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Research Article

A Gallai's Theorem type result for the edge stability of graphs

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Abstract

For an arbitrary invariant $\rho(G)$ of a graph G the ρ-edge stability number $es_\rho(G)$ of G is the minimum number of edges of G whose removal results in a graph $H \subseteq G$ with $\rho(H) \neq \rho(G)$. If such an edge set does not exist, then $es_\rho(G) = \infty$. Gallai's Theorem states that $\alpha'(G) + \beta'(G) = n(G)$ for a graph G without isolated vertices, where $\alpha'(G)$ is the matching number, $\beta'(G)$ the edge covering number, and $n(G)$ the order of G. We prove a corresponding result for invariants that are based on the edge stability number $es_\rho(G)$.

Keywords: edge stability number; graph invariant; Gallai's Theorem.

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1. Introduction

We consider finite simple graphs $G = (V(G), E(G))$ and denote the class of finite simple graphs by $\mathcal{I}$. An empty graph is a graph with empty edge set.

Definition 1.1. *A (graph) invariant $\rho(G)$ is a function $\rho : \mathcal{I} \to \mathbb{R}_0^+ \cup \{\infty\}$. An invariant is integer-valued if $\rho(\mathcal{I}) \subseteq \mathbb{N}_0$. An invariant $\rho(G)$ is monotone increasing if $H \subseteq G$ implies $\rho(H) \leq \rho(G)$, and monotone decreasing if $H \subseteq G$ implies $\rho(H) \geq \rho(G)$; $\rho(G)$ is monotone if it is monotone increasing or monotone decreasing. If the conditions hold for certain classes of subgraphs (for example, induced or spanning subgraphs), then we say that $\rho(G)$ is monotone (increasing or decreasing) with respect to the class.*

Definition 1.2. *If H_1 and H_2 are disjoint graphs, then an invariant is called additive if $\rho(H_1 \cup H_2) = \rho(H_1) + \rho(H_2)$ and maxing if $\rho(H_1 \cup H_2) = \max\{\rho(H_1), \rho(H_2)\}$.*

For example, the maximum degree $\Delta(G)$ of a graph G is integer-valued, monotone increasing, and maxing. The minimum degree $\delta(G)$ is integer-valued, not monotone, but monotone increasing with respect to spanning subgraphs, not additive, and not maxing. The independence number $\alpha(G)$ is integer-valued, not monotone, but monotone increasing with respect to induced subgraphs and monotone decreasing with respect to spanning subgraphs, and additive. The chromatic number $\chi(G)$ is integer-valued, monotone increasing, and maxing. The domination number $\gamma(G)$ is integer-valued, not monotone, and additive.

It is an interesting topic to determine the stability of an arbitrary invariant $\rho(G)$ of a graph G with respect to specific graph operations such as removing vertices of G, or removing edges, or subdividing edges. The stability with respect to removing edges from G leads to the following invariant.

Definition 1.3. *The ρ-edge stability number $es_\rho(G)$ of a graph G is the minimum number of edges of G whose removal results in a graph $H \subseteq G$ with $\rho(H) \neq \rho(G)$. If such an edge set does not exist, then we set $es_\rho(G) = \infty$.*

In [3] the ρ-edge stability number is also defined and called ρ-line-stability. This paper contains just some basic results on this topic.

For some specific invariants $\rho(G)$ the problem of determining the ρ-edge stability number was already considered, for example for the chromatic number $\chi(G)$, for the chromatic index $\chi'(G)$, for the total chromatic number $\chi''(G)$, and particularly for the domination number $\gamma(G)$.

The χ-edge stability number or chromatic edge stability number $es_\chi(G)$ was introduced in [2, 11] and also studied in [1, 3, 4, 6–8, 10]. The χ'-edge stability number or chromatic edge stability index $es_{\chi'}(G)$ was considered, among others,

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in [7] and the χ''-edge stability number or total chromatic edge stability number $es_{\chi''}(G)$ in [6]. The increase of the domination number $\gamma(G)$ with respect to edge removal was extensively studied (see e.g. [3] or [12] for a survey). The so-called bondage number $b(G)$ coincides with the γ-edge stability number $es_\gamma(G)$.

Let us mention that in our previous papers on this topic [6–8, 10] we used a different definition for the second (trivial) case.

Two observations on $es_\rho(G)$ are that if $\rho(G) \neq \rho(G-E')$, then $es_\rho(G) \leq |E'|$, and if $\rho(G-E') \neq \rho(G-E'')$, then $es_\rho(G) \leq \max\{|E'|, |E''|\}$, where $E', E'' \subseteq E(G)$. Moreover, $es_\rho(G) \leq es_\rho(G-E') + |E'|$.

In [7] we proved several general results on the ρ-edge stability number $es_\rho(G)$.

Theorem 1.1. [7] Let $\rho(G)$ be additive, $G = H_1 \cup \cdots \cup H_k$ a graph whose subgraphs $H_1, \ldots, H_k$ and the integer $s \geq 0$ are defined such that $\rho(H_i)$ can be changed by edge deletion if and only if $1 \leq i \leq s$. Then $es_\rho(G) = \infty$ if $s = 0$ and $es_\rho(G) = \min\{es_\rho(H_i) : 1 \leq i \leq s\}$ if $s \neq 0$.

For maxing invariants we proved the following result.

Theorem 1.2. [7] Let $\rho(G)$ be maxing and monotone increasing, $G = H_1 \cup \cdots \cup H_k$ a graph whose subgraphs $H_1, \ldots, H_k$ and the integer $s \geq 1$ are defined such that $\rho(H_i) = \rho(G)$ if and only if $1 \leq i \leq s$. Then $es_\rho(G) = \infty$ if there is a subgraph H_j, $1 \leq j \leq s$, such that $\rho(H_j)$ cannot be changed by edge deletions, and $es_\rho(G) = \sum_{i=1}^{s} es_\rho(H_i)$ otherwise.

Theorems 1.1 and 1.2 imply that $es_\rho(G)$ can be computed by the ρ-edge stability numbers of the components of G if the invariant is additive or if it is maxing and monotone increasing. Therefore, it is sufficient to consider connected graphs G in these cases.

The following results provide lower bounds for $es_\rho(G)$.

Theorem 1.3. [7] Let $\rho(G)$ be monotone and let G be a nonempty graph with $\rho(G) = k$. If G contains s nonempty subgraphs $G_1, \ldots, G_s$ with $\rho(G_1) = \cdots = \rho(G_s) = k$ such that $a \geq 0$ is the number of edges that occur in at least two of these subgraphs and $q \geq 1$ is the maximum number of these subgraphs with a common edge, then both $es_\rho(G) \geq \frac{1}{q}\sum_{i=1}^{s} es_\rho(G_i) \geq s/q$ and $es_\rho(G) \geq \sum_{i=1}^{s} es_\rho(G_i) - a(q-1)$ hold.

Corollary 1.1. [7] Let $\rho(G)$ be monotone and let G be a nonempty graph with $\rho(G) = k$. If G contains s nonempty subgraphs $G_1, \ldots, G_s$ with $\rho(G_1) = \cdots = \rho(G_s) = k$ and pairwise disjoint edge sets, then $es_\rho(G) \geq \sum_{i=1}^{s} es_\rho(G_i) \geq s$.

In 1959 Gallai proved the following results [5]. Let G be a graph of order $n(G)$ without isolated vertices, $\alpha(G)$ be the independence number, that is, the maximum number of mutually non-adjacent vertices of G, $\beta(G)$ the vertex covering number, that is, the minimum number of vertices of G such that every edge of G is incident to at least one of these vertices, $\alpha'(G)$ the edge independence number or matching number, that is, the maximum number of mutually non-adjacent edges of G, and $\beta'(G)$ the edge covering number, that is, the minimum number of edges of G such that every vertex of G is incident to at least one of these edges. Then $\alpha(G) + \beta(G) = n(G)$ and $\alpha'(G) + \beta'(G) = n(G)$. The latter equation nowadays is known as Gallai's Theorem. We prove a corresponding result for invariants that depend on the edge stability number $es_\rho(G)$.

2. Results

The following results are based on Gallai's Theorem [5]. We define two invariants $\alpha'_\rho(G)$ and $\beta'_\rho(G)$ as follows.

Definition 2.1. *If $\rho(G)$ is an invariant, then $\alpha'_\rho(G)$ is defined to be the maximum number of edges of a spanning subgraph H of G with $\rho(H) \neq \rho(G)$. If such a subgraph does not exist (that is, if $\rho(H)$ is constant for all spanning subgraphs H of G), then we set $\alpha'_\rho(G) = \infty$. Let $\beta'_\rho(G)$ be the minimum number of edges of G that cover all nonempty spanning subgraphs H of G with $\rho(H) = \rho(G)$, that is, each such subgraph must contain at least one edge of the covering set.*

Note that $0 \leq \beta'_\rho(G) \leq m(G)$ where $m(G)$ is the size $|E(G)|$ of G. If $\rho(H)$ is constant for all spanning subgraphs H of G, then $es_\rho(G) = \alpha'_\rho(G) = \infty$ by the definitions and $\beta'_\rho(G) = m(G)$ (including the case that G is empty) by considering the spanning subgraphs that contain a single edge $e \in E(G)$.

In the following we require that $\rho(H)$ is not constant for all spanning subgraphs H of G which is equivalent to requiring that $es_\rho(G) < \infty$.

Lemma 2.1. *If $es_\rho(G) < \infty$, then $es_\rho(G) = m(G) - \alpha'_\rho(G)$.*

Proof. Since $\rho(G)$ can be changed by edge deletions, there are sets $E' \subseteq E(G)$ with $\rho(G-E') \neq \rho(G)$. If $|E'| = es_\rho(G)$ is in addition minimal, then the size of the spanning subgraph $G - E'$ is maximal, and vice versa. This implies that $\alpha'_\rho(G) = m(G) - es_\rho(G)$, that is, $es_\rho(G) = m(G) - \alpha'_\rho(G)$. □

Theorem 2.1. *If $\rho(G)$ is monotone with respect to spanning subgraphs and $es_\rho(G) < \infty$, then $\alpha'_\rho(G) + \beta'_\rho(G) = m(G)$.*

Proof. Note that $es_\rho(G) < \infty$ implies $\alpha'_\rho(G) < \infty$.

Let $G' = (V(G), E')$ be a spanning subgraph of G with $E' \subsetneq E(G)$, $|E'| = \alpha'_\rho(G)$, and $\rho(G') \neq \rho(G)$. Then the complement $\overline{E'} = E(G) \setminus E'$ covers all nonempty spanning subgraphs H of G with $\rho(H) = \rho(G)$. Suppose not, then there is a nonempty spanning subgraph H of G with $\rho(H) = \rho(G)$ that contains no edge of $\overline{E'}$, that is, $E(H) \subseteq E'$ and H is a spanning subgraph of G'. But $\rho(G)$ is monotone with respect to spanning subgraphs, so either $\rho(H) \leq \rho(G') < \rho(G)$, or $\rho(H) \geq \rho(G') > \rho(G)$, that is, $\rho(H) \neq \rho(G)$, a contradiction.

This implies $\beta'_\rho(G) \leq |\overline{E'}| = m(G) - \alpha'_\rho(G)$ by the minimality of $\beta'_\rho(G)$, that is, $\alpha'_\rho(G) + \beta'_\rho(G) \leq m(G)$.

Conversely, let $E'' \subseteq E(G)$ be a set of $\beta'_\rho(G)$ edges that covers all nonempty spanning subgraphs H of G with $\rho(H) = \rho(G)$. Consider the complement $\overline{E''} = E(G) \setminus E''$ and $G'' = (V(G), \overline{E''})$. If G'' is empty, then $\rho(G'') \neq \rho(G)$ by the monotonicity and $es_\rho(G) < \infty$. Otherwise, G'' is a nonempty spanning subgraph of G with no edge of the covering set E''. Then $\rho(G'') \neq \rho(G)$ since otherwise G'' would contain an edge of the covering set E'', a contradiction. By the maximality, $\alpha'_\rho(G) \geq |\overline{E''}| = m(G) - \beta'_\rho(G)$, that is, $\alpha'_\rho(G) + \beta'_\rho(G) \geq m(G)$ and thus equality follows. □

Corollary 2.1. *If $\rho(G)$ is monotone with respect to spanning subgraphs and $es_\rho(G) < \infty$, then $es_\rho(G) = \beta'_\rho(G)$.*

Proof. By Lemma 2.1 and Theorem 2.1, $es_\rho(G) = m(G) - \alpha'_\rho(G) = \beta'_\rho(G)$. □

These results imply that only one of the invariants $es_\rho(G)$, $\alpha'_\rho(G)$, $\beta'_\rho(G)$ needs to be determined in order to know also the other two invariants if $\rho(G)$ is monotone with respect to spanning subgraphs. Moreover, known bounds for $es_\rho(G)$ can also be applied to the other two invariants. We give some examples considering the chromatic number $\chi(G)$ and the chromatic index $\chi'(G)$ of a graph G, which are monotone increasing invariants (see [7, 10]).

Example 2.1.

(1). If G is a non-empty bipartite graph, then $\chi(G) = 2$ and all edges of G must be removed in order to lower the chromatic number. Therefore, $es_\chi(G) = |E(G)| = m(G)$ and $\beta'_\chi(G) = m(G)$ by Corollary 2.1. This can also be shown directly since all subgraphs induced by a single edge have the same chromatic number as G and must be covered. Note that $\alpha'_\chi(G) = 0$ since the only subgraph of G with a lower chromatic number is empty.

(2). Consider the Petersen graph P with chromatic number $\chi(P) = 3$. There are 12 cycles C_5 in P, and each edge $e = uv$ is contained in 4 of them: The end-vertices u, v have 2 neighbors each that do not belong to e, so there are $2 \cdot 2 = 4$ paths P_4 with e as middle edge, and their end-vertices are connected by a path of length 2 which forms a C_5. This shows that at least $12/4 = 3$ edges are needed to cover all odd cycles of P, that is, $\beta'_\chi(P) \geq 3$. On the other hand, consider an independent vertex set S of P of cardinality 4. The subgraph $P - S$ contains 3 edges e_1, e_2, e_3 such that $P - \{e_1, e_2, e_3\}$ is isomorphic to a complete graph K_4 with vertex set S and each edge subdivided once (the subdivision vertices are the end-vertices of e_1, e_2, e_3). This is a bipartite graph with partition sets S and the set of subdivision vertices, which implies that $es_\chi(P) \leq 3$. Therefore, $es_\chi(P) = \beta'_\chi(P) = 3$ by Corollary 2.1 and $\alpha'_\chi(P) = 2 \cdot 6 = 12$ by Lemma 2.1.

(3). Consider the complete r-partite graph $K = K_{n_1,n_2,\ldots,n_r}$ with $n_1 \leq n_2 \leq \cdots \leq n_r$, $r \geq 3$. Removing all $n_1 n_2$ edges between the two smallest partition sets gives a complete $(r-1)$-partite subgraph $K_{n_1+n_2,n_3,\ldots,n_r}$. On the other hand, removing less than $n_1 n_2$ arbitrary edges gives a subgraph with a K_r (see [10]). This shows that the largest $(r-1)$-partite subgraph is $K_{n_1+n_2,n_3,\ldots,n_r}$, that is, $\alpha'_\chi(K) = m(K_{n_1+n_2,n_3,\ldots,n_r})$. Therefore, $es_\chi(K) = \beta'_\chi(K) = m(K) - \alpha'_\chi(K) = n_1 n_2$ by Lemma 2.1 and Corollary 2.1.

Example 2.2. For complete graphs K_n, $n \geq 2$, it holds that $\chi'(K_n) = n - 1$ if n is even and $\chi'(K_n) = n$ if n is odd. Removing the edges of an arbitrary color in a $\chi'(K_n)$-edge coloring of K_n (that is, a perfect matching if n is even or a near-perfect matching if n is odd) reduces the chromatic index, so $es_{\chi'}(K_n) \leq \lfloor n/2 \rfloor$ follows. If n is even, then removing less than $n/2$ edges leaves a vertex of degree $n - 1$ which implies that the chromatic index of the subgraph is the same as that of K_n. If n is odd, then an edge coloring with $n - 1$ colors may only properly color $(n-1)(n-1)/2$ edges of a subgraph of K_n, which implies that at least $n(n-1)/2 - (n-1)(n-1)/2 = (n-1)/2$ edges must be removed in order to reduce the chromatic index. In both cases it follows $es_{\chi'}(K_n) \geq \lfloor n/2 \rfloor$. Therefore, $es_{\chi'}(K_n) = \lfloor n/2 \rfloor$ (see also [7]) and $\beta'_\chi(K_n) = \lfloor n/2 \rfloor$ by Corollary 2.1. Thus, by Lemma 2.1, the largest subgraph of K_n with chromatic index less than $\chi'(K_n)$ is obtained by removing $\lfloor n/2 \rfloor$ independent edges, that is, the complete multipartite graph with $n/2$ partition sets of size 2 if n is even or with 1 partition set of size 1 and $(n-1)/2$ partition sets of size 2 if n is odd.

If $\rho(G)$ is monotone with respect to spanning subgraphs and $es_\rho(G) < \infty$, then Theorem 1.3 and Corollary 1.1 also give bounds for $\beta'_\rho(G)$ in terms of $\beta'_\rho(G_i)$ by Corollary 2.1, where the G_i are nonempty subgraphs of G with the same value of the invariant ρ.

3. Remarks

If we remove vertices instead of edges, then we obtain the following related invariant (see [9]).

Definition 3.1. *The ρ-vertex stability number $vs_\rho(G)$ of a graph G is the minimum number of vertices of G whose removal results in a graph $H \subseteq G$ with $\rho(H) \neq \rho(G)$. If such a vertex set does not exist, then we set $vs_\rho(G) = \infty$.*

In [9] we proved a corresponding Gallai's Theorem type result for the vertex stability of graphs.

Definition 3.2. *If $\rho(G)$ is an invariant, then $\alpha_\rho(G)$ is defined to be the maximum number of vertices of an induced subgraph H of G with $\rho(H) \neq \rho(G)$. If such a subgraph does not exist (that is, if $\rho(H)$ is constant for all induced subgraphs H of G), then we set $\alpha_\rho(G) = \infty$. Let $\beta_\rho(G)$ be the minimum number of vertices that cover all induced subgraphs H of G with $\rho(H) = \rho(G)$, that is, each such subgraph must contain at least one vertex of the covering set.*

Note that $1 \leq \beta_\rho(G) \leq n(G)$ since G is an induced subgraph of itself. If $\rho(H)$ is constant for all induced subgraphs H of G (including K_1), then $vs_\rho(G) = \alpha_\rho(G) = \infty$ and $\beta_\rho(G) = n(G)$ by the definitions.

We proved in [9] that $vs_\rho(G) = n(G) - \alpha_\rho(G)$ if $vs_\rho(G) < \infty$. Moreover, if $\rho(G)$ is monotone with respect to induced subgraphs and $vs_\rho(G) < \infty$, then $\alpha_\rho(G) + \beta_\rho(G) = n(G)$, and therefore $vs_\rho(G) = \beta_\rho(G)$.

References

[1] S. Akbari, S. Klavžar, N. Movarraei, M. Nahvi, Nordhaus-Gaddum and other bounds for the chromatic edge-stability number, *European J. Combin.* **84** (2020) #103042.

[2] S. Arumugam, I. Sahul Hamid, A. Muthukamatchi, Independent domination and graph colorings, *Ramanujan Math. Soc. Lect. Notes Ser. Math.* **7** (2008) 195–203.

[3] D. Bauer, F. Harary, J. Nieminen, C. L. Suffel, Domination alteration sets in graphs, *Discrete Math.* **47** (1983) 153–161.

[4] B. Brešar, S. Klavžar, N. Movarraei, Critical graphs for the chromatic edge-stability number, *Discrete Math.* **343** (2020) #111845.

[5] T. Gallai, Über extreme Punkt- und Kantenmengen (German), *Ann. Univ. Sci. Budapest. Eőtvős Sect. Math.* **2** (1959) 133–138.

[6] A. Kemnitz, M. Marangio, On the total chromatic edge stability number and the total chromatic subdivision number of graphs, *Discrete Math. Lett.* **10** (2022) 1–8.

[7] A. Kemnitz, M. Marangio, On the ρ-edge stability number of graphs, *Discuss. Math. Graph Theory* **42** (2022) 249–262.

[8] A. Kemnitz, M. Marangio, On the ρ-subdivision number of graphs, *Discuss. Math. Graph Theory* **43** (2023) 979–997.

[9] A. Kemnitz, M. Marangio, On the vertex stability number of graphs, Submitted.

[10] A. Kemnitz, M. Marangio, N. Movarraei, On the chromatic edge stability number of graphs, *Graphs Combin.* **34** (2018) 1539–1551.

[11] W. Staton, Edge deletions and the chromatic number, *Ars Combin.* **10** (1980) 103–106.

[12] J.-M. Xu, On bondage numbers of graphs: A survey with some comments, *Int. J. Combin.* **2013** (2013) #595210.