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Research Article

Generating trees for 0021-avoiding inversion sequences and a conjecture of Hong and Li

Toufik Mansour*

Department of Mathematics, University of Haifa, Haifa 3498838, Israel

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Abstract

An *inversion sequence* of length n is a word $e = e_0 \cdots e_n$ which satisfies, for each $i \in [n] = \{0, 1, \ldots, n\}$, the inequality $0 \le e_i \le i$. In this paper, by generating tree tools, an explicit formula is found for the generating function for the number of inversion sequences of length n that avoid 0021, which resolves the conjecture of Hong and Li posed in the recent paper [*Electron. J. Combin.* **29** (2022) #4.37].

Keywords: inversion sequences; generating trees; 0021-avoiding inversion sequences.

2020 Mathematics Subject Classification: 05A05, 05A15.

1. Introduction

An *inversion sequence* [5, 10] of length n is a word $e = e_0 \cdots e_n$ which satisfies for each $i \in [n] = \{0, 1, \ldots, n\}$ the inequality $0 \le e_i \le i$. The set of inversion sequences of length n is denoted by $\mathbf{I}_n$.

We say that a word $x = x_1 \cdots x_n$ is *order-isomorphic* to a word $y = y_1 \cdots y_n$ if for every pair of indices $i, j \in [n]$, we have $x_i < x_j$ if and only if $y_i < y_j$. We say that a word $w = w_1 \cdots w_n$ *contains* a word $p = p_1 \cdots p_m$ if w contains a (not necessarily consecutive) subsequence of length m which is order-isomorphic to p. Otherwise, we say that w *avoids* p. In such context, p is usually called a *pattern*. We denote the set of all inversion sequences in $\mathbf{I}_n$ that avoid a pattern p by $\mathbf{I}_n(p)$.

The systematic study of pattern avoidance for inversion sequences is initiated around 2015 [5, 10]. Several aspects of pattern-avoidance for inversion sequences have been considered (for example, see [1–3, 6–9, 11–14] and references therein). Recently, Hong and Li [6] conjectured that the generating function $A(x) = \sum_{n\ge0} \mathbf{I}_n(0021)x^{n+1}$ satisfies the relation

$$(1 - A(x))(1 + A(x))^2 = \frac{1}{1 - x}.$$

The aim of this paper is to prove this conjecture, namely, we aim to show the following result.

Theorem 1.1. *The generating function* $A(x) = \sum_{n\ge0} |\mathbf{I}_n(0021)|x^{n+1}$ *for the number of inversion sequences of length* n *that avoid* 0021 *satisfies*

$$(1 - A(x))(1 + A(x))^2 = \frac{1}{1 - x}.$$

Moreover,

$$\begin{aligned} A(x) &= \frac{4}{3}\sin\left(\frac{1}{3}\arccos\left(\frac{11+16x}{16(1-x)}\right)+\frac{\pi}{6}\right)-\frac{1}{3} \\ &= x + 2x^2 + 6x^3 + 23x^4 + 101x^5 + 480x^6 + 2400x^7 + 12434x^8 + 66142x^9 + 359112x^{10} \\ &+ 1981904x^{11} + 11085198x^{12} + 62696874x^{13} + 357970472x^{14} + 2060459256x^{15} \\ &+ 11943445311x^{16} + 69656978837x^{17} + 408466559630x^{18} + 2406825745010x^{19} \\ &+ 14243262687023x^{20} + \cdots . \end{aligned}$$

Note that the conjecture of Hong and Li has been simultaneously proved by Chern, Fu, and Lin [4]. Here, by using generating trees tools for the set of inversion sequences that avoid 0021 and then translating to generating functions, we obtain an explicit formula for the generating function $\sum_{n\ge0} \mathbf{I}_n(0021)x^{n+1}$. We end the paper, by considering the generating function $\sum_{n\ge0} \mathbf{I}_n(0011)x^{n+1}$.

*E-mail address: tmansour@univ.haifa.ac.il

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2. Proof of Theorem 1.1

Fix $\tau = 0021$. As a first step, we recall the generating trees for pattern avoidance in inversion sequences as described in [12]. Let $\mathbf{I} = \cup_{n=0}^{\infty} \mathbf{I}_n(\tau)$. We will construct a pattern-avoidance tree $\mathcal{T}$ for the class of pattern-avoiding inversion sequences $\mathbf{I}$. The root is 0 (inversion sequence with one letter), that is, $0 \in \mathcal{T}$. Starting with this root which stays at level 0, the nodes at level $n+1$ of tree $\mathcal{T}$ can be constructed from the nodes at level n such a way that the children of $e = e_0 \cdots e_n \in \mathbf{I}_n(0021)$ are $e' = e_0 \cdots e_n j$ with $j = 0, 1, \ldots, n+1$ such that $e' \in \mathbf{I}_{n+1}(0021)$.

Now, we relabel the vertices of the tree $\mathcal{T}$ as follows. Define $\mathcal{T}(e)$ to be the subtree consisting of the inversion sequence e as the root and its descendants in $\mathcal{T}$. We say that e is *equivalent* to e', denoted by $e \sim e'$, if and only if $\mathcal{T}(e) \cong \mathcal{T}(e')$ (in the sense of plain trees). Let $\mathcal{T}'$ be the same tree $\mathcal{T}$ where we replace each node e by the first node $e' \in \mathcal{T}$ from top to bottom and from left to right in $\mathcal{T}$ such that $\mathcal{T}(e) \cong \mathcal{T}(e')$. For instance, $\mathcal{T}(001) \cong \mathcal{T}(000)$, $\mathcal{T}(002) \cong \mathcal{T}(00)$, and $\mathcal{T}(010) \cong \mathcal{T}(000)$.

Lemma 2.1. *The generating tree $\mathcal{T}'$ is given by root 0 and the following succession rules*

$$\begin{aligned} r_k &\rightsquigarrow t_{0,k+2}, t_{1,k+1}, \ldots, t_{k,2}, r_{k+1}, \qquad k \geq 0, \\ t_{d,m} &\rightsquigarrow t_{0,m+d+1}, t_{1,m+d}, \ldots, t_{d,m+1},\; t_{d,m+1}, t_{d,m}, \ldots, t_{d,2}, \qquad m \geq 2,\ d \geq 0, \end{aligned}$$

where $r_k = 01 \cdots k$ and $t_{d,m} = 01 \cdots (d-1)d^m$.

Proof. We label the inversion sequences $0 \in \mathbf{I}_0(0021)$ by r_0. Thus, $r_0 \rightsquigarrow t_{0,2} r_1$. Thus, it remains to show that the rules are holding. By the definitions, the children of $r_k \in \mathbf{I}_k(0021)$ are $01 \cdots k0, 01 \cdots k1, \cdots, 01 \cdots kk, 012 \cdots k(k+1)$. By reordering the letters of any inversion sequence $r_k \pi' \in \mathbf{I}_n(0021)$, we have that $\mathcal{T}(01 \cdots kd) \cong \mathcal{T}(01 \cdots (d-1)d^{k+2-d})$. Thus,

$$r_k \rightsquigarrow t_{0,m+d+1}, t_{1,m+d}, \ldots, t_{d,m+1}\; t_{d,m+1}, t_{d,m}, \ldots, t_{d,2},$$

for all $d \geq 0$ and $m \geq 2$.

By the definitions, the children of $t_{d,m} \in \mathbf{I}_{m+d}(0021)$ are $t_{d,m} j$ with $0 \leq j \leq d+m$. By reordering the letters of any inversion sequence, we have that $\mathcal{T}(t_{d,m} j) \cong \mathcal{T}(t_{j,m+d+1-j})$ whenever $j = 0, 1, \ldots, d$ and $\mathcal{T}(t_{d,m} j) \cong \mathcal{T}(t_{d,m+2+d-j})$ whenever $j = d+1, \ldots, d+m$. Since similarity, let us explain only the case $j = d+m$ as follows. Let $d \geq 0$, $m \geq 2$, and let $\pi = 01 \cdots (d-1)d^m(d+m)\pi'$ to be any inversion sequence that avoids 0021. Since the subword $dd(d+m)$ plays as 002 in 0021, we have that π' does not contain any letter between d and $d+m$. So by reducing the letters, we see that $\pi \in \mathbf{I}_n(0021)$ if and only if $01 \cdots (d-1)d^2\pi'' \in \mathbf{I}_{n+1-m}(0021)$, where π'' obtain from π' by subtracting $m-1$ from each letter greater than or equal to $d+m$ in π'. Thus, $\mathcal{T}(t_{d,m}(d+m)) \cong \mathcal{T}(t_{d,2})$. Hence,

$$t_{d,m} \rightsquigarrow t_{0,m+d+1}, t_{1,m+d}, \ldots, t_{d,m+1},\; t_{d,m+1}, t_{d,m}, \ldots, t_{d,2}, \qquad m \geq 2,\ d \geq 0,$$

which completes the proof. □

To find an explicit formula for the generating function $A(x) = \sum_{n\geq 0} |\mathbf{I}_n(0021)| x^{n+1}$, we define $R_k(x)$ (respectively, $T_{d,m}(x)$) to be the generating function for the number of nodes at level $n \geq 1$ for the tree of $\mathcal{T}(r_k)$ (respectively, $\mathcal{T}(t_{d,m})$), where its root stays at level 0. Clearly, $A(x) = R_0(x)$. By Lemma 2.1, we have

$$\begin{aligned} R_k(x) &= x + x\sum_{j=0}^{k} T_{j,k+2-j}(x) + xR_{k+1}(x), \quad k \geq 0, \\ T_{d,m}(x) &= x + x\sum_{j=0}^{d} T_{j,d+m+1-j}(x) + x\sum_{j=2}^{m+1} T_{d,j}(x), \quad d \geq 0, m \geq 2. \end{aligned}$$

Define $R(x,v) = \sum_{k\geq 0} R_k(x) v^k$ and $T(x,v,u) = \sum_{d\geq 0}\sum_{m\geq 2} T_{d,m}(x) v^d u^{m-2}$. Then, the recurrence relations can be written as

$$\begin{aligned} R(x,v) &= \frac{x}{1-v} + xT(x,v,v) + \frac{x}{v}(R(x,v) - R(x,0)), \\ T(x,v,u) &= \frac{x}{(1-v)(1-u)} + \frac{x}{u-v}(T(x,v,u) - T(x,v,v)) + \frac{x}{u(1-u)}T(x,v,u) - \frac{x}{u}T(x,v,0). \end{aligned}$$

In particular, by taking $v = x$, we obtain

$$R(x,0) = \frac{x}{1-x} + xT(x,x,x), \tag{1}$$

$$\left(1 - \frac{x}{u-x} - \frac{x}{u(1-u)}\right) T(x,x,u) = \frac{x}{(1-x)(1-u)} - \frac{x}{u-x}T(x,x,x) - \frac{x}{u}T(x,x,0). \tag{2}$$

So, to complete the proof, we have to find an explicit formula for the generating function $T(x,x,x)$.

Let $K(u) = 1 - \frac{x}{u-x} - \frac{x}{u(1-u)}$ be the kernel of (2). For the kernel equation, namely $K(u) = 0$, we have three roots

$$u_j(x) = 2\sqrt{Q}\cos\left(\frac{1}{3}\arccos\left(\frac{R}{Q\sqrt{Q}}\right) + \frac{2\pi j}{3}\right) + \frac{1+2x}{3}, \quad j = 0, 1, 2, \tag{3}$$

where

$$Q = \frac{(1-x)(1-4x)}{9} \quad \text{and} \quad R = \frac{(1-x)(2-13x-16x^2)}{54}.$$

From now, we are interested in the roots

$$u_1(x) = \frac{3-\sqrt{5}}{2}x + \frac{5-2\sqrt{5}}{5}x^2 + \left(\frac{7}{2} - \frac{77\sqrt{5}}{50}\right)x^3 + \left(\frac{29}{2} - \frac{1617\sqrt{5}}{250}\right)x^4 + \cdots,$$

$$u_2(x) = \frac{3+\sqrt{5}}{2}x + \frac{5+2\sqrt{5}}{5}x^2 + \left(\frac{7}{2} + \frac{77\sqrt{5}}{50}\right)x^3 + \left(\frac{29}{2} + \frac{1617\sqrt{5}}{250}\right)x^4 + \cdots.$$

By substituting $u = u_1(x)$ and $u = u_2(x)$ into (2), then solving the obtaining system of equations for $T(x,x,x)$ and $T(x,x,0)$, we obtain

$$\begin{aligned} T(x,x,0) &= \frac{u_1(x)u_2(x)}{x(1-u_1(x))(1-u_2(x))} \qquad (4)\\ &= x^2 + 4x^3 + 16x^4 + 70x^5 + 330x^6 + 1640x^7 + 8461x^8 + \cdots, \\ T(x,x,x) &= -\frac{(u_1(x)-x)(u_2(x)-x)}{x(1-x)(1-u_1(x))(1-u_2(x))} \qquad (5)\\ &= x + 5x^2 + 22x^3 + 100x^4 + 479x^5 + 2399x^6 + 12433x^7 + 66141x^8 + \cdots. \end{aligned}$$

By (1), we have

$$R(x,0) = \frac{x}{1-x} + x\,T(x,x,x). \tag{6}$$

Lemma 2.2. *We have*

$$A(x) = R(x,0) = \frac{x}{u_0(x) - x} = \frac{x}{2\sqrt{Q}\cos\left(\frac{1}{3}\arccos(\frac{R}{Q\sqrt{Q}})\right) + \frac{1-x}{3}},$$

where

$$Q = \frac{(1-x)(1-4x)}{9} \quad \text{and} \quad R = \frac{(1-x)(2-13x-16x^2)}{54}.$$

Moreover, the generating function $A(x)$ *satisfies that* $(1-A(x))(1+A(x))^2 = \frac{1}{1-x}$.

Proof. Since $u_0(x), u_1(x), u_2(x)$ are roots of $K(u) = 0$ (see (3)), then we have $u_1(x) + u_2(x) = 1 + 2x - u_0(x)$ and $u_1(x)u_2(x) = x^2/u_0(x)$. Hence, by (5), we have

$$T(x,x,x) = \frac{1-u_0(x)}{(1-x)(u_0(x)-x)},$$

which, by (6), implies

$$A(x) = R(x,0) = \frac{x}{u_0(x) - x}.$$

Note that $K(u_0(x)) = 0$ that is $u_0^3(x) - (1+2x)u_0^2(x) + 3xu_0(x) - x^2 = 0$. Hence,

$$\begin{aligned} (1-A(x))(1+A(x))^2 &= \frac{(u_0(x)-2x)u_0^2(x)}{(u_0(x)-x)^3} \\ &= \frac{(1+2x)u_0^2(x) - 3xu_0(x) + x^2 - 2xu_0^2(x)}{(1+2x)u_0^2(x) - 3xu_0(x) + x^2 - 3xu_0^2(x) + 3x^2u_0(x) - x^3} \\ &= \frac{u_0^2(x) - 3xu_0(x) + x^2}{(1-x)(u_0^2(x) - 3xu_0(x) + x^2)} \\ &= \frac{1}{1-x}, \end{aligned}$$

which completes the proof. □

By Lemma 2.2, we see that the generating function $A(x)$ satisfies

$$A^3(x)+A^2(x)-A(x)+\frac{1}{1-x}=0.$$

Note that for this equation there are three roots

$$a_j=\frac{4}{3}cos\left(\frac{1}{3}arccos\left(\frac{11+16x}{16(-1+x)}\right)+\frac{2\pi j}{3}\right)-\frac{1}{3}.$$

Since $A(x)$ is a power series with positive coefficients, we have that

$$A(x)=a_3=\frac{4}{3}\sin\left(\frac{1}{3}\arccos\left(\frac{11+16x}{16(1-x)}\right)+\frac{\pi}{6}\right)-\frac{1}{3},$$

which completes the proof of Theorem 1.1.

3. Further results

Similarly, as in the proof of Lemma 2.1, we have the following result for the set of inversion sequences in $\mathbf{I}_n(0011)$.

Lemma 3.1. *The generating tree $\mathcal{T}'$ is given by root* 0 *and the following succession rules*

$$\begin{aligned} r_k &\rightsquigarrow t_{0,k+2}t_{1,k+1}\cdots t_{k,2}r_{k+1}, && k\geq 0,\\ t_{d,m} &\rightsquigarrow t_{0,m+d+1}t_{1,m+d}\cdots t_{d,m+1}(t_{d,m})^m, && m\geq 2,\ d\geq 0, \end{aligned}$$

where $r_k=01\cdots k$, $t_{d,m}=01\cdots(d-1)d^m$, *and* $t_{d,m}^m=\underbrace{t_{d,m},\dots,t_{d,m}}_{m\ times}$.

Define $R_k(x)$ (respectively, $T_{d,m}(x)$) to be the generating function for the number of nodes at level $n\geq 1$ for the tree of $\mathcal{T}(r_k)$ (respectively, $\mathcal{T}(t_{d,m})$), where its root stays at level 1. By Lemma 3.1, we have

$$\begin{aligned} R_k(x)&=x+x\sum_{j=0}^{k}T_{j,k+2-j}(x)+xR_{k+1}(x), \quad k\geq 0,\\ T_{d,m}(x)&=x+x\sum_{j=0}^{d}T_{j,d+m+1-j}(x)+mxT_{d,m}(x), \quad d\geq 0, m\geq 2. \end{aligned}$$

Define $R(x,v)=\sum_{k\geq 0}R_k(x)v^k$ and $T(x,v,u)=\sum_{d\geq 0}\sum_{m\geq 2}T_{d,m}(x)v^du^{m-2}$. Then, the above recurrence relations can be written as

$$\begin{aligned} R(x,v)&=\frac{x}{1-v}+xT(x,v,v)+\frac{x}{v}(R(x,v)-R(x,0)),\\ T(x,v,u)&=\frac{x}{(1-v)(1-u)}+\frac{x}{u-v}(T(x,v,u)-T(x,v,v))+ux\frac{\partial}{\partial u}T(x,v,u)+2xT(x,v,u). \end{aligned}$$

We failed to obtain an explicit formula for $R(x,0)$ from this system of equations, but we still can use this system to generate easily the coefficients of $R(x,0)$. For instance, we have

$$\begin{aligned} R(x,0)=\ &x+2x^2+6x^3+22x^4+92x^5+428x^6+2184x^7+12096x^8+72104x^9+459440x^{10}\\ &+3111616x^{11}+22292592x^{12}+168263312x^{13}+1333377904x^{14}+11059335280x^{15}\\ &+95753379216x^{16}+863373139824x^{17}+8089902823120x^{18}+78625465178608x^{19}\\ &+791248858589264x^{20}+\cdots. \end{aligned}$$

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