A Gallai’s Theorem type result for the edge stability of graphs

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Abstract

For an arbitrary invariant $\rho(G)$ of a graph $G$, the $\rho$-edge stability number $es_\rho(G)$ of $G$ is the minimum number of edges of $G$ whose removal results in a graph $H \subseteq G$ with $\rho(H) \neq \rho(G)$. If such an edge set does not exist, then $es_\rho(G) = \infty$. Gallai’s Theorem states that $\alpha'(G) + \beta'(G) = n(G)$ for a graph $G$ without isolated vertices, where $\alpha'(G)$ is the matching number, $\beta'(G)$ the edge covering number, and $n(G)$ the order of $G$. We prove a corresponding result for invariants that are based on the edge stability number $es_\rho(G)$.

Keywords: edge stability number; graph invariant; Gallai’s Theorem.

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1. Introduction

We consider finite simple graphs $G = (V(G), E(G))$ and denote the class of finite simple graphs by $\mathcal{I}$. An empty graph is a graph with empty edge set.

Definition 1.1. A (graph) invariant $\rho(G)$ is a function $\rho: \mathcal{I} \to \mathbb{R}^+_0 \cup \{\infty\}$. An invariant is integer-valued if $\rho(\mathcal{I}) \subseteq \mathbb{N}_0$. An invariant $\rho(G)$ is monotone increasing if $H \subseteq G$ implies $\rho(H) \leq \rho(G)$, and monotone decreasing if $H \subseteq G$ implies $\rho(H) \geq \rho(G)$; $\rho(G)$ is monotone if it is monotone increasing or monotone decreasing. If the conditions hold for certain classes of subgraphs (for example, induced or spanning subgraphs), then we say that $\rho(G)$ is monotone (increasing or decreasing) with respect to the class.

Definition 1.2. If $H_1$ and $H_2$ are disjoint graphs, then an invariant is called additive if $\rho(H_1 \cup H_2) = \rho(H_1) + \rho(H_2)$ and maxing if $\rho(H_1 \cup H_2) = \max\{\rho(H_1), \rho(H_2)\}$.

For example, the maximum degree $\Delta(G)$ of a graph $G$ is integer-valued, monotone increasing, and maxing. The minimum degree $\delta(G)$ is integer-valued, not monotone, but monotone increasing with respect to spanning subgraphs, not additive, and not maxing. The independence number $\alpha(G)$ is integer-valued, not monotone, but monotone increasing with respect to induced subgraphs and monotone decreasing with respect to spanning subgraphs, and additive. The chromatic number $\chi(G)$ is integer-valued, monotone increasing, and maxing. The domination number $\gamma(G)$ is integer-valued, not monotone, and additive.

It is an interesting topic to determine the stability of an arbitrary invariant $\rho(G)$ of a graph $G$ with respect to specific graph operations such as removing vertices of $G$, or removing edges, or subdividing edges. The stability with respect to removing edges from $G$ leads to the following invariant.

Definition 1.3. The $\rho$-edge stability number $es_\rho(G)$ of a graph $G$ is the minimum number of edges of $G$ whose removal results in a graph $H \subseteq G$ with $\rho(H) \neq \rho(G)$. If such an edge set does not exist, then we set $es_\rho(G) = \infty$.

In [3] the $\rho$-edge stability number is also defined and called $\rho$-line-stability. This paper contains just some basic results on this topic.

For some specific invariants $\rho(G)$ the problem of determining the $\rho$-edge stability number was already considered, for example for the chromatic number $\chi(G)$, for the chromatic index $\chi'(G)$, for the total chromatic number $\chi''(G)$, and particularly for the domination number $\gamma(G)$.

The $\chi$-edge stability number or chromatic edge stability number $es_\chi(G)$ was introduced in [2, 11] and also studied in [1, 3, 4, 6–8, 10]. The $\chi'$-edge stability number or chromatic edge stability index $es_{\chi'}(G)$ was considered, among others,
in [7] and the $\chi''$-edge stability number or total chromatic edge stability number $es_{\chi''}(G)$ in [6]. The increase of the domination number $\gamma(G)$ with respect to edge removal was extensively studied (see e.g. [3] or [12] for a survey). The so-called bondage number $b(G)$ coincides with the $\gamma$-edge stability number $es_{\gamma}(G)$.

Let us mention that in our previous papers on this topic [6–8,10] we used a different definition for the second (trivial) case.

Two observations on $es_p(G)$ are that if $\rho(G) \neq \rho(G - E')$, then $es_p(G) \leq |E'|$, and if $\rho(G - E') \neq \rho(G - E'')$, then $es_p(G) \leq \max(|E'|, |E''|)$, where $E', E'' \subseteq E(G)$. Moreover, $es_p(G) \leq es_p(G - E') + |E'|$.

In [7] we proved several general results on the $p$-edge stability number $es_p(G)$.

**Theorem 1.1.** [7] Let $\rho(G)$ be additive, $G = H_1 \cup \cdots \cup H_k$ a graph whose subgraphs $H_1, \ldots, H_k$ and the integer $s \geq 0$ are defined such that $\rho(H_i)$ can be changed by edge deletion if and only if $1 \leq i \leq s$. Then $es_p(G) = \infty$ if $s = 0$ and $es_p(G) = \min\{es_p(H_i) : 1 \leq i \leq s\}$ if $s \neq 0$.

For maxing invariants we proved the following result.

**Theorem 1.2.** [7] Let $\rho(G)$ be maxing and monotone increasing, $G = H_1 \cup \cdots \cup H_k$ a graph whose subgraphs $H_1, \ldots, H_k$ and the integer $s \geq 1$ are defined such that $\rho(H_i) = \rho(G)$ if and only if $1 \leq i \leq s$. Then $es_p(G) = \infty$ if there is a subgraph $H_j$, $1 \leq j \leq s$, such that $\rho(H_j)$ cannot be changed by edge deletions, and $es_p(G) = \sum_{i=1}^s es_p(H_i)$ otherwise.

Theorems 1.1 and 1.2 imply that $es_p(G)$ can be computed by the $p$-edge stability numbers of the components of $G$ if the invariant is additive or if it is maxing and monotone increasing. Therefore, it is sufficient to consider connected graphs $G$ in these cases.

The following results provide lower bounds for $es_p(G)$.

**Theorem 1.3.** [7] Let $\rho(G)$ be monotone and let $G$ be a nonempty graph with $\rho(G) = k$. If $G$ contains $s$ nonempty subgraphs $G_1, \ldots, G_s$ such that $\rho(G_1) = \cdots = \rho(G_s) = k$ such that $a \geq 0$ is the number of edges that occur in at least two of these subgraphs and $q \geq 1$ is the maximum number of these subgraphs with a common edge, then both $es_p(G) \geq \frac{1}{q} \sum_{i=1}^s es_p(G_i) \geq s/q$ and $es_p(G) \geq \sum_{i=1}^s es_p(G_i) - a(q - 1)$ hold.

**Corollary 1.1.** [7] Let $\rho(G)$ be monotone and let $G$ be a nonempty graph with $\rho(G) = k$. If $G$ contains $s$ nonempty subgraphs $G_1, \ldots, G_s$ with $\rho(G_1) = \cdots = \rho(G_s) = k$ and pairwise disjoint edge sets, then $es_p(G) \geq \sum_{i=1}^s es_p(G_i) \geq s$.

In 1959 Gallai proved the following results [5]. Let $G$ be a graph of order $\alpha(G)$ without isolated vertices, $\alpha(G)$ be the independence number, that is, the maximum number of mutually non-adjacent vertices of $G$, $\beta(G)$ the vertex covering number, that is, the minimum number of vertices of $G$ such that every edge of $G$ is incident to at least one of these vertices, $\alpha'(G)$ the edge independence number or matching number, that is, the maximum number of mutually non-adjacent edges of $G$, and $\beta'(G)$ the edge covering number, that is, the minimum number of edges of $G$ such that every vertex of $G$ is incident to at least one of these edges. Then $\alpha(G) + \beta(G) = m(G)$ and $\alpha'(G) + \beta'(G) = m(G)$. The latter equation nowadays is known as Gallai’s Theorem. We prove a corresponding result for invariants that depend on the edge stability number $es_p(G)$.

2. Results

The following results are based on Gallai’s Theorem [5]. We define two invariants $\alpha'_p(G)$ and $\beta'_p(G)$ as follows.

**Definition 2.1.** If $\rho(G)$ is an invariant, then $\alpha'_p(G)$ is defined to be the maximum number of edges of a spanning subgraph $H$ of $G$ with $\rho(H) \neq \rho(G)$. If such a subgraph does not exist (that is, if $\rho(H)$ is constant for all spanning subgraphs $H$ of $G$), then we set $\alpha'_p(G) = \infty$. Let $\beta'_p(G)$ be the minimum number of edges of $G$ that cover all nonempty spanning subgraphs $H$ of $G$ with $\rho(H) = \rho(G)$, that is, each such subgraph must contain at least one edge of the covering set.

Note that $0 \leq \beta'_p(G) \leq m(G)$ where $m(G)$ is the size $|E(G)|$ of $G$. If $\rho(H)$ is constant for all spanning subgraphs $H$ of $G$, then $es_p(G) = \alpha'_p(G) = \infty$ by the definitions and $\beta'_p(G) = m(G)$ (including the case that $G$ is empty) by considering the spanning subgraphs that contain a single edge $e \in E(G)$.

In the following we require that $\rho(H)$ is not constant for all spanning subgraphs $H$ of $G$ which is equivalent to requiring that $es_p(G) < \infty$.

**Lemma 2.1.** If $es_p(G) < \infty$ then $es_p(G) = m(G) - \alpha'_p(G)$.

**Proof.** Since $\rho(G)$ can be changed by edge deletions, there are sets $E' \subseteq E(G)$ with $\rho(G - E') \neq \rho(G)$. If $|E'| = es_p(G)$ is in addition minimal, then the size of the spanning subgraph $G - E'$ is maximal, and vice versa. This implies that $\alpha'_p(G) = m(G) - es_p(G)$, that is, $es_p(G) = m(G) - \alpha'_p(G)$.

$\square$
Theorem 2.1. If \( \rho(G) \) is monotone with respect to spanning subgraphs and \( es_\rho(G) < \infty \), then \( \alpha'_\rho(G) + \beta'_\rho(G) = m(G) \).

Proof. Note that \( es_\rho(G) < \infty \) implies \( \alpha'_\rho(G) < \infty \).

Let \( G' = (V(G), E') \) be a spanning subgraph of \( G \) with \( E' \subseteq E(G) \), \( |E'| = \alpha'_\rho(G) \), and \( \rho(G') \neq \rho(G) \). Then the complement \( G' = G \setminus E' \) covers all nonempty spanning subgraphs \( H \) of \( G \) with \( \rho(H) = \rho(G) \). Suppose not, then there is a nonempty spanning subgraph \( H \) of \( G \) with \( \rho(H) = \rho(G) \) that contains no edge of \( E' \), that is, \( E(H) \subseteq E' \) and \( H \) is a spanning subgraph of \( G' \). But \( \rho(G) \) is monotone with respect to subgraphs, so either \( \rho(H) \leq \rho(G') < \rho(G) \), or \( \rho(H) \geq \rho(G') > \rho(G) \), that is, \( \rho(H) \notin \rho(G) \), a contradiction.

This implies \( \beta'_\rho(G) \leq |E'| = m(G) - \alpha'_\rho(G) \) by the minimality of \( \beta'_\rho(G) \), that is, \( \alpha'_\rho(G) + \beta'_\rho(G) \leq m(G) \).

Conversely, let \( E'' \subseteq E(G) \) be a set of \( \beta'\rho(G) \) edges that covers all nonempty spanning subgraphs \( H \) of \( G \) with \( \rho(H) = \rho(G) \).

Consider the complement \( G'' = G \setminus E'' \) and \( G'' = (V(G), \overline{E''}) \). If \( G'' \) is empty, then \( \rho(G'') \neq \rho(G) \) by the monotonicity and \( es_\rho(G) < \infty \). Otherwise, \( G'' \) is a nonempty spanning subgraph of \( G \) with no edge of the covering set \( E'' \).

Then \( \rho(G'') \neq \rho(G) \) since otherwise \( G'' \) would contain an edge of the covering set \( E'' \), a contradiction. By the maximality, \( \alpha'_\rho(G) \geq |E''| = m(G) - \beta'_\rho(G) \), that is, \( \alpha'_\rho(G) + \beta'_\rho(G) \geq m(G) \) and thus equality follows.

\( \square \)

Corollary 2.1. If \( \rho(G) \) is monotone with respect to spanning subgraphs and \( es_\rho(G) < \infty \), then \( es_\rho(G) = \beta'_\rho(G) \).

Proof. By Lemma 2.1 and Theorem 2.1, \( es_\rho(G) = m(G) - \alpha'_\rho(G) = \beta'_\rho(G) \).

These results imply that only one of the invariants \( es_\rho(G), \alpha'_\rho(G), \beta'_\rho(G) \) needs to be determined in order to know also the other two invariants if \( \rho(G) \) is monotone with respect to spanning subgraphs. Moreover, known bounds for \( es_\rho(G) \) can also be applied to the other two invariants. We give some examples considering the chromatic number \( \chi(G) \) and the chromatic index \( \chi'(G) \) of a graph \( G \), which are monotone increasing invariants (see [7, 10]).

Example 2.1.

(1). If \( G \) is a non-empty bipartite graph, then \( \chi(G) = 2 \) and all edges of \( G \) must be removed in order to lower the chromatic number. Therefore, \( es_\chi(G) = |E(G)| = m(G) \) and \( \beta'_\chi(G) = m(G) \) by Corollary 2.1. This can also be shown directly since all subgraphs induced by a single edge have the same chromatic number as \( G \) and must be covered. Note that \( \alpha'_\chi(G) = 0 \) since the only subgraph of \( G \) with a lower chromatic number is empty.

(2). Consider the Petersen graph \( P \) with chromatic number \( \chi(P) = 3 \). There are 12 cycles \( C_5 \) in \( P \), and each edge \( e = uv \) is contained in 4 of them: The end-vertices \( u, v \) have 2 neighbors each that do not belong to \( e \), so there are 2 · 2 = 4 paths \( P_4 \) with \( e \) as middle edge, and their end-vertices are connected by a path of length 2 which forms a \( C_5 \). This shows that at least \( 12/4 = 3 \) edges are needed to cover all odd cycles of \( P \), that is, \( \beta'_\chi(P) \geq 3 \). On the other hand, consider an independent vertex set \( S \) of \( P \) of cardinality 4. The subgraph \( P - S \) contains 3 edges, \( e_1, e_2, e_3 \) such that \( P - \{e_1, e_2, e_3\} \) is isomorphic to a complete graph \( K_4 \) with vertex set \( S \) and each edge subdivided once (the subdivision vertices are the end-vertices of \( e_1, e_2, e_3 \)). This is a bipartite graph with partition sets \( S \) and the set of subdivision vertices, which implies that \( es_\chi(P) \leq 3 \). Therefore, \( es_\chi(P) = \beta'_\chi(P) = 3 \) by Corollary 2.1 and \( \alpha'_\chi(P) = 2 · 6 = 12 \) by Lemma 2.1.

(3). Consider the complete \( r \)-partite graph \( K = K_{n_1,n_2,\ldots,n_r} \) with \( n_1 \leq n_2 \leq \cdots \leq n_r \), \( r \geq 3 \). Removing all \( n_1 \) edges between the two smallest partition sets gives a complete \( (r-1) \)-partite subgraph \( K_{n_1+n_2,n_3,\ldots,n_r} \). On the other hand, removing less than \( n_1 \) arbitrary edges gives a subgraph with a \( K_r \) (see [10]). This shows that the largest \( (r-1) \)-partite subgraph is \( K_{n_1,n_2,n_3,\ldots,n_r} \), that is, \( \alpha'_\chi(K) = m(K_{n_1+n_2,n_3,\ldots,n_r}) \). Therefore, \( es_\chi(K) = \beta'_\chi(K) = m(K) - \alpha'_\chi(K) = n_1 \) by Lemma 2.1 and Corollary 2.1.

Example 2.2. For complete graphs \( K_n \), \( n \geq 2 \), it holds that \( \chi'(K_n) = n - 1 \) if \( n \) is even and \( \chi'(K_n) = n \) if \( n \) is odd. Removing the edges of an arbitrary color in a \( \chi'(K_n) \)-edge coloring of \( K_n \) (that is, a perfect matching if \( n \) is even or a near-perfect matching if \( n \) is odd) reduces the chromatic index, so \( es_{\chi'}(K_n) \leq \lfloor n/2 \rfloor \) follows. If \( n \) is even, then removing less than \( n/2 \) edges leaves a vertex of degree \( n - 1 \) which implies that the chromatic index of the subgraph is the same as that of \( K_n \). If \( n \) is odd, then an edge coloring with \( n - 1 \) colors may only properly color \( (n - 1)(n - 1)/2 \) edges of a subgraph of \( K_n \), which implies that at least \( n(n - 1)/2 - (n - 1)(n - 1)/2 = (n - 1)/2 \) edges must be removed in order to reduce the chromatic index. In both cases it follows \( es_{\chi'}(K_n) \geq \lfloor n/2 \rfloor \). Therefore, \( es_{\chi'}(K_n) = \lfloor n/2 \rfloor \) (see also [7]) and \( \beta'_\chi(K_n) = \lfloor n/2 \rfloor \) by Corollary 2.1. Thus, by Lemma 2.1, the largest subgraph of \( K_n \) with chromatic index less than \( \chi'(K_n) \) is obtained by removing \( \lfloor n/2 \rfloor \) independent edges, that is, the complete multipartite graph with \( n/2 \) partition sets of size 2 if \( n \) is even or with 1 partition set of size 1 and \( (n - 1)/2 \) partition sets of size 2 if \( n \) is odd.

If \( \rho(G) \) is monotone with respect to spanning subgraphs and \( es_\rho(G) < \infty \), then Theorem 1.3 and Corollary 1.1 also give bounds for \( \beta'_\rho(G) \) in terms of \( \beta'_\rho(G_i) \) by Corollary 2.1, where the \( G_i \) are nonempty subgraphs of \( G \) with the same value of the invariant \( \rho \).
3. Remarks

If we remove vertices instead of edges, then we obtain the following related invariant (see [9]).

**Definition 3.1.** The $\rho$-vertex stability number $\text{vs}_\rho(G)$ of a graph $G$ is the minimum number of vertices of $G$ whose removal results in a graph $H \subseteq G$ with $\rho(H) \neq \rho(G)$. If such a vertex set does not exist, then we set $\text{vs}_\rho(G) = \infty$.

In [9] we proved a corresponding Gallai’s Theorem type result for the vertex stability of graphs.

**Definition 3.2.** If $\rho(G)$ is an invariant, then $\alpha_\rho(G)$ is defined to be the maximum number of vertices of an induced subgraph $H$ of $G$ with $\rho(H) = \rho(G)$, that is, each such subgraph must contain at least one vertex of the covering set.

Note that $1 \leq \beta_\rho(G) \leq n(G)$ since $G$ is an induced subgraph of itself. If $\rho(H)$ is constant for all induced subgraphs $H$ of $G$ (including $K_1$), then $\text{vs}_\rho(G) = \alpha_\rho(G) = \infty$ and $\beta_\rho(G) = n(G)$ by the definitions.

We proved in [9] that $\text{vs}_\rho(G) = n(G) - \alpha_\rho(G)$ if $\text{vs}_\rho(G) < \infty$. Moreover, if $\rho(G)$ is monotone with respect to induced subgraphs and $\text{vs}_\rho(G) < \infty$, then $\alpha_\rho(G) + \beta_\rho(G) = n(G)$, and therefore $\text{vs}_\rho(G) = \beta_\rho(G)$.

**References**