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Research Article

Chimneys in compositions and bargraphs

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Abstract

Motivated by cut points in graph theory, we consider a similar notion in compositions and bargraphs. This is equivalent to counting r-chimneys (a single column extending beyond its immediate neighbours by at least r cells in a bargraph). We establish generating functions for compositions that avoid or count 2-chimneys. Thereafter, in the case of bargraphs we provide two methods for obtaining these generating functions as well as asymptotic estimates for the more general r-chimneys where $r \geq 1$.

Keywords: generating function; compositions; bargraphs; chimneys.

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1. Introduction

A *composition* of the positive integer n is a representation of n as an ordered sum of the positive integers $a_1, a_2, \dots, a_m$: $n = a_1 + a_2 + \cdots + a_m$. Each such a_i is called the i-th part of the composition (for example, see [9]). Any composition may be represented graphically as a sequence of columns where the height of the i-th column is equal to the size of the i-th part. We say that a composition of n has a horizontal cut point if removing two vertical edges from the graphical representation separates the composition into two disjoint smaller compositions. For example, 253 has a cut point which is constituted by the vertical edges of the fourth cell from the bottom in the second column of size five. Removing these separates the original into compositions 1 and 233. A 2-chimney is any part that is at least two higher than its left and right neighbour (if they exist). It is easy to see that having a cut point is equivalent to having a 2-chimney. We want to count the number of compositions of n that do not have any 2-chimneys. Or, complementarily, the number of compositions of n that have at least one 2-chimney.

On the other hand a bargraph is a non-intersecting lattice path in $\mathbb{N}_0^2$ with 3 allowed types of steps; up $(0, 1)$, down $(0, -1)$ and horizontal $(1, 0)$. An up step may not immediately follow a down step nor visa versa. They start at the origin with an up step and terminate immediately upon return to the x-axis and to qualify for the term bargraph, the generating function for these should track the number of horizontal steps (usually by x) and the number of up steps (usually by y). An r-chimney in bargraphs is any part that is at least r higher than its left and right neighbour (if they exist). In contrast, an exact r-chimney is any part that is exactly r higher than its left and right neighbour (if they exist). We find generating functions for bargraphs that avoid r-chimneys and others that enumerate these for each r. For previous examples of the methods employed in bargraph statistics, see [1–4]. The earliest papers on bargraphs were in a Physics setting [11, 12] and the first combinatorial paper was [6]. A predecessor of the latter was unpublished, see [8]. Papers straddling the domains of Physics and Mathematics are [5, 10].

The generating function that counts all bargraphs is given by

$$B(x, y) = \frac{1 - x - y - xy - \sqrt{(1 - x - y - xy)^2 - 4x^2 y}}{2x},$$

see [6], where x counts the number of horizontal steps and y counts the number of vertical up steps. The asymptotics of the coefficient of x^n in $B(x, x)$ has been considered, and in order to compute it, the dominant singularity ρ is the positive root of $1 - 4x + 2x^2 + x^4 = 0$. By singularity analysis (for example, see [7]) we have

$$[x^n]B(x, x) \sim \frac{1}{2}\sqrt{\frac{1 - \rho - \rho^3}{\pi \rho n^3}}\rho^{-n} \tag{1}$$

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with

$$\rho = \frac{1}{3}\left(-1 - \frac{2^{8/3}}{(13+3\sqrt{3})^{1/3}} + 2^{1/3}(13+3\sqrt{33})^{1/3}\right) \approx 0.295598\cdots. \tag{2}$$

2. Chimneys in compositions

2.1. The generating function for compositions that avoid 2-chimneys

We first obtain the relevant functional equations as follows:

Consider compositions with no 2-chimneys. More specifically, we define $F_{a_1a_2\cdots a_s}(x,y)$ to be the generating function for the number of such compositions $\pi_1\pi_2\cdots\pi_m$ of n with m parts such that $\pi_1\pi_2\cdots\pi_s = a_1a_2\cdots a_s$. For $s=0$, we define $F(x,y)$ to be the generating function for these compositions of n with m parts (x marks n and y marks m). From the definitions, it follows that $F(x,y) = 1 + \sum_{a\geq 1} F_a(x,y)$. We also have

$$\begin{aligned}
&F_{a(a-1)}(x,y) = xF_{(a-1)(a-1)}(x,y),\\
&F_{a(a+1)}(x,y) = \frac{1}{x}F_{(a+1)(a+1)}(x,y),\\
&F_{aj}(x,y) = 0 \text{ for } a\geq 3 \text{ and } 1\leq j\leq a-2 \text{ and},\\
&F_{aj}(x,y) = x^a y F_j(x,y) \text{ for all } j\geq a+2\geq 3.
\end{aligned}$$

So,

$$\begin{aligned}
F_1(x,y) &= xy + F_{11}(x,y) + F_{12}(x,y) + \sum_{j\geq 3} F_{1j}(x,y)\\
&= xy + F_{11}(x,y) + \frac{1}{x}F_{22}(x,y) + xy\sum_{j\geq 3} F_j(x,y), \qquad (3)\\
F_2(x,y) &= x^2y + F_{21}(x,y) + F_{22}(x,y) + F_{23}(x,y) + \sum_{j\geq 4} F_{2j}(x,y)\\
&= x^2y + xF_{11}(x,y) + F_{22}(x,y) + \frac{1}{x}F_{33}(x,y) + x^2y\sum_{j\geq 4} F_j(x,y) \qquad (4)
\end{aligned}$$

and for all $a\geq 3$,

$$\begin{aligned}
F_a(x,y) &= F_{a(a-1)}(x,y) + F_{aa}(x,y) + F_{a(a+1)}(x,y) + \sum_{j\geq a+2} F_{aj}(x,y)\\
&= xF_{(a-1)(a-1)}(x,y) + F_{aa}(x,y) + \frac{1}{x}F_{(a+1)(a+1)}(x,y) + x^a y \sum_{j\geq a+2} F_j(x,y). \qquad (5)
\end{aligned}$$

Define $F(x,y,v) := \sum_{a\geq 1} F_a(x,y)v^{a-1}$ and $G(x,y,v) := \sum_{a\geq 1} F_{aa}(x,y)v^{a-1}$. Hence, by (3)-(5), we have

$$\sum_{a\geq 1} F_a(x,y)v^{a-1} = xy + x^2yv + (xv+1)\sum_{a\geq 1} F_{aa}(x,y) + \frac{1}{xv}\sum_{a\geq 2} F_{aa}(x,y) + \sum_{a\geq 1} x^a y v^{a-1}\sum_{j\geq a+2} F_j(x,y),$$

from which it follows that

$$\begin{aligned}
F(x,y,v) &= xy + x^2yv + (xv+1)G(x,y,v) + \frac{G(x,y,v)-G(x,y,0)}{xv}\\
&\quad + \frac{xy(F(x,y,1)-F(x,y,0))}{1-xv} - \frac{y(F(x,y,xv)-F(x,y,0))}{v(1-xv)}. \qquad (6)
\end{aligned}$$

Similarly for $a\geq 3$,

$$\begin{aligned}
F_{aa}(x,y) &= x^{2a}y^2 + \sum_{j=1}^{a+1} F_{aaj}(x,y) + \sum_{j\geq a+2} F_{aaj}(x,y)\\
&= x^{2a}y^2 + \sum_{j=1}^{a+1} x^{2a-j} y F_{jj}(x,y) + x^{2a}y^2 \sum_{j\geq a+2} F_j(x,y).
\end{aligned}$$

By (5), we have

$$F_{aa}(x,y)=x^{2a}y^2+x^ayF_a(x,y)+\sum_{j=1}^{a-2}x^{2a-j}yF_{jj}(x,y).$$

We now multiply by v^{a-1} and sum over $a\geq 3$, using the initial values $F_{jj}(x,y)=x^jyF_j(x,y)$ for $j=1,2$, to obtain

$$G(x,y,v)=\frac{x^6y^2v^2}{1-x^2v}+xyF(x,y,xv)+\frac{x^5yv^2}{1-x^2v}G(x,y,xv). \tag{7}$$

By employing (7) with $v=0$, (6) can be simplified to

$$F(x,y,v)=xy(1+xv)+\big(xv+1+\frac{1}{xv}\big)G(x,y,v)+\frac{y(xvF(x,y,1)-F(x,y,xv))}{v(1-xv)}. \tag{8}$$

Now, write (7) and (8) in the matrix form, to obtain

$$\begin{pmatrix} 1 & -xv-1-\frac{1}{xv}\\ 0 & 1\end{pmatrix}\begin{pmatrix}F(x,y,v)\\ G(x,y,v)\end{pmatrix}=\begin{pmatrix}-\frac{y}{v(1-xv)} & 0\\ xy & \frac{x^5yv^2}{1-x^2v}\end{pmatrix}\begin{pmatrix}F(x,y,xv)\\ G(x,y,xv)\end{pmatrix}+\begin{pmatrix}xy(1+xv)+\frac{xyF(x,y,1)}{1-xv}\\ \frac{x^6y^2v^2}{1-x^2v}\end{pmatrix},$$

which is equivalent to

$$\begin{pmatrix}F(x,y,v)\\ G(x,y,v)\end{pmatrix}=\mathbf{A}(v)\begin{pmatrix}F(x,y,xv)\\ G(x,y,xv)\end{pmatrix}+\mathbf{b}(v)+F(x,y,1)\mathbf{c}(v). \tag{9}$$

with

$$\mathbf{A}(v)=\begin{pmatrix}-\frac{x^3yv^2}{1-xv} & \frac{x^4yv(1+xv+x^2v^2)}{1-x^2v}\\ xy & \frac{x^5yv^2}{1-x^2v}\end{pmatrix},\quad \mathbf{b}(v)=\begin{pmatrix}xy(1+xv)-\frac{(x^3v^3-1)x^5y^2v}{(xv-1)(x^2v-1)}\\ \frac{x^6y^2v^2}{1-x^2v}\end{pmatrix}\text{ and }\mathbf{c}(v)=\begin{pmatrix}\frac{xy}{1-xv}\\ 0\end{pmatrix}.$$

In order to solve (9), we need the following lemma which follows immediately using induction.

Lemma 2.1.1. *For all $m\geq 0$,*

$$\mathbf{B}_{2m}(v):=\prod_{j=0}^{2m}\mathbf{A}(x^jv)=\frac{x^{m^2+5m+1}y^{2m+1}v^m}{\prod_{i=1}^{2m+1}(x^iv-1)}\begin{pmatrix}x^2v^2 & -\frac{x^{m+3}v(x^{3m+3}v^3-1)}{x^{2m+2}v-1}\\ xv-1 & -\frac{x^{4m+4}v^2(xv-1)}{x^{2m+2}v-1}\end{pmatrix}$$

and

$$\mathbf{B}_{2m+1}(v):=\prod_{j=0}^{2m+1}\mathbf{A}(x^jv)=\frac{x^{m^2+6m+5}y^{2m+2}v^{m+1}}{\prod_{i=1}^{2m+2}(x^iv-1)}\begin{pmatrix}1 & -\frac{x^{m+3}(x^{3m+3}-1)v^2}{x^{2m+3}v-1}\\ 0 & \frac{x^{m+1}(xv-1)}{x^{2m+3}v-1}\end{pmatrix}.$$

Next, we assume that $|x|,|y|<1$. By iterating (9), we obtain

$$\begin{pmatrix}F(x,y,v)\\ G(x,y,v)\end{pmatrix}=\sum_{j\geq 0}\left(\prod_{i=0}^{j-1}\mathbf{A}(x^iv)\right)(\mathbf{b}(x^jv)+F(x,y,1)\mathbf{c}(x^jv)).$$

Employing the notation in Lemma 2.1.1, we have

$$\begin{aligned}\begin{pmatrix}F(x,y,v)\\ G(x,y,v)\end{pmatrix}&=\sum_{j\geq 0}\mathbf{B}_{j-1}(v)\mathbf{b}(x^jv)+F(x,y,1)\sum_{j\geq 0}\mathbf{B}_{j-1}(v)\mathbf{c}(x^jv)\\ &=\sum_{j\geq 0}\mathbf{B}_{2j-1}(v)\mathbf{b}(x^{2j}v)+\sum_{j\geq 0}\mathbf{B}_{2j}(v)\mathbf{b}(x^{2j+1}v)\\ &\quad+F(x,y,1)\sum_{j\geq 0}\mathbf{B}_{2j-1}(v)\mathbf{c}(x^{2j}v)+F(x,y,1)\sum_{j\geq 0}\mathbf{B}_{2j}(v)\mathbf{c}(x^{2j+1}v).\end{aligned}$$

And again from Lemma 2.1.1, we obtain

$$\begin{aligned}\begin{pmatrix}F(x,y,1)\\ G(x,y,1)\end{pmatrix}&=\sum_{j\geq 0}\frac{x^{j(j+4)+1}y^{2j+1}}{\prod_{i=1}^{2j}(x^i-1)}\begin{pmatrix}1+x^{2j+1}+\frac{x^{2j+4}(1-x^{3j+3})y}{(x^{2j+1}-1)(x^{2j+2}-1)}\\ -\frac{x^{5j+5}(x-1)y}{(x^{2j+1}-1)(x^{2j+2}-1)}\end{pmatrix}\\ &\quad+\sum_{j\geq 0}\frac{x^{j(j+5)+2}y^{2j+2}}{\prod_{i=1}^{2j+1}(x^i-1)}\begin{pmatrix}x^2(1+x^{2j+2})+\frac{x^{2j+7}(1-x^{3j+3})y}{(x^{2j+2}-1)(x^{2j+3}-1)}\\ (1+x^{2j+2})(x-1)-\frac{x^{2j+5}(x-1)y}{(x^{2j+2}-1)(x^{2j+3}-1)}\end{pmatrix}\\ &\quad-F(x,y,1)\sum_{j\geq 0}\frac{x^{j(j+4)+1}y^{2j+1}}{\prod_{i=1}^{2j+1}(x^i-1)}\begin{pmatrix}1\\ 0\end{pmatrix}-F(x,y,1)\sum_{j\geq 0}\frac{x^{j(j+5)+2}y^{2j+2}}{\prod_{i=1}^{2j+2}(x^i-1)}\begin{pmatrix}x^2\\ x-1\end{pmatrix}.\end{aligned}$$

Solving for $F(x,y,1)$, we obtain the next result.

Theorem 2.1.1. *The generating function $F(x,y,1)$ for the number of compositions of n with m parts and no 2-chimneys is given by*

$$\frac{\sum\limits_{j\ge0}\frac{x^{j(j+4)+1}y^{2j+1}}{\prod_{i=1}^{2j}(x^i-1)}\left(1+x^{2j+1}+\frac{x^{2j+4}(1-x^{3j+3})y}{(x^{2j+1}-1)(x^{2j+2}-1)}\right)+\sum\limits_{j\ge0}\frac{x^{j(j+5)+4}y^{2j+2}}{\prod_{i=1}^{2j+1}(x^i-1)}\left(1+x^{2j+2}+\frac{x^{2j+5}(1-x^{3j+3})y}{(x^{2j+2}-1)(x^{2j+3}-1)}\right)}{1+\sum\limits_{j\ge0}\frac{x^{j(j+4)+1}y^{2j+1}}{\prod_{i=1}^{2j+1}(x^i-1)}+\sum\limits_{j\ge0}\frac{x^{j(j+5)+4}y^{2j+2}}{\prod_{i=1}^{2j+2}(x^i-1)}}.$$

The denominator of $F(x,1,1)$ has a dominant simple zero at $\eta=0.541219\cdots$. Using singularity analysis again (see [7]) we deduce the next result.

Corollary 2.1.1. *With η as above, the number of compositions of n with no 2-chimneys is asymptotic to $c(\eta)\eta^{-n-1}$ where $c(\eta)=0.252779\cdots$ as $n\to\infty$.*

2.2. The total number of two-chimneys over all compositions of n

Consider the following three cases.

1. For compositions with one part, 2-chimneys are counted by

$$f_1(x):=\frac{x^2}{1-x}.$$

2. The first part or the last part is a 2-chimney:

$$f_2(x):=2\frac{1-x}{1-2x}\sum_{i\ge1}\sum_{j\ge i+2}x^{i+j}.$$

3. Internal 2-chimneys: To count internal 2-chimneys we set up a bijection with compositions with a marked internal 2-chimney. Eg: 31**5**142 or 3151**4**2 where the internal 2-chimneys are indicated in bold.

   Let ijk be the three columns that make up the marked internal 2-chimney. The cases $k<i$ and $i<k$ are equivalent. The internal 2-chimney generating function is

$$f_3(x):=\left(\frac{1-x}{1-2x}\right)^2\left(2\sum_{i\ge1}\sum_{j\ge i+2}\sum_{k=1}^{i-1}x^{i+j+k}+\sum_{i\ge1}\sum_{j\ge i+2}x^{2i+j}\right).$$

   We precede and follow the marked 2-chimney with arbitrary unmarked compositions.

The generating function for the total number of 2-chimneys is

$$f_1(x)+f_2(x)+f_3(x)=\frac{x^2(1-2x+2x^4+x^5)}{1-3x+3x^3+3x^4-4x^6}.$$

By singularity analysis, the total number of 2-chimneys over all compositions of n is asymptotic to $\frac{1}{441}2^{n-3}(105n+116)$ as $n\to\infty$.

Remark 2.2.1. *The generating function for the number of 2-chimneys can be extended to 3-chimneys by observing that a 2-chimney can be extended to a 3-chimney by adding one cell, so the number of 2-chimneys for a composition of n is the number of 3-chimneys for a composition of $n-1$, and so on. That means the generating function for the total number of cut points is just the generating function for the number of 2-chimneys minus x^2, with an extra factor of $1-x$ in the denominator, which agrees with the final formula in Section 2.3 as derived below.*

2.3. The total number of cut points over all compositions of n

The number of cut points corresponds to the total number of r-chimneys summed over all $r\ge2$. Corresponding to the three cases above we have

1. For compositions with one part, cut points are counted by

$$f_1(x):=\sum_{j\ge3}(j-2)x^j.$$

2. The first part or the last part has cut points:

$$f_2(x) := 2\frac{1-x}{1-2x}\sum_{i\geq 1}\sum_{j\geq i+2}(j-i-1)x^{i+j}.$$

3. Internal cut points: In this case the corresponding generating function is

$$f_3(x) := \left(\frac{1-x}{1-2x}\right)^2\left(2\sum_{i\geq 1}\sum_{j\geq i+2}\sum_{k=1}^{i-1}(j-i-1)x^{i+j+k} + \sum_{i\geq 1}\sum_{j\geq i+2}(j-i-1)x^{2i+j}\right).$$

The generating function for the total number of cut points is

$$f_1(x)+f_2(x)+f_3(x) = \frac{x^3-3x^5-x^6+x^7+4x^8}{(1-2x)^2(1-x)^2(1+x)\,(1+x+x^2)}.$$

By singularity analysis, the total number of cut points over all compositions of n is asymptotic to $\frac{1}{441}2^{n-2}(105n+11)$ as $n\to\infty$.

3. Counting exact r-chimneys in bargraphs

We say that a bargraph B contains an exact r-chimney if the path B' without its first up step has a factor $uu\cdots uhdd\cdots d = u^rhd^r$; that is there exist B'', B''' such that $B' = B''u^rhd^rB'''$, where B'' does not end with an up step and B''' does not start with a d step. For example, the bargraph

$$uuuhddhhuuuuhddddhhuhdhuhhdd$$

contains the factors u^2hd^2, u^4hd^4 and uhd, so it contains exact 2-chimneys, 4-chimneys and 1-chimneys. We use two different methods to obtain the same result.

3.1. Using an iterative decomposition

Define $F(x,y) = F(x,y,q_1,q_2,\ldots)$ to be the generating function for the number of bargraphs according to the number of horizontal steps, up steps, exact 1-chimneys, exact 2-chimneys, …, where x marks the number of horizontal steps, y marks number of up steps and for all $i\geq 1$, q_i marks the number of exact i-chimneys. To facilitate obtaining an expression for $F(x,y)$, we refine this by also defining $F_j(x,y) = F_j(x,y,q_1,q_2,\ldots)$ to be the generating function for the number of bargraphs B with the trackers as before but where u^jBd^j has no horizontal step at line $y=0,1,\ldots,j-1$.

Immediately from the definitions,

$$F(x,y) = 1 + F_1(x,y). \tag{10}$$

Now let us write an equation for $F_1(x,y)$. Note that each bargraph uBd that has no horizontal step at line $y=0$ can be decomposed as $uB^{(0)}hB^{(1)}\cdots hB^{(s)}d$ with $B^{(i)}$ having no horizontal step at lines $y=0,1$. Thus, the contribution of the case s is given by $F_2(x,y)$ and $yx^s(F_2(x,y)/y+1)^{s+1}$, where $s=0$ and $s\geq 1$, respectively. Hence

$$F_1(x,y) = F_2(x,y) + \frac{yx(F_2(x,y)/y+1)^2}{1-x(F_2(x,y)/y+1)}. \tag{11}$$

Next let us write an equation for $F_j(x,y)$ where $j\geq 2$. Note that each bargraph u^jBd^j that has no horizontal step at line $y=0,1,\ldots,j-1$ can be decomposed as $u^jB^{(0)}hB^{(1)}\cdots hB^{(s)}d^j$ with $B^{(i)}$ having no horizontal step at lines $y=0,1,\ldots,j$. The contribution of the case $s=0$ is $F_{j+1}(x,y)$. The case $s=1$ can be considered by looking at the bargraphs u^jhd^j, $u^jB'hd^j$, $u^jhB''d^j$ and $u^jB'hB''d^j$ with B', B'' non empty and with no horizontal step at line $y=j$. Thus, the contribution of the case $s=1$ is

$$y^jxQ_{j-1} + y^jxF_2(x,y)/y + y^jxF_2(x,y)/y + y^jxF_2^2(x,y)/y^2,$$

where $Q_i = q_1q_2\cdots q_i$. The contribution of the case $s\geq 2$ is $y^jx^s(F_2(x,y)/y+1)^{s+1}$. Hence, by adding all the contributions, we obtain

$$F_j(x,y) = F_{j+1}(x,y) + y^jx(Q_{j-1}-1) + y^jx(F_2(x,y)/y+1)^2 + \sum_{s\geq 2}y^jx^s(F_2(x,y)/y+1)^{s+1},$$

which is equivalent to

$$F_j(x,y) = F_{j+1}(x,y) + y^jx(Q_{j-1}-1) + \frac{y^jx(F_2(x,y)/y+1)^2}{1-x(F_2(x,y)/y+1)}.$$

Now summing for $j \geq 2$ while using the fact that $F(x,y) = 1 + F_1(x,y)$, we obtain

$$F_2(x,y) = \sum_{j\geq 2} y^j x(Q_{j-1} - 1) + \frac{y^2 x(F_2(x,y)/y+1)^2}{(1-y)(1-x(F_2(x,y)/y+1))}.$$

Therefore,

$$F_2(x,y) = \frac{y(1+x+xH - \sqrt{(1+x+xH)^2 - \frac{4x}{1-y}(1+H)})}{\frac{2x}{1-y}} - y,$$

where $H = \sum_{j\geq 2} y^{j-1}x(q_1q_2\cdots q_{j-1} - 1)$. Hence, by (10)-(11), we have the following result.

Theorem 3.1.1. *The generating function $F(x,y,q_1,q_2,\ldots)$ defined at the start of this section is given by*

$$\frac{1+x-xH - \sqrt{(1+x+xH)^2 - \frac{4x}{1-y}(1+H)}}{\frac{2x}{1-y}}.$$

where $H = \sum_{j\geq 2} y^{j-1}x(q_1q_2\cdots q_{j-1} - 1)$.

3.2. Counting all bargraphs that avoid chimneys (using the wasp-waist decomposition)

As stated at the beginning of this section, we present a second method to obtain the generating function enumerating bargraphs which avoid r-chimneys. This is based on a first return to level one decomposition colloquially known as the wasp-waist decomposition. Here is a symbolic sketch of this decomposition (see [6]) which has 5 cases. Cases 4 and 5 represent the first return to level one.

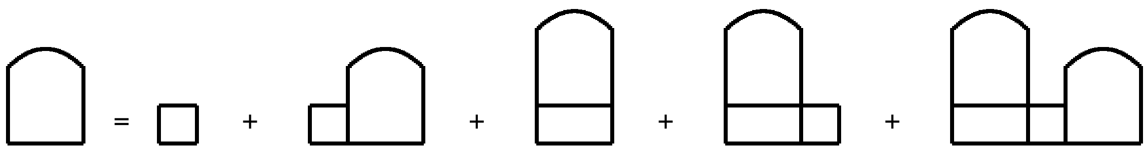

Figure 1: Wasp-waist decomposition of bargraphs.

Recall that we set x as a horizontal step and y as an up step and here we use $f(x,y)$ as the generating function for counting chimneys in bargraphs. We use q to track all r chimneys (for any r), q_1 to tracks 1-chimneys, q_2 to track 2-chimneys etc. Following the order of the wasp-waist decomposition, we obtain:

$$f(x,y,q) = yx + xf(x,y,q) + y(f(x,y,q)+h) + yx(f(x,y,q)+h) + xf(x,y,q)(f(x,y,q)+h)$$

where

$$h = \sum_{j\geq 2} y^{j-1}x(q_1q_2q_3\ldots q_{j-2}(q_{j-1}-1)).$$

Solving for f yields

$$f(x,y,q) = \frac{-hx-xy-y-x+1}{2x} - \frac{\sqrt{(-hx-xy-y-x+1)^2 + 4x(-hyx-hy-yx)}}{2x}.$$

Note that for H as defined in Theorem 3.1.1 and h as in this subsection, we have $(1-y)H = h$. Hence the two generating functions agree except for the constant term. This is because the wasp-waist method excludes the case of the empty bargraph.

For the rest of this section, we will use the generating function obtained in Theorem 3.1.1.

Example 3.2.1. *Applying the above theorem for $q_2 = 0$ and $q_j = 1$ where $j \neq 2$, we obtain the generating function for the number of bargraphs with no 2-chimneys according to number of horizontal steps and up steps to be given by*

$$\begin{aligned} G_2(x,y) &:= F(x,y,1,0,1,1,\ldots) \\ &= \frac{(1-y)(1+x) + y^2x^2 - \sqrt{((1-y)(1+x) - y^2x^2)^2 - 4x(1-y-y^2x)}}{2x}. \end{aligned}$$

In particular, the generating function for the number of bargraphs with no 2*-chimneys according to semi-perimeter is given by*

$$G_2(x,x)=\frac{1-x^2+x^4-\sqrt{(1-x^2-x^4)^2-4x(1-x-x^3)}}{2x}.$$

Hence, by singularity analysis (for example, see [7]) we have

$$[x^n]G_2(x,x)=\frac{\sqrt{-2\psi^7-3\psi^5-3\psi^3-\psi+1}}{2\sqrt{\pi\psi n^3}}\psi^{-n},$$

where ψ *is the smallest positive root of the polynomial* $(1-x^2-x^4)^2-4x(1-x-x^3)=x^8+2x^6+3x^4+2x^2-4x+1$.

3.3. Avoiding r-chimneys

Applying Theorem 3.1.1 using $q_r=0$ and $q_j=1$ for $j\neq r$, the generating function $G_r(x,y)$ for the number of bargraphs without r-chimneys according to number of horizontal steps and up steps is given by

$$G_r(x,y)=\frac{(1-y)(1+x)+y^rx^2-\sqrt{((1-y)(1+x)-y^rx^2)^2-4x(1-y-y^rx)}}{2x}.$$

Thus, the generating function for the number of bargraphs without r-chimneys according to semi-perimeter is given by $G_r(x,x)$. By singularity analysis we have $[x^n]G_r(x,x)=$

$$\frac{\left(\frac{1}{n}\right)^{3/2}\psi^{-n-\frac{1}{2}}\sqrt{-2r\psi^{r+1}-4\psi^{r+1}-2r\psi^{r+3}-8\psi^{r+3}-2r\psi^{2r+3}-4\psi^{2r+3}-4\psi^3-4\psi+4}}{4\sqrt{\pi}},$$

where ψ is the smallest positive root of the polynomial $(1-x^2-x^{r+2})^2-4x(1-x-x^{r+1})$.

3.4. Average of r-chimneys

Applying Theorem 3.1.1 for $q_r=q$ and $q_j=1$ for $j\neq r$, the generating function $G_r(x,y,q)$ for the number of bargraphs according to the number of horizontal steps, up steps and number of r-chimneys is

$$\frac{(1-y)(1+x)+(1-q)y^rx^2-\sqrt{((1-y)(1+x)-(1-q)y^rx^2)^2-4x(1-y-(1-q)y^rx)}}{2x}.$$

Thus, the generating function for the number of bargraphs according to semi-perimeter and number of r-chimneys is given by $G_r(x,x,q)$ and the total number of r-chimneys has generating function

$$\frac{\partial G_r(x,x,q)}{\partial q}|_{q=1}=\frac{x^{1+r}\left(1+x^2-\sqrt{1-4x+2x^2+x^4}\right)}{2\sqrt{1-4x+2x^2+x^4}}.$$

Hence, by singularity analysis the total number of r-chimneys in bargraphs of semi-perimeter n is asymptotic to

$$\frac{\sqrt{\frac{1}{n}}\rho^{\frac{1}{2}-n+r}\left(1+\rho^2\right)}{2\sqrt{\pi}\sqrt{4-4\rho-4\rho^3}}$$

as $n\to\infty$, where ρ is given by (2).

3.5. Average number of cut points

By applying Theorem 3.1.1 for $q_2=q_3\cdots=q$ and $q_1=1$, we obtain that the generating function $A(x,y,q)$ for the number of bargraphs according to number of horizontal steps, up steps and number of cut points (equivalently, all r-chimneys for $r\geq 2$) is given by

$$A(x,y,q)=\frac{1+x+\frac{y^2x^2(1-q)}{(1-yq)(1-y)}-\sqrt{(1+x-\frac{y^2x^2(1-q)}{(1-yq)(1-y)})^2-\frac{4x}{1-y}(1-\frac{y^2x(1-q)}{(1-yq)(1-y)})}}{\frac{2x}{1-y}}.$$

Thus, the generating function for the number of bargraphs according to semi-perimeter and number of cut points is given by $A(x,x,q)$ and the generating function for the total number of cut points is

$$\frac{\partial A(x,x,q)}{\partial q}|_{q=1}=\frac{x^3\left(x-1+\frac{(x-1)(1+x^2)}{\sqrt{1-4x+2x^2+x^4}}\right)}{2(1-x)^2}.$$

Hence, by singularity analysis the total number of cut points in bargraphs of semi-perimeter n is asymptotic to

$$\frac{\sqrt{\frac{1}{n}}\rho^{\frac{5}{2}-n}\left(1+\rho^2\right)}{4\sqrt{\pi}(1-\rho)\sqrt{1-\rho-\rho^3}}$$

as $n\to\infty$, where ρ is given by (2).

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