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Research Article

The eccentricity spread of weak-friendship graphs

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Abstract

A weak-friendship graph is a connected induced subgraph of a friendship graph. The unique graphs attaining the first two smallest eccentricity spread in the class of weak-friendship graphs of given order are determined in this paper.

Keywords: cacti graphs; friendship graphs; eccentricity spread; eccentricity eigenvalues.

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1. Introduction

Let $G = (V_G, E_G)$ be a graph with order $n = |V_G|$ and size $m = |E_G|$, and let $M(G)$ be a corresponding $n \times n$ real symmetric or Hermitian complex matrix defined in a prescribed way. The M-*spectrum* of G is the multiset $\mathrm{sp}_M(G)$ consisting of the M-*eigenvalues* $\lambda_1^M(G) \geqslant \cdots \geqslant \lambda_n^M(G)$ of $M(G)$, i.e. the roots of the M-*characteristic polynomial* $p_M(G, x) := \det(xI_n - M(G))$. The M-*spread* of G is defined as

$$S_M(G) := \lambda_1^M(G) - \lambda_n^M(G).$$

This algebraic invariant has applications in combinatorial optimization problems (see for instance [7]).

The *distance* between two vertices u and v of V_G, i.e. the minimum length of the paths joining them, is denoted by $d_G(u, v)$. Let $D(G) = (d_{uv})$ be the *distance matrix* of G, where $d_{uv} = d_G(u, v)$. The *eccentricity* $e_G(u)$ of a vertex $u \in V_G$ is given by $e_G(u) = \max\{d_{uv} \mid v \in V_G\}$. The distance spread $S_D(G)$ is also known as the spectral diameter of the distance matrix of G, and it is used as a molecular descriptor in chemoinformatics (see, e.g., [5, 9]).

The matrix $\mathcal{E}(G) = (\epsilon_{uv})$, where

$$\epsilon_{uv} = \begin{cases} d_G(u,v) & \text{if } d_G(u,v) = \min\{e_G(u), e_G(v)\}, \\ 0 & \text{otherwise,} \end{cases}$$

is known as the *eccentricity matrix* of G (see for instances [10, 11, 17, 22–26]). The matrix $\mathcal{E}(G)$ can be obtained from the distance matrix $D(G)$ by retaining the largest distances in each row and each column and replacing the remaining entries with zeros. The locutions D_{MAX} *matrix* and *anti-adjacency matrix* are alternative names assigned in the literature to $\mathcal{E}(G)$. For the eigenvalue $\lambda_i^{\mathcal{E}}(G)$ we adopt the lighter notation $\lambda_i(G)$.

We recall that the friendship graph with $2p+1$ vertices is the graph with $3p$ edges consisting of $p(\geqslant 1)$ disjoint triangles that meet in one vertex. Following [20], a weak-friendship graph is a connected induced subgraph of a friendship graph. Weak-friendship graphs are also known as butterfly-graphs [1, 21]. They are a peculiar type of bundles (i.e. cacti whose cycles all have a common vertex), firefly graphs [1] and butterfly-like graphs [16].

Let n be a positive integer. For $0 \leqslant k \leqslant \left\lfloor \frac{n-1}{2} \right\rfloor$, we denote by $\mathrm{WF}_{n,k}$ the (unique) weak-friendship graph with n vertices and k triangles. Alternatively, $\mathrm{WF}_{n,k}$ can be described as the graph obtained fron the star $K_{1,n-1}$ with n vertices by adding k independent edges, or as the join between P_1 and $kP_2 \cup (n-2k-1)P_1$ (see Fig. 1). We set

$$\mathcal{WF}_n := \left\{ \mathrm{WF}_{n,k} \;\middle|\; 0 \leqslant k \leqslant \left\lfloor \frac{n-1}{2} \right\rfloor \right\} \quad \text{and} \quad \mathcal{WF} := \bigcup_{n \in \mathbb{N}} \mathcal{WF}_n.$$

One of the reasons making the graphs in $\mathcal{WF}$ quite interesting is that specific weak-friendship graphs turned out to be extremal with respect to several and very different spectral invariants (see, for instance, [1–3, 6, 8, 12, 13, 19]).

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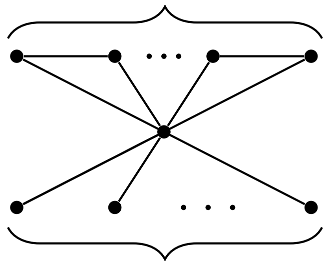

Figure 1: The weak friendship graph $\mathrm{WF}_{n,k}$.

As usual, we respectively denote by P_n and by C_n (if $n \geqslant 3$) the path and the cycle with n vertices. Let H be an induced subgraph of a graph G. Unlike what happens for other well-studied graph matrices, for $M \in \{D, \mathcal{E}\}$ the M-eigenvalues of H and G are not necessarily interlaced and it may happen that $S_M(H) > S_M(G)$. For instance,

$$S_D(P_5) \approx 13.52492 > S_D(C_6) = 13, \qquad \text{and} \qquad S_{\mathcal{E}}(P_5) \approx 12.1882 > S_{\mathcal{E}}(C_6) = 6.$$

Nevertheless, there are many results about the distance eigenvalues and the distance spread [3,4,14,15,18,27,28], whereas the literature on the eccentricity spread is still scarce. In this paper, we determine the unique graphs with the first two smallest eccentricity spreads in $\mathcal{WF}_n$.

2. The least eccentricity eigenvalue

Throughout the paper, we assume that the vertices in $V_{\mathrm{WF}_{n,k}} = \{v_1, \dots, v_n\}$ have been labelled in such a way that v_1 is the unique dominating vertex, and v_{2i} is adjacent to v_{2i+1} for $1 \leqslant i \leqslant k$.

Let $\mathcal{M}_{m\times n}$ be the set of real matrices with m rows and n columns. In order to write down the eccentricity matrix of $\mathrm{WF}_{n,k}$, we adopt the following standard notation: $J_{n\times m} \in \mathcal{M}_{m\times n}$ and $O_{n\times m} \in \mathcal{M}_{m\times n}$ are the all-ones matrix and the zero matrix respectively. We also set $J_n = J_{n\times n}$, $O_n = O_{n\times n}$, $(J-I)_n = J_n - I_n$ and $\mathbf{1}_n = J_{n\times 1}$. We also recall that the Kronecker product $R \otimes S$ of $R = (r_{ij}) \in \mathcal{M}_{m\times n}$ and $S = (S_{hk}) \in \mathcal{M}_{p\times q}$ is the $mp \times nq$ matrix obtained from R by replacing each entry r_{ij} of R with $r_{ij}S$.

It is somehow instructive to check that

$$\mathcal{E}(\mathrm{WF}_{6,2}) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 & 2 & 2 \\ 1 & 2 & 2 & 0 & 0 & 2 \\ 1 & 2 & 2 & 0 & 0 & 2 \\ 1 & 2 & 2 & 2 & 2 & 0 \end{bmatrix}.$$

More generally,

$$\mathcal{E}(\mathrm{WF}_{n,k}) = \begin{bmatrix} 0 & \mathbf{1}_{2k}^T & \mathbf{1}_{n-2k-1}^T \\ \mathbf{1}_{2k} & 2\left(J_{2k} - I_k \otimes J_2\right) & 2J_{2k\times(n-2k-1)} \\ \mathbf{1}_{n-2k-1} & 2J_{(n-2k-1)\times 2k} & 2(J-I)_{n-2k-1} \end{bmatrix} \quad \text{for } n-2k-1 > 0. \tag{1}$$

Instead, the eccentricity matrix of the friendship graph $\mathrm{WF}_{2k+1,k}$ consists of the upper-left 2×2 block matrix in (1).

Along the proof of Proposition 2.1, whose techniques resemble those adopted to prove Lemma 2.3 in [13], we make use of the well-known Schur formula for computing the determinant of a 2×2-block matrix: if Q is an invertible square matrix, then

$$\det \begin{bmatrix} M & N \\ P & Q \end{bmatrix} = \det Q \cdot \det \left[M - NQ^{-1}P \right]. \tag{2}$$

Additionally, we use the symbols $\kappa_i \rightsquigarrow \kappa_i + q\kappa_j$ (respectively, $\kappa^i \rightsquigarrow \kappa^i + q\kappa^j$) to denote the operation consisting in adding q-times the j-th row (respectively, j-th column) of a matrix to its i-th row (respectively, i-th column).

Proposition 2.1. *For $n \geqslant 5$ and $2 \leqslant k \leqslant \dfrac{n-1}{2}$, $\lambda_n(\mathrm{WF}_{n,k}) = -4$.*

Proof. We split the proof in two cases, dealing first with the case $n - 2k - 1 > 0$.

Case 1: $2 \leqslant k < (n-1)/2$. From (1) we obtain

$$p_{\mathcal{E}}(\mathrm{WF}_{n,k}, x) = \det \begin{bmatrix} x & -\mathbf{1}_{2k}^T & -\mathbf{1}_{n-2k-1}^T \\ -\mathbf{1}_{2k} & I_k \otimes (2J_2 + xI_2) - 2J_{2k} & -2J_{2k\times(n-2k-1)} \\ -\mathbf{1}_{n-2k-1} & -2J_{(n-2k-1)\times 2k} & (x+2)I_{n-2k-1} - 2J_{n-2k-1} \end{bmatrix}$$

If we perform the operations $\kappa_i \rightsquigarrow \kappa_i - 2\kappa_1$, for $2 \leqslant i \leqslant n$, on the lines of the matrix $xI_n - \mathcal{E}(\mathrm{WF}_{n,k})$, followed by $\kappa_{2i+1} \rightsquigarrow \kappa_{2i+1} - \kappa_{2i}$, for $1 \leqslant i \leqslant k$, and finally $\kappa^{2i} \rightsquigarrow \kappa^{2i} + \kappa^{2i+1}$, for $1 \leqslant i \leqslant k$, we arrive at

$$p_{\mathcal{E}}(\mathrm{WF}_{n,k}, x) = \det \begin{bmatrix} x & -\mathbf{1}_k^T \otimes A & -\mathbf{1}_{n-2k-1}^T \\ -\mathbf{1}_k \otimes B & I_k \otimes C & O_{2k\times(n-2k-1)} \\ -(2x+1)\mathbf{1}_{n-2k-1} & O_{(n-2k-1)\times 2k} & (x+2)I_{n-2k-1} \end{bmatrix},$$

where

$$A = \begin{bmatrix} 2 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2x+1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad C = \begin{bmatrix} x+4 & 2 \\ 0 & x \end{bmatrix}.$$

We now set

$$M = \begin{bmatrix} x & -\mathbf{1}_k^T \otimes A \\ -\mathbf{1}_k \otimes B & I_k \otimes C \end{bmatrix}, \qquad N = \begin{bmatrix} -\mathbf{1}_{n-2k-1}^T \\ O_{2k\times(n-2k-1)} \end{bmatrix},$$

$$P = \begin{bmatrix} -(2x+1)\mathbf{1}_{n-2k-1} & O_{(n-2k-1)\times 2k} \end{bmatrix} \qquad \text{and} \qquad Q = (x+2)I_{n-2k-1}.$$

Since

$$NQ^{-1}P = \begin{bmatrix} \frac{(n-2k-1)(2x+1)}{x+2} & O_{1\times 2k} \\ O_{2k\times 1} & O_{2k} \end{bmatrix},$$

we can write

$$\begin{aligned} \det(M - NQ^{-1}P) &= \det \begin{bmatrix} x - \frac{(n-2k-1)(2x+1)}{x+2} & -\mathbf{1}_k^T \otimes A \\ -\mathbf{1}_{2k} \otimes B & \mathbf{1}_{2k} \otimes C \end{bmatrix} \\ &= x^k (x+4)^{k-1} \left(x(x+4) - \frac{(n-2k-1)(x+4)(2x+1)}{x+2} - 2k(2x+1) \right). \end{aligned}$$

Therefore, using (2),

$$\begin{aligned} p_{\mathcal{E}}(\mathrm{WF}_{n,k}, x) &= \det Q \cdot det(M - NQ^{-1}P) \\ &= (x+2)^{n-2k-2} \cdot x^k \cdot (x+4)^{k-1} \cdot q_{n,k}(x), \end{aligned} \tag{3}$$

where

$$q_{n,k}(x) = x^3 - 2(n-4)x^2 - (9n - 8k - 17)x - 4(n-k-1). \tag{4}$$

Let $x_1 \geqslant x_2 \geqslant x_3$ be the three roots of $q_{n,k}(x)$. Since $n > 2k+1$ and $k \geqslant 2$, we have

$$\begin{aligned} q_{n,k}(2n) &= 2(7n^2 + 8nk + 15n + 2k + 2) > 0, \\ q_{n,k}(0) &= -4(n-k-1) < 0, \\ q_{n,k}(-1) &= 3n - 4k - 6 \geqslant 3(2k+2) - 4k - 6 = 2k > 0, \\ q_{n,k}(-2) &= 6(n - 2k - 1) > 0, \\ q_{n,k}(-4) &= -28k < 0. \end{aligned}$$

It follows that $x_1 \in (0, 2n)$, $x_2 \in (-1, 0)$ and $x_3 \in (-4, -2)$; moreover, from (3), we deduce that $\mathcal{E}(\mathrm{WF}_{n,k})$ has six pairwise distinct eigenvalues, namely $x_1 > 0 > x_2 > -2 > x_3 > -4$. Hence, $\lambda_n(\mathrm{WF}_{n,k}) = -4$.

Case 2: $k = (n-1)/2$. We are now dealing with the friendship graph $\mathrm{WF}_{2k+1,k}$, whose $\mathcal{E}$-characteristic polynomial is

$$\begin{aligned} p_{\mathcal{E}}(\mathrm{WF}_{2k+1,k}, x) &= \det \begin{bmatrix} x & -\mathbf{1}_{2k}^T \\ -\mathbf{1}_{2k} & I_k \otimes (2J_2 + xI_2) - 2J_{2k} \end{bmatrix} \\ &= x^k \cdot (x+4)^{k-1} \cdot r_k(x), \end{aligned}$$

where $r_k(x) = x^2 - 4(k-1)x - 2k$. Let $x_1 \geqslant x_2$ be the two roots of $r_k(x)$. Note that

$$r_k(4k) = 14k > 0, \quad r_k(0) = -2k < 0, \quad \text{and} \quad r_k(-1) = 2k - 3 > 0 \quad (\text{since } k \geqslant 2).$$

Thus, $x_1 \in (0, 4k)$ and $x_2 \in (-1, 0)$. This time, the pairwise distinct eccentricity eigenvalues of $\mathrm{WF}_{n,k}$ are $x_1 > 0 > x_2 > -4$, implying that the least eccentricity eigenvalue $\lambda_{2k+1}(\mathrm{WF}_{2k+1,k})$ is -4 as in the previous case. □

By gathering the spectral results achieved along the proof of Proposition 2.1, we can easily prove Propositions 2.2 and 2.3. The statement of the latter involves the polynomials

$$f_n(x) = x^3 - 2(n-4)x^2 - (5n-9)x - 2n \quad \text{and} \quad g_n(x) = x^2 - 2(n-3)x - (n-1), \tag{5}$$

defined for every $n \in \mathbb{N}$.

Proposition 2.2. *Let $n \geqslant 7$ and $2 \leqslant k < (n-1)/2$. $\lambda_1(\mathrm{WF}_{n,k})$ is the unique positive root of the polynomial $q_{n,k}(x)$ in (4). Moreover, $\lambda_1(\mathrm{WF}_{n,k}) < 2n$.*

Proposition 2.3. *Let $n \geqslant 5$.*

(i) If n is even, $\lambda_1(\mathrm{WF}_{n,\frac{n}{2}-1})$ is the largest root of the polynomial $f_n(x)$ in (5);

(ii) If n is odd, $\lambda_1(\mathrm{WF}_{n,\frac{n-1}{2}})$ is equal to $n-3+\sqrt{n^2-5n+8}$, the positive root of the polynomial $g_n(x)$ in (5);

(iii) $p_{\mathcal{E}}(\mathrm{WF}_{n,1},x) = x(x+2)^{n-4}h_n(x)$, where $h_n(x) = x^3 - 2(n-4)x^2 - (9n-25)x - 4(n-2)$.

3. Weak friendship graphs with minimum eccentricity spread

Lemma 3.1. [29, Theorem 3.9] *Let G be a tree of order $n \geqslant 3$. Then, $S_D(G) \geq n + \sqrt{n^2-3n+3}$, with equality if and only if $G \cong \mathrm{WF}_{n,0} = K_{1,n-1}$.*

Actually, the restriction $n \geqslant 3$ is absent both in [29, Theorem 3.9] and [13, Lemma 2.1]. Yet, it is clear that $S_D(\mathrm{WF}_{1,0}) = 0$ and $S_D(\mathrm{WF}_{2,0}) = 2$; in fact, the proof of [29, Theorem 3.9] uses the fact that -2 is the least distance eigenvalue of $\mathrm{WF}_{n,0}$, and this is only true for $n \geqslant 3$.

Proposition 3.1. *For $n \geqslant 5$, $S_{\mathcal{E}}(\mathrm{WF}_{n,0}) < S_{\mathcal{E}}(\mathrm{WF}_{n,\lfloor\frac{n-1}{2}\rfloor})$.*

Proof. Since $n \geqslant 5$ and $D(K_{1,n-1}) = \mathcal{E}(K_{1,n-1})$, by Proposition 2.1 and Lemma 3.1 we have

$$\lambda_n\left(\mathrm{WF}_{n,\lfloor\frac{n-1}{2}\rfloor}\right) = -4 \quad \text{and} \quad S_{\mathcal{E}}(\mathrm{WF}_{n,0}) = n + \sqrt{n^2-3n+3}.$$

To prove the result, we only need to show that $\lambda_1 > \xi_n$, where

$$\lambda_1 := \lambda_1\left(\mathrm{WF}_{n,\lfloor\frac{n-1}{2}\rfloor}\right) \quad \text{and} \quad \xi_n := n - 4 + \sqrt{n^2-3n+3}.$$

We distinguish two cases depending on the parity of n.

Case 1: n is even. In this case $n \geqslant 6$ and, consequently, $\xi_n > 6$. By Proposition 2.3(i), λ_1 is the largest root of $f_n(x)$ in (5). One verifies that $f_n(\xi_n) = -2(n+2\xi_n) < 0$; therefore, $\lambda_1 > \xi_n$.

Case 2: n is odd. Since we are assuming $n \geqslant 5$, then $\xi_n > 4$. By Proposition 2.3(ii), λ_1 is the largest root of $g_n(x)$ in (5). Now $g(\xi_n) = -2(6-2n+\xi) < -2(10-2n) \leqslant 0$, implying $\lambda_1 > \xi_n$, as claimed. □

Proposition 3.2. *For $n \geqslant 5$, $S_{\mathcal{E}}(\mathrm{WF}_{n,\lfloor\frac{n-1}{2}\rfloor}) < S_{\mathcal{E}}(\mathrm{WF}_{n,1})$.*

Proof. The claimed inequality holds for $n \in \{5,6,7,8\}$; in fact, by a direct computation,

$$\begin{aligned} &S_{\mathcal{E}}(\mathrm{WF}_{5,2}) \approx 8.8284 < 8.9484 \approx S_{\mathcal{E}}(\mathrm{WF}_{5,1}), && S_{\mathcal{E}}(\mathrm{WF}_{6,2}) \approx 11.1648 < 11.2245 \approx S_{\mathcal{E}}(\mathrm{WF}_{6,1}),\\ &S_{\mathcal{E}}(\mathrm{WF}_{7,3}) \approx 12.6904 < 12 + \sqrt{2} = S_{\mathcal{E}}(\mathrm{WF}_{7,1}), && S_{\mathcal{E}}(\mathrm{WF}_{8,3}) \approx 14.9613 < 15.5534 \approx S_{\mathcal{E}}(\mathrm{WF}_{8,1}).\end{aligned}$$

Suppose now $n \geqslant 9$. The $\mathcal{E}$-characteristic polynomial of $\mathrm{WF}_{n,1}$ can be read in Proposition 2.3(iii). We consider the following evaluations:

$$\begin{aligned} h_n(2n-3) &= 2n^2 + 13n - 22 > 0, & h_n(-2) &= 6(n-3) > 0,\\ h_n(2n-4) &= -2(n-2)(n-7) < 0, & h_n(-7/2) &= \frac{3}{8}(8n-65) > 0\\ h_n(0) &= 8 - 4n < 0, & h_n(-4) &= -28 < 0.\end{aligned} \tag{6}$$

Denoted by $x_1 \geqslant x_2 \geqslant x_3$ the three roots of $h_n(x)$, from (6) one deduces $x_1 \in (2n-4, 2n-3)$, $x_2 \in (-2, 0)$ and $x_3 \in (-4, -7/2)$; we also see that $\mathrm{sp}_{\mathcal{E}}(\mathrm{WF}_{n,1})$ has five pairwise distinct eigenvalues, namely $x_3, -2, x_2, 0$ and x_1, implying $-4 < \lambda_n(\mathrm{WF}_{n,1}) < -7/2$, and $S_{\mathcal{E}}(\mathrm{WF}_{n,1}) > \mu_1 + 7/2$, where $\mu_1 := \lambda_1(\mathrm{WF}_{n,1})$.

By Proposition 2.1, we have $\lambda_n(\mathrm{WF}_{n,k}) = -4$. In order to finish the proof, it will be enough to prove the inequality

$$\lambda_1 := \lambda_1\left(\mathrm{WF}_{n,\lfloor\frac{n-1}{2}\rfloor}\right) < \mu_1 - \frac{1}{2}. \tag{7}$$

Case 1: n is even (and $\geqslant 10$). Recall that λ_1 is the largest root of $f_n(x)$ in (5) by Proposition 2.3. If we show that $f_n(\mu_1 - 1/2)$ is positive, Inequality (7) will be proved. We set

$$\zeta(x) := f\left(x - \frac{1}{2}\right) = x^3 + \frac{4(13-4n)x^2 + 2(7-12n)x - 21}{8}.$$

Since the first derivative $\zeta'(x)$ is quadratic and has just one positive root, a standard calculus argument based on the positivity of $\zeta'(2n-4) = 4n^2 - 9n - 9/4$ shows that the function $\zeta(x)$ is strictly increasing in the interval $(2n-4, +\infty)$. Hence,

$$f\left(\mu_1 - \frac{1}{2}\right) = \zeta(\mu_1) > \zeta(2n-4) = f\left(2n - \frac{9}{2}\right) = 4n^2 - \frac{49}{2}n + \frac{243}{8} > 0,$$

as claimed.

Case 2: n is odd (and $\geqslant 9$). This time, by Proposition 2.3(ii), λ_1 is the largest root of $g_n(x)$ in (5). Similarly to the previous case, we consider the function

$$\vartheta(x) = g_n\left(x - \frac{1}{2}\right) = x^2 - (2n-5)x - \frac{7}{4}.$$

from $\vartheta'(2n-4) = 2n-3 > 0$ we deduce that the function $\vartheta(x)$ is strictly increasing in the interval $(2n-4, 2n-3)$. Therefore,

$$g_n\left(\mu_1 - \frac{1}{2}\right) = \vartheta(\mu_1) > \vartheta(2n-4) = g_n\left(2n - \frac{9}{2}\right) = 2n - \frac{23}{4} > 0,$$

proving that $\mu_1 - 1/2$ is larger than λ_1, the only positive root of $g_n(x)$.Thus, (7) is proved and the proof is over. □

Theorem 3.1. $S_{\mathcal{E}}(\mathrm{WF}_{n,k}) < S_{\mathcal{E}}(\mathrm{WF}_{n,k-1})$ *for all* $n \geqslant 7$ *and* $3 \leqslant k \leqslant \left\lfloor \frac{n-1}{2} \right\rfloor$.

Proof. Since, by Proposition 2.1, $\lambda_n(\mathrm{WF}_{n,k}) = -4$, it will suffice to prove the inequality

$$\lambda_1(\mathrm{WF}_{n,k}) < \lambda_1(\mathrm{WF}_{n,k-1}) \qquad \text{for } n \geqslant 7 \text{ and } 3 \leqslant k \leqslant \left\lfloor \frac{n-1}{2} \right\rfloor. \tag{8}$$

We distinguish two cases depending whether $\mathrm{WF}_{n,k}$ is a friendship graph or not.

Case 1: n is odd and $k = \frac{n-1}{2}$. By Propositions 2.2 and 2.3, we have

$$\lambda_1(\mathrm{WF}_{n,k}) = n - 3 + \sqrt{n^2 - 5n + 8},$$

and $\lambda_1(\mathrm{WF}_{n,k-1})$ is the only positive root of the polynomial $q_{n,\frac{n-3}{2}}(x)$ defined in (4). Inequality (8) comes from

$$q_{n,\frac{n-3}{2}}\left(n - 3 + \sqrt{n^2-5n+8}\right) = -4\left(2n - 5 + 2\sqrt{n^2-5n+8}\right) < 0.$$

Case 2: $3 \leqslant k < (n-1)/2$. Let $q_{n,k}(x)$ be polynomial defined in (4). We immediately see that

$$q_{x,k}(x) - q_{x,k-1}(x) = 4(2x+1) > 0 \qquad \text{for } x > 0. \tag{9}$$

From (9) and Proposition 2.2 we obtain (8) as wanted. □

We are now ready to detect the $\mathcal{E}$-spread minimizers in $\mathcal{WF}_n$.

Theorem 3.2. *For all* $n \in \mathbb{N} \setminus \{4\}$, $\mathrm{WF}_{n,0}$ *is the only graph in* $\mathcal{WF}_n$ *attaining the minimum* $\mathcal{E}$*-spread.*

Proof. Since $\mathcal{WF}_1$ and $\mathcal{WF}_2$ are singletons, the first nontrivial case occurs for $n = 3$. Clearly, $\mathcal{WF}_3 = \{P_3, C_3\}$, and

$$S_{\mathcal{E}}(\mathrm{WF}_{3,0}) = S_{\mathcal{E}}(P_3) = 2\sqrt{2} < 3 = S_{\mathcal{E}}(C_3) = S_{\mathcal{E}}(\mathrm{WF}_{3,1}).$$

From Lemma 2.1 and a direct computation, we see that $\mathrm{WF}_{4,0}$ does not attain the minimum $\mathcal{E}$-spread of $\mathcal{WF}_4$. In fact, the cardinality of $\mathcal{WF}_4$ is 2, and

$$S_{\mathcal{E}}(\mathrm{WF}_{4,0}) = 4 + \sqrt{7} \approx 6.6457 > 6.4982 \approx S_{\mathcal{E}}(\mathrm{WF}_{4,1}).$$

The sets $\mathcal{WF}_5$ and $\mathcal{WF}_6$ both contains three weak friendship graphs. By Propositions 3.1 and 3.2

$$S_{\mathcal{E}}(\mathrm{WF}_{5,0}) < S_{\mathcal{E}}(\mathrm{WF}_{5,2}) < S_{\mathcal{E}}(\mathrm{WF}_{5,1}) \quad \text{and} \quad S_{\mathcal{E}}(\mathrm{WF}_{6,0}) < S_{\mathcal{E}}(\mathrm{WF}_{6,2}) < S_{\mathcal{E}}(\mathrm{WF}_{6,1}).$$

Let now $n \geqslant 7$. Using Proposition 3.1 and Theorem 3.1, we arrive at

$$S_{\mathcal{E}}(\mathrm{WF}_{n,0}) < S_{\mathcal{E}}(\mathrm{WF}_{n,\lfloor\frac{n-1}{2}\rfloor}) < S_{\mathcal{E}}(\mathrm{WF}_{n,\lfloor\frac{n-1}{2}\rfloor-1}) < \cdots < S_{\mathcal{E}}(\mathrm{WF}_{n,3}) < S_{\mathcal{E}}(\mathrm{WF}_{n,2}).$$

The proof ends by combining the above inequalities with $S_{\mathcal{E}}(\mathrm{WF}_{n,\lfloor\frac{n-1}{2}\rfloor}) < S_{\mathcal{E}}(\mathrm{WF}_{n,1})$, coming from Proposition 3.2. □

The results achieved along the proof in Theorem 3.2 lead to the following theorem.

Theorem 3.3. *For* $n = 3$ *or* $n \geqslant 5$, $\mathrm{WF}_{n,\lfloor\frac{n-1}{2}\rfloor}$ *is the unique graph in* $\mathcal{WF}_n$ *attaining the second smallest* $\mathcal{E}$*-spread.*

4. Conclusions

In this paper we have proved that for $n \geqslant 5$, the graphs $\mathrm{WF}_{n,0}$ and $\mathrm{WF}_{n,\lfloor \frac{n-1}{2} \rfloor}$ respectively attain the smallest and the second smallest $\mathcal{E}$-spread in the set $\mathcal{WF}_n$ containing the weak friendship graphs with n vertices. Let $\mathcal{B}_n$ (respectively, $\mathcal{C}_n$) be the set of bundles (respectively, cacti graph) with n vertices. In Section 1, we already noted that the sequence of inclusions $\mathcal{WF}_n \subseteq \mathcal{B}_n \subseteq \mathcal{C}_n$ holds. By comparing $S_{\mathcal{E}}(\mathrm{WF}_{n,0}) = n + \sqrt{n^2 - 3n + 3}$ with

$$S_{\mathcal{E}}(C_n) = \begin{cases} n & \text{if } n \text{ is even;} \\ 4 + 2k\cos\left(\dfrac{\pi}{2k+1}\right) & \text{if } n \text{ is odd and } k = (n-1)/2 \end{cases}$$

(computed with the aid of [24, Theorem 3.2]), we observe that for $n \geqslant 5$, in both $\mathcal{B}_n$ and $\mathcal{C}_n$ the weak fiendship graph $\mathrm{WF}_{n,0}$ does not attain the minimum $\mathcal{E}$-spread; in fact, $S_{\mathcal{E}}(\mathrm{WF}_{n,0}) > S_{\mathcal{E}}(C_n)$. For this reason, we end this paper by proposing the following problem (which actually comprises six interrelated issues):

Problem 4.1. *For $n \geqslant 5$, find the graphs attaining the minimum and the maximum $\mathcal{E}$-spread in $\mathcal{B}_n$, $\mathcal{C}_n$, $\mathcal{B}_n \setminus \{C_n\}$ and $\mathcal{C}_n \setminus \{C_n\}$.*

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