$ be the region of G whose boundary is the cycle $(u_{2p-1}, u_{2p}, \cdots, u_k, v_k, v_{k-1}, \cdots, v_{2p}, v_{2p-1}, u_{2p-1})$. If $n \geq 14$, $p \geq 6$, and $2 \leq j \leq (p-2)/2$, then let R_j be the region of G whose boundary is the 4-cycle $(u_{2j-1}, u_{2j}, v_{2j}, v_{2j-1}, u_{2j-1})$. Let B_j be the boundary of R_j for $1 \leq j \leq p/2$. Since these boundaries are pairwise disjoint and $\bigcup_{i=1}^{\frac{p}{2}} V(B_i) = V(C) - \{x\}$, it follows by Lemma 4.1 that G is not zonal. □

By Corollary 2.1 and Theorem 3.2, all 2-connected plane graphs with maximum degree 3 having at most one vertex of degree 2 are inner zonal. By Theorem 3.3, if G is a 2-connected inner zonal graph with $\Delta(G) \leq 3$, then any graph obtained by subdividing each edge in a set of edges of G at least twice is also inner zonal. Furthermore, every nonzonal graph constructed in the proof of Theorem 4.1 is inner zonal. In addition, it is straightforward to verify that if G is a 2-connected plane graph with maximum degree 3 having exactly two vertices x and y of degree 2 such that either $xy \in E(G)$ or $G + xy$ is planar, then G is inner zonal. In fact, we are not aware of any 2-connected plane graph with maximum degree 3 that is not inner zonal. Consequently, we conclude with the following conjecture.

Conjecture 4.1. *Every 2-connected plane graph with maximum degree 3 is inner zonal.*

Acknowledgments

We are grateful to Professor Gary Chartrand for suggesting the concept of inner zonal labelings to us and kindly providing useful information on this topic. Furthermore, we thank the anonymous referees whose valuable suggestions resulted in an improved paper.

References

[1] K. Appel, W. Haken, Every planar map is four colorable, *Bull. Amer. Math. Soc.* **82** (1976) 711–712.
[2] A. Bowling, P. Zhang, Absolutely and conditionally zonal graphs, *Electron. J. Math.* **4** (2022) 1–11.
[3] A. Bowling, P. Zhang, Inner zonality in graphs, *Int. J. Comput. Math: Computer Systems Theory* **7** (2022) 192–206.
[4] A. Bowling, P. Zhang, Zonal graphs of small cycle rank. *Electron. J. Graph Theory Appl.* **11** (2023) 1–14.
[5] G. Chartrand, C. Egan, P. Zhang, *How to Label a Graph*, Springer, New York, 2019.
[6] G. Chartrand, C. Egan, P. Zhang, Zonal graphs revisited, *Bull. Inst. Combin. Appl.* **99** (2023), To appear.
[7] G. Chartrand, P. Zhang, *Chromatic Graph Theory*, Second Edition, CRC Press, Boca Raton, 2020.
[8] K. Kuratowski, Sur le problème des courbes gauches en topologie, *Fund. Math.* **15** (1930) 271–283.
[9] P. G. Tait, Remarks on the colouring of maps, *Proc. Royal Soc. Edinburgh* **10** (1880) 729.
[10] R. Wilson, *Four Colors Suffice: How the Map Problem Was Solved*, Princeton University Press, Princeton, 2002.