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Research Article

Hagen–Rothe convolution identities through Lagrange interpolations

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Abstract

New proofs of Hagen–Rothe identities concerning binomial convolutions are presented through Lagrange interpolations.

Keywords: Chu–Vandermonde formula; Hagen–Rothe identities; finite difference; Lagrange interpolation.

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1. Introduction and Outline

The Chu–Vandermonde convolution formula is one of the most popular binomial identities:

$$\sum_{k=0}^{n}\binom{x}{k}\binom{y}{n-k}=\binom{x+y}{n}.$$

Three classical generalizations of this formula can be reproduced as follows:

$$\sum_{k=0}^{n}\frac{x}{x+k\lambda}\binom{x+k\lambda}{k}\binom{y-k\lambda}{n-k}=\binom{x+y}{n}, \tag{1}$$

$$\sum_{k=0}^{n}\frac{x}{x+k\lambda}\binom{x+k\lambda}{k}\frac{y-n\lambda}{y-k\lambda}\binom{y-k\lambda}{n-k}=\frac{x+y-n\lambda}{x+y}\binom{x+y}{n}, \tag{2}$$

$$\sum_{k=0}^{n}\binom{x+k\lambda}{k}\binom{y-k\lambda}{n-k}=\sum_{\ell=0}^{n}\binom{x+y-\ell}{n-\ell}\lambda^{\ell}. \tag{3}$$

The first two were discovered by Hagen and Rothe (cf. Comtet [10, §3.1] and Graham et al. [14, §5.4]), while the third identity is due to Jensen [15]. Their limiting cases result in the well–known Abel identities (cf. [1, 8]). When $\lambda=0$, all of them become the Chu–Vandermonde formula. Analogously for $\lambda=1$, it is routine to check that the corresponding formulae are equivalent to the Chu–Vandermonde formula.

These convolution identities have wide applications to enumerative combinatorics and number theory. The reader can refer to Strehl [20] for a historical note. The typical proofs for these convolution identities are highlighted as follows:

- Generating function method: Gould [12, 13] (see Chu [3] also).
- Series rearrangement and finite differences: Chu [7].
- Gould–Hsu inverse series relations: Chu and Hsu [2, 9].
- The classical Lagrange expansion formula: Riordan [18, §4.5].
- The Cauchy residue method of integral representation: Egorychev [11, §2.1].
- Lattice path combinatorics: Mohanty [16, §4.5] and Narayana [17, Appendix].
- Riordan arrays (equivalent to the Lagrange expansion formula): Sprugnoli [19].

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The objective of this article is to present proofs of Identities (1), (2), and (3), by means of Lagrange interpolations and finite difference method. Let Δ be the difference operator with unit increment

$$\Delta^0 f(x) := f(x) \quad \text{and} \quad \Delta f(x) := f(x+1) - f(x).$$

For a natural number n, the differences of order n is given by

$$\Delta^n f(x) := \Delta\big\{\Delta^{n-1} f(x)\big\},$$

which is expressed by the following Newton–Gregory formula

$$\Delta^n f(x) = \sum_{k=0}^{n} (-1)^{n+k}\binom{n}{k} f(x+k).$$

For an indeterminate x and an integer n, recall that the Pochhammer symbol $(x)_n$ is defined by

$$(x)_n = \begin{cases} x(x+1)\cdots(x+n-1), & n = 1, 2, \cdots; \\ 1, & n = 0; \\ \dfrac{1}{(x-1)(x-2)\cdots(x+n)}, & n = -1, -2, \cdots. \end{cases}$$

When $p_m(x)$ is a polynomial of degree $m \le n$ with the leading coefficient c_m, the following properties (cf. [4–6,8]) are quite useful:

$$\Delta^n p_m(x) = n! c_n \chi(m = n) \quad \text{and} \quad \Delta^n \frac{p_m(x)}{x-\tau} = (-1)^n \frac{n! p_m(\tau)}{(x-\tau)_{n+1}},$$

where χ stands for the logical function with $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$. The former equality is well–known. The latter can be justified easily as follows. First when $p_m(x) \equiv 1$, it is trivial to check it by the induction principle. Observing that $\frac{p_m(x)-p_m(\tau)}{x-\tau}$ is a polynomial of degree $m-1$ with its nth difference equal to zero, we have immediately

$$\Delta^n \frac{p_m(x)}{x-\tau} = \Delta^n \frac{p_m(\tau)}{x-\tau} = (-1)^n \frac{n! p_m(\tau)}{(x-\tau)_{n+1}}.$$

In addition, we fix $\Delta_0^n f(x) = \Delta^n f(x)\big|_{x=0}$ for the differences starting at $x = 0$.

2. Proofs of three formulae

Now, we are in a position to present detailed proofs for the three identities announced in the introduction. To be precise, we assume that λ is a variable subject to $\lambda \neq 1$ throughout this section, even though three identities (1), (2) and (3) are also valid in this case as declared in the introduction.

Proof of Identity (1)

Denote by $\mathcal{P}(x)$ the binomial sum in (1), which is obviously a polynomial of degree n. Its value at $x = m - y$ can be reformulated as

$$\begin{aligned} \mathcal{P}(m-y) &= \sum_{k=0}^{n} \frac{m-y}{m-y+k\lambda}\binom{m-y+k\lambda}{k}\binom{y-k\lambda}{n-k} \\ &= \frac{m-y}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(1+m-k+k\lambda-y)_{k-1}(k\lambda-y)_{n-k}. \end{aligned}$$

By making use of the two equalities

$$\begin{aligned} &(k\lambda-y)_{n-k} = (k\lambda-y)_{1+m-k}(1+m-k+k\lambda-y)_{n-m-1}, \\ &(k\lambda-y)_{1+m-k}(1+m-k+k\lambda-y)_{k-1} = (k\lambda-y)_m; \end{aligned}$$

we have the following alternative expression

$$\mathcal{P}(m-y) = \frac{m-y}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(k\lambda-y)_m(1+m-k+k\lambda-y)_{n-m-1}.$$

Then $\mathcal{P}(m-y)$ vanishes for $0 \le m < n$, because it results in the nth difference of the polynomial $(x\lambda - y)_m(1+m-x-y+x\lambda)_{n-m-1}$ of degree $n-1$. When $m=n$, we can evaluate

$$\begin{aligned}\mathcal{P}(n-y) &= \frac{n-y}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{(k\lambda-y)_n}{n-k-y+k\lambda}\\ &= \frac{n-y}{n!}\Delta_0^n\frac{(x\lambda-y)_n}{n-x-y+x\lambda}\\ &= \frac{n-y}{n!(\lambda-1)}\Delta_0^n\frac{(\frac{y-n\lambda}{\lambda-1})_n}{x-\frac{y-n}{\lambda-1}}\\ &= (-1)^n\frac{n-y}{\lambda-1}\frac{(\frac{y-n\lambda}{\lambda-1})_n}{(-\frac{y-n}{\lambda-1})_{n+1}} = 1.\end{aligned}$$

Therefore, the first convolution identity (1) follows from the Lagrange polynomial of $\mathcal{P}(x)$ for the interpolation points $\{x_m := m-y\}_{m=0}^{n}$, which contains the only surviving term with $m=n$:

$$\mathcal{P}(x) = \sum_{m=0}^{n}\mathcal{P}(x_m)\prod_{\substack{\ell=0\\ \ell\neq m}}^{n}\frac{x-x_\ell}{x_m-x_\ell} = \binom{x+y}{n}. \qquad \square$$

Proof of Identity (2)

Let $\mathcal{Q}(x)$ be the binomial sum in (2), which is again a polynomial of degree n. Its value at $x=m-y$ reads as

$$\begin{aligned}\mathcal{Q}(m-y) &= \sum_{k=0}^{n}\frac{m-y}{m-y+k\lambda}\binom{m-y+k\lambda}{k}\frac{y-n\lambda}{y-k\lambda}\binom{y-k\lambda}{n-k}\\ &= \frac{(y-m)(y-n\lambda)}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(1+m-k+k\lambda-y)_{k-1}(1+k\lambda-y)_{n-k-1}\\ &= \frac{(y-m)(y-n\lambda)}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(1+k\lambda-y)_{m-1}(1+m-k+k\lambda-y)_{n-m-1}.\end{aligned}$$

For $1 \le m < n$, it is clear that $\mathcal{Q}(m-y)$ vanishes because it results in the nth difference of the polynomial $(1-y+x\lambda)_{m-1}(1+m-x-y+x\lambda)_{n-m-1}$ of degree $n-2$. In addition, we can further evaluate

$$\begin{aligned}\mathcal{Q}(0-y) &= \frac{y(y-n\lambda)}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{(1-k-y+k\lambda)_{n-1}}{k\lambda-y}.\\ &= \frac{y(y-n\lambda)}{n!}\Delta_0^n\frac{(1-x+x\lambda-y)_{n-1}}{x\lambda-y}\\ &= \frac{y(y-n\lambda)}{n!\lambda}\Delta^n\frac{(1-y/\lambda)_{n-1}}{x-y/\lambda}\Big|_{x=0}\\ &= (-1)^n\frac{y(y-n\lambda)}{\lambda}\frac{(1-y/\lambda)_{n-1}}{(-y/\lambda)_{n+1}} = (-1)^n\lambda\end{aligned}$$

and

$$\begin{aligned}\mathcal{Q}(n-y) &= \frac{(y-n)(y-n\lambda)}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{(1+k\lambda-y)_{n-1}}{n-k+k\lambda-y}\\ &= \frac{(y-n)(y-n\lambda)}{n!}\Delta_0^n\frac{(1+x\lambda-y)_{n-1}}{n-x+x\lambda-y}\\ &= \frac{(y-n)(y-n\lambda)}{n!(\lambda-1)}\Delta^n\frac{(1+\frac{y-n\lambda}{\lambda-1})_{n-1}}{x-\frac{y-n}{\lambda-1}}\Big|_{x=0}\\ &= (-1)^n\frac{(y-n)(y-n\lambda)}{\lambda-1}\frac{(1+\frac{y-n\lambda}{\lambda-1})_{n-1}}{(-\frac{y-n}{\lambda-1})_{n+1}} = 1-\lambda.\end{aligned}$$

Consequently, the second convolution identity (2) follows from the Lagrange polynomial of $\mathcal{Q}(x)$ for the interpolation points $\{x_m := m-y\}_{m=0}^n$, which consists of the only two terms corresponding to $m=0$ and $m=n$:

$$\begin{aligned}\mathcal{Q}(x) &= \sum_{m=0}^{n} \mathcal{Q}(x_m) \prod_{\substack{\ell=0\\ \ell\neq m}}^{n} \frac{x-x_\ell}{x_m-x_\ell}\\ &= \lambda\binom{x+y-1}{n} + (1-\lambda)\binom{x+y}{n}\\ &= \frac{x+y-n\lambda}{x+y}\binom{x+y}{n}. \qquad \square\end{aligned}$$

Proof of Identity (3)

Assume that $\mathcal{R}(x)$ stands for the left sum of (3). Its value at $x=m-y$ can be expressed as

$$\begin{aligned}\mathcal{R}(m-y) &= \sum_{k=0}^{n}\binom{m-y+k\lambda}{k}\binom{y-k\lambda}{n-k}\\ &= \sum_{k=0}^{n}\frac{(-1)^{n-k}}{n!}\binom{n}{k}(1-k+m-y+k\lambda)_k(k\lambda-y)_{n-k}\\ &= \sum_{k=0}^{n}\frac{(-1)^{n-k}}{n!}\binom{n}{k}(k\lambda-y)_{m+1}(1+m-k-y+k\lambda)_{n-m-1}.\end{aligned}$$

When $0\leq m<n$, we have that $\mathcal{R}(m-y)$ results in the nth difference of the polynomial $(x\lambda-y)_{m+1}(1+m-x-y+x\lambda)_{n-m-1}$ of degree n:

$$\mathcal{R}(m-y) = \lambda^{m+1}(\lambda-1)^{n-m-1},$$

which coincides with the right sum of (3) at $x=m-y$:

$$\begin{aligned}\sum_{\ell=0}^{n}\binom{m-\ell}{n-\ell}\lambda^\ell = \sum_{k=0}^{n}\binom{m-n+k}{k}\lambda^{n-k} &= \sum_{k=0}^{n-m-1}(-1)^k\binom{n-m-1}{k}\lambda^{n-k}\\ &= \lambda^n(1-\lambda^{-1})^{n-m-1} = \lambda^{m+1}(\lambda-1)^{n-m-1}.\end{aligned}$$

Furthermore, we can evaluate

$$\begin{aligned}\mathcal{R}(n-y) &= \sum_{k=0}^{n}\frac{(-1)^{n-k}}{n!}\binom{n}{k}\frac{(k\lambda-y)_{n+1}}{n-k-y+k\lambda} = \frac{\Delta_0^n}{n!(\lambda-1)}\frac{(x\lambda-y)_{n+1}}{x-\frac{y-n}{\lambda-1}}\\ &= \frac{1}{n!(\lambda-1)}\Delta_0^n\left\{\frac{(x\lambda-y)_{n+1}-(\frac{y-n\lambda}{\lambda-1})_{n+1}}{x-\frac{y-n}{\lambda-1}}+\frac{(\frac{y-n\lambda}{\lambda-1})_{n+1}}{x-\frac{y-n}{\lambda-1}}\right\}\\ &= \frac{1}{\lambda-1}\left\{\lambda^{n+1}+(-1)^n\frac{(\frac{y-n\lambda}{\lambda-1})_{n+1}}{(-\frac{y-n}{\lambda-1})_{n+1}}\right\} = \frac{\lambda^{n+1}-1}{\lambda-1}.\end{aligned}$$

This coincides again with the right sum of (3) at $x=n-y$:

$$\sum_{\ell=0}^{n}\binom{n-\ell}{n-\ell}\lambda^\ell = \sum_{\ell=0}^{n}\lambda^\ell = \frac{\lambda^{n+1}-1}{\lambda-1}.$$

Since both sides of (3) are polynomials of the same degree n and have the same values at the $n+1$ distinct points $\{m-y\}_{m=0}^n$, they have the same Lagrange polynomial for the interpolation points $\{x_m := m-y\}_{m=0}^n$:

$$\sum_{m=0}^{n}\mathcal{R}(x_m)\prod_{\substack{\ell=0\\ \ell\neq m}}^{n}\frac{x-x_\ell}{x_m-x_\ell} = \binom{x+y}{n}\frac{\lambda^{n+1}-1}{\lambda-1}+\sum_{m=0}^{n-1}\lambda^{m+1}\binom{x+y}{m}\binom{n-x-y}{n-m}(\lambda-1)^{n-m-1}.$$

This not only confirms Jensen's convolution identity (3), but also gives, as a bonus, the third expression

$$\mathcal{R}(x) = \sum_{m=0}^{n}\{\lambda^{m+1}-\chi(m=n)\}\binom{x+y}{m}\binom{n-x-y}{n-m}(\lambda-1)^{n-m-1}.$$

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