Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **11** (2023) 99–110
DOI: 10.47443/dml.2022.179

Review Article

Recent results on hyperbolicity on unitary operators on graphs

Jesús A. Méndez[1], Rosalio Reyes[2], José M. Rodríguez[3,*], José M. Sigarreta[1]

[1]*Facultad de Matemáticas, Universidad Autónoma de Guerrero, Chilpancingo, Mexico*
[2]*Instituto de Física, Benemérita Universidad Autónoma de Puebla, Puebla, Mexico*
[3]*Departamento de Matemáticas, Universidad Carlos III de Madrid, Getafe, Spain*

(Received: 18 October 2022. Received in revised form: 26 November 2022. Accepted: 12 December 2022. Published online: 3 March 2023.)

Abstract

For a geodesic metric space X and for $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ in X. The space X is δ*-hyperbolic* (in Gromov sense) if any side of T is contained in a δ-neighborhood of the union of the two other sides, for every geodesic triangle T in X. If X is hyperbolic, we denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e., $\delta(X) := \sup\{\delta(T) : T \text{ is a geodesic triangle in } X\}$. In this paper, we collect previous results and prove new theorems on the hyperbolic constant of some important unitary operators on graphs.

Keywords: geodesics; Gromov hyperbolicity; hyperbolicity constant; hyperbolic graph; hyperbolic space; central graph.

2020 Mathematics Subject Classification: 05C76.

1. Introduction

Gromov introduced hyperbolic spaces in [18]. These spaces have many interesting applications in computer science and in various fields of mathematics, including geometric group theory, topology, and analysis (see [2, 10, 17, 18, 44, 47]). In geometric group theory, hyperbolic groups (i.e., groups whose Cayley graphs are hyperbolic) play a central role, as they have many desirable properties, such as finiteness properties, rigidity properties, and algorithmic properties. Hyperbolic spaces also arise in the study of mapping class groups, Teichmuller spaces, and other moduli spaces of geometric objects.

In geometry and topology, hyperbolic spaces are important in the study of manifolds with negative sectional curvature, as they provide a way to approximate such manifolds by discrete models. Hyperbolic spaces also have interesting properties related to their boundaries, such as the Gromov boundary, which can be used to study the large-scale geometry of the space.

The concept of Gromov hyperbolicity grasps the essence of negatively curved spaces like the classical hyperbolic space, Riemannian manifolds of negative sectional curvature bounded away from 0, and of discrete spaces like trees and the Cayley graphs of many finitely generated groups.

The relation between graphs and metric spaces is an important point to consider when studying hyperbolicity. In fact, the notion of hyperbolicity was originally introduced in the context of graphs, and later extended to geodesic metric spaces. The relationship between hyperbolicity in these two settings is well understood, and there are many results that establish equivalence between the two notions (see [10]). Understanding the relationship between hyperbolicity in graphs and geodesic metric spaces is important because we can gain insights into the behavior of these structures and develop new techniques for analyzing them. In particular, the hyperbolicity of a geodesic metric space is equivalent to hyperbolicity of a graph related to it [10]. This result indicates that the study of hyperbolicity can be reduced to the analysis of certain graphs associated with the metric space, which may simplify many computations and proofs.

In a geodesic metric space, a geodesic is a curve that minimizes distance between its endpoints, and is equipped with an arc-length parametrization. A metric space is said to be geodesic if every pair of points in the space can be connected by a geodesic. In general, there may be multiple geodesics between two points x and y in a geodesic metric space, but we can use the notation $[xy]$ to denote any one of them; this notation is ambiguous, but it is very convenient. In the case of a graph, we can use the notation uv to denote the edge that connects vertices u and v.

Let X be a geodesic metric space, and let $x_1, x_2, x_3 \in X$ be three points. The *geodesic triangle* T with vertices x_1, x_2, x_3 is defined to be the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$, and $[x_3x_1]$ in X. We say that X is δ*-hyperbolic* if for every geodesic triangle T any point on a geodesic between two points in T is within distance δ of the other sides of T. If X is hyperbolic, we denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e., $\delta(X) := \inf\{\delta : X \text{ is } \delta\text{-hyperbolic}\}$; if X is not

*Corresponding author (jomaro@math.uc3m.es).

hyperbolic, we define $\delta(X) = \infty$. We say that X is *hyperbolic* if X is δ-hyperbolic for some $\delta \geq 0$. If X has connected components $\{X_i\}_{i\in I}$, then we define $\delta(X) := \sup_{i\in I} \delta(X_i)$, and we say that X is hyperbolic if $\delta(X) < \infty$.

Gromov hyperbolicity can be formulated in various equivalent ways. The book [17] provides several definitions of Gromov hyperbolicity, each with its own strengths and weaknesses. The authors have chosen the above definition based on its deep geometric meaning [17].

Along this work, $G = (V, E) = (V(G), E(G))$ will denote a simple (without loops and multiple edges) graph (not necessarily connected) such that $V \neq \emptyset$ and every edge has length 1. In order to consider a connected graph G as a geodesic metric space, we identify any edge $uv \in E(G)$ with the real interval $[0, 1]$. Hence, the points in G are the points in the interior of any edge in $E(G)$ and the vertices in $V(G)$. In this way, any connected graph G has a natural distance defined as the length of the shortest path connecting two points in the graph, and we can see G as a metric graph. We denote by d_G or d this distance. If two points are in different connected components of the graph, their distance is defined to be infinity. With this construction, any connected component of a graph is a geodesic metric space in its own right. This construction allows us to apply geometric concepts and tools from metric spaces to the study of graphs, and vice versa.

The study of hyperbolic graphs is a subject of increasing interest in Discrete Mathematics and its applications, for example, networks and algorithms (see [27]), random graphs (see, [40–42]), etc. In fact, many real networks are hyperbolic (see [1, 28, 31, 45]). Hyperbolicity in graphs has also been used in issues such as the secure transmission of information through the network (see [23], [25]). Other problems that have been addressed are sensor networks, distance estimation, traffic flow, congestion minimization (see [3, 24]). The hyperbolicity constant has been successfully applied to the study of chemical structures, (see [33]) and DNA study (see [11]).

The study of hyperbolic graphs, from a mathematical point of view, has three main objectives:

1.– Study the hyperbolicity constant of some kinds of graphs.

2.– Obtain relationships between the hyperbolicity constant and other parameters of a graph.

3.– Study the invariance of the hyperbolicity constant under transformations.

In this work, in order to achieve the stated objectives, we collect the previous results and prove new theorems on the hyperbolic constant of some important unitary operators on graphs.

2. Hyperbolicity on unitary operators

In [26], J. Krausz introduced the concept of graph operators. A graph operator is a mapping $F : \Gamma \to \Gamma'$, where Γ and Γ' are families of graphs. Different kinds of graph operators have been investigated in the studies on graph dynamics (see [19,32]) and topological indices (see [9,15,16,34,35,48]). Some large graphs are composed from some existing smaller ones by using graph operators, and many properties of such large graphs are strongly associated with that of the corresponding smaller ones. Motivated from the above works, we study here the hyperbolicity constant of some graph operators.

Given an edge $e = uv \in E(G)$ with endpoints u and v, we write $V(e) = \{u, v\}$. Next, we recall the definition of some of the main graph operators.

The *line graph*, denoted by $\mathcal{L}(G)$, is the graph whose vertices correspond to the edges of G with two vertices being adjacent if and only if the corresponding edges in G have a vertex in common.

The *complement* of a graph G, denoted by $\overline{G}$, is the graph whose vertices correspond to $V(G)$ but whose edge set consists of the edges not present in G.

The *subdivision graph*, denoted by $S(G)$, is the graph obtained from G by replacing each of its edge by a path of length two, or equivalently, by inserting an additional vertex into each edge of G.

The *para-line graph* of G, denoted by $\mathcal{P}(G)$, is the line graph of the subdivision graph of G, i.e. $\mathcal{P}(G) = \mathcal{L}(S(G))$.

The *total graph*, denoted by $T(G)$, has as its vertices the edges and vertices of G. Adjacency in $T(G)$ is defined as adjacency or incidence for the corresponding elements of G.

The graph $R(G)$ is obtained from G by adding a new vertex corresponding to each edge of G, then joining each new vertex to the end vertices of the corresponding edge. Another way to describe $R(G)$ is to replace each edge of G by a triangle.

The graph $Q(G)$ is the graph obtained from G by inserting a new vertex into each edge of G and by joining edges those pairs of these new vertices which lie on adjacent edges of G.

The *central graph* of G, denoted by $C(G)$, is obtained by subdividing every edge of G exactly once and joining all non-adjacent vertices of G in $C(G)$.

Operator $R(G)$ is referred to by some authors as *semi-total point* or *semi-total vertex* and operator $Q(G)$ as *semi-total line*, *semi-total edge* or *middle graph* of G (see [4, 20, 39, 46]).

In this section we prove some results relating the hyperbolicity constants of a graph G and its operators $S(G)$, $\mathcal{P}(G)$ and $C(G)$. For more information about the $R(G)$, $Q(G)$ and $T(G)$ operators, see [29].

Definition 2.1. *Let (X, d) be a metric space and $x, y \in X$. The* Gromov product *between x and y, with base point $w \in X$, is defined as $(x, y)_w := \frac{1}{2}\big(d(x, w) + d(y, w) - d(x, y)\big) \geq 0$. We say that the metric space (X, d) is* δ-hyperbolic *with respect to the Gromov product, for some constant $\delta \geq 0$, if*

$$(x, z)_w \geq \min\big\{(x, y)_w, (y, z)_w\big\} - \delta \tag{1}$$

for every $x, y, z, w \in X$.

The following result reported in [2,17] establishes a relationship between the definition given by the Rips condition and the Gromov product.

Theorem 2.1. *Let X be a geodesic metric space:*

1. *If X is δ-hyperbolic with respect to the Gromov product, then X is 3δ-hyperbolic.*
2. *If X is δ-hyperbolic, then X is 4δ-hyperbolic with respect to the Gromov product.*

Let us consider now the hyperbolicity constant with respect to the Gromov product (see Definition 2.1). We denote by $\delta^*(G)$ the sharp constant for the inequality (1), *i.e.*,

$$\delta^*(G) := \sup\big\{\min\big\{(x, y)_w, (y, z)_w\big\} - (x, z)_w : x, y, z, w \in G\big\}.$$

Theorem 2.1 gives $\delta^*(G) \leq 4\delta(G)$ and $\delta(G) \leq 3\delta^*(G)$. In [43] we found the following improvement of the previous inequality: $\delta^*(G) \leq 2\delta(G)$. We denote by $\delta_v^*(G)$ the constant of hyperbolicity of the Gromov product restricted to the vertices of G, i.e.,

$$\delta_v^*(G) := \sup\big\{\min\big\{(x, y)_w, (y, z)_w\big\} - (x, z)_w : x, y, z, w \in V(G)\big\}.$$

The following result given in [29] relates $\delta^*(G)$ and $\delta_v^*(G)$.

Proposition 2.1. *Let G be a graph. Then $\delta_v^*(G) \leq \delta^*(G) \leq \delta_v^*(G) + 3$.*

Subdivision and para-line operators

The following result is immediate from the definition of $S(G)$.

Proposition 2.2. *[29, Proposition 1] Let G be a graph. Then $\delta(S(G)) = 2\delta(G)$, $\quad \delta^*(S(G)) = 2\delta^*(G)$.*

We remark that the equality is not true for $\delta_v^*(G)$ (e.g., $S(C_5) = C_{10}$ but $2\delta_v^*(C_5) = 1 \neq 2 = \delta_v^*(S(C_5))$), but there are inequalities. In order to obtain these inequalities, we need the following result [14].

Theorem 2.2. *Let $B = (V_0 \cup V_1, E)$ be a bipartite graph. We have $\delta_B(V_i) \leq \delta_v^*(B) \leq \delta_B(V_i) + 2$, where*

$$\delta_B(V_i) = \sup\{\min\big\{(x, y)_w, (y, z)_w\big\} - (x, z)_w : x, y, z, w \in V_i\}$$

for every $i \in \{1, 2\}$.

Proposition 2.3. *[29, Corollary 1] Let G be a graph. Then $2\delta_v^*(G) \leq \delta_v^*(S(G)) \leq 2\delta_v^*(G) + 2$.*

The hyperbolicity of the line graph has been studied previously (see [12–14]). The line graph of G is interesting in the theory of geometric graphs, since it is the intersection graph of $E(G)$.

Theorem 2.3. *[12, Corollary 3.12] Let G be a graph. Then $\delta(G) \leq \delta(\mathcal{L}(G)) \leq 5\delta(G) + 5/2$. Furthermore, the first inequality is sharp: the equality is attained by every cycle graph.*

Proposition 2.2 and Theorem 2.3 have the following consequence.

Corollary 2.1. *Let G be a graph. Then*

$$2\delta(G) \leq \delta(\mathcal{P}(G)) \leq 10\delta(G) + \frac{5}{2}.$$

Theorem 2.4. *[14, Theorem 6] Let G be a graph. Then $\delta_v^*(G) - 1 \leq \delta_v^*(\mathcal{L}(G)) \leq \delta_v^*(G) + 1$.*

Proposition 2.2, Corollary 2.3 and Theorem 2.4 give the following result:

Corollary 2.2. *Let G be a graph. Then $2\delta_v^*(G) - 1 \leq \delta_v^*(\mathcal{P}(G)) \leq 2\delta_v^*(G) + 3$.*

Theorem 2.5. *[29, Theorem 4] Let G be a graph. Then $\delta^*(G) - 4 \le \delta^*(\mathcal{L}(G)) \le \delta^*(G) + 4$.*

The following result follows from Proposition 2.2 and Theorem 2.5

Corollary 2.3. *Let G be a graph. Then $2\delta^*(G) - 4 \le \delta^*(\mathcal{P}(G)) \le 2\delta^*(G) + 4$.*

Proposition 2.2, and Theorems 2.3 and 2.5 have the following consequence.

Corollary 2.4. *[29, Corollary 3] Let G be a graph. Then*

$$\delta(S(G)) \le 2\delta(\mathcal{L}(G)) \le 5\delta(S(G)) + 5,$$
$$\delta^*(S(G)) - 8 \le 2\delta^*(\mathcal{L}(G)) \le \delta^*(S(G)) + 8.$$

Note that Theorem 2.5 improves the inequality $\delta^*(\mathcal{L}(G)) \le \delta^*(G) + 6$ reported in [13]. Given a graph G, we define

$$\operatorname{diam} V(G) := \sup\{d_G(v,w) \mid v,w \in V(G)\},$$
$$\operatorname{diam} G := \sup\{d_G(x,y) \mid x,y \in G\}.$$

The following result appears in [38].

Theorem 2.6. *For any graph G the inequalities*

$$\operatorname{diam} V(G) \le \operatorname{diam} G \le \operatorname{diam} V(G) + 1,$$
$$\delta(G) \le \frac{1}{2}\operatorname{diam} G \le \frac{1}{2}(\operatorname{diam} V(G) + 1),$$

are fulfilled.

From [38] we have the following result.

Theorem 2.7. *The following graphs with edges of length 1 have these precise values of δ.*
- *The* path graphs *P_n satisfies $\delta(P_n) = 0$ for every $n \ge 1$.*
- *The* cycle graphs *C_n satisfies $\delta(C_n) = n/4$ for every $n \ge 3$.*
- *The* complete graphs *K_n satisfies $\delta(K_1) = \delta(K_2) = 0$, $\delta(K_3) = 3/4$ and $\delta(K_n) = 1$ for every $n \ge 4$.*

Let us denote by $V(E)$ the set of vertices generated by subdividing the edges of G. If $v_iv_j \in E(G)$, we denote by $v^{i,j}$ its associated vertex in $V(E)$.

Theorem 2.8. *The following graphs have the following values of δ:*

i. *The path graph satisfies $\delta(\mathcal{P}(P_n)) = 0$ for every $n \ge 1$.*

ii. *The cycle graph satisfies $\delta(\mathcal{P}(C_n)) = n/2$ for every $n \ge 3$.*

iii. *The complete graph satisfies $\delta(\mathcal{P}(K_n)) = 2$ for every $n \ge 4$.*

iv. *The star graph S_k with k leaves satisfies $\delta(\mathcal{P}(S_3)) = 3/4$ and $\delta(\mathcal{P}(S_k)) = 1$ for every $k \ge 4$.*

v. *The wheel graph satisfies $\delta(\mathcal{P}(W_5)) = 9/4$, $\delta(\mathcal{P}(W_6)) = 5/2$ and $\delta(\mathcal{P}(Wn)) = 3$ for every $n \ge 7$.*

vi. *The Petersen graph satisfies $\delta(\mathcal{P}(P)) = 3$.*

Proof. If $e = v_iv_j \in E(G)$, then we denote by $v_{v_iv_j}$ the vertex in $\mathcal{L}(G)$ corresponding to e.

i. We have $\mathcal{P}(P_n) = P_{2n-2}$, therefore $\delta(\mathcal{P}(P_n)) = 0$.

ii. We have $\mathcal{P}(C_n) = C_{2n}$, therefore $\delta(\mathcal{P}(C_n)) = n/2$.

iii. Note that $\mathcal{P}(K_n)$ has n clique subgraphs of $(n-1)$ vertices associated with the n vertices of G and $\operatorname{diam} V(\mathcal{P}(K_n)) = 3$. Let x and y be the midpoints of $v_{v_4v^{3,4}}v_{v_4v^{1,4}}$ and $v_{v_2v^{1,2}}v_{v_2v^{2,3}}$, respectively. Consider P^* and P' two geodesics joining x and y such that $P^* \cap V(\mathcal{P}(K_n)) = \{v_{v_4v^{1,4}}, v_{v_1v^{1,4}}, v_{v_1v^{1,2}}, v_{v_2v^{1,2}}\}$ and $P' \cap V(\mathcal{P}(K_n)) = \{v_{v_4v_{3,4}}, v_{v_3v^{3,4}}, v_{v_3v^{2,3}}, v_{v_2v^{2,3}}\}$. Let z and p be the midpoints of P^* and P', respectively, and consider the geodesic triangle $T = \{[xz], [yz], P'\}$. We have

$$\delta(\mathcal{P}(K_n)) \ge \delta(T) \ge d_{\mathcal{P}(K_n)}(p, [xz] \cup [yz]) = 2 = \frac{\operatorname{diam} V(\mathcal{P}(K_n)) + 1}{2} \ge \delta(\mathcal{P}(K_n)).$$

iv. Note that $\mathcal{P}(S_n)$ is the complete graph K_{n-1} with a leaf at each vertex.

v. If $n=5$, then $\operatorname{diam} V(\mathcal{P}(W_5))=4$ and $\operatorname{diam}\mathcal{P}(W_5)=9/2$. Let x and y be $v_{v_1v^{0,1}}$ and the midpoint of $v_{v_3v^{3,4}}v_{v^{2,3}v_3}$, respectively. Consider P^* and P' two geodesics joining x and y such that $P^*\cap V(\mathcal{P}(W_5))=\{v_{v_1v^{0,1}},v_{v_0v^{0,1}},v_{v_0v^{0,3}},v_{v_3v^{0,3}},v_{v_3v^{3,4}}\}$ and $P'\cap V(\mathcal{P}(W_5))=\{v_{v_1v_{0,1}},v_{v_1v^{1,2}},v_{v_2v^{1,2}},v_{v_2v^{2,3}},v_{v_3v^{2,3}}\}$. Let z and p be the midpoints of P^* and P', respectively, and consider the geodesic triangle $T=\{[xz],[yz],P'\}$. We have

$$\delta(\mathcal{P}(W_5))\ge\delta(T)\ge d_{\mathcal{P}(W_5)}(p,[xz]\cup[yz])=\frac{9}{4}=\frac{\operatorname{diam}(\mathcal{P}(W_5))}{2}\ge\delta(\mathcal{P}(W_5)).$$

If $n=6$, then $\operatorname{diam}\mathcal{P}(W_6)=\operatorname{diam}V(\mathcal{P}(W_6))=5$. Let x and y be $v_{v_1v^{1,2}}$ and $v_{v_4v^{3,4}}$, respectively. Consider P^* and P' two geodesics joining x and y such that $P^*\cap V(\mathcal{P}(W_6))=\{v_{v_1v^{1,2}},v_{v_1v^{0,1}},v_{v_0v^{0,1}},v_{v_0v^{0,4}},v_{v_4v^{0,4}},v_{v_4v^{3,4}}\}$ and $P'\cap V(\mathcal{P}(W_6))=\{v_{v_1v^{1,2}},v_{v_2v^{1,2}},v_{v_2v^{2,3}},v_{v_3v^{2,3}},v_{v_3v^{3,4}},v_{v_4v^{3,4}}\}$. Let z and p be the midpoints of P^* and P', respectively, and consider the geodesic triangle $T=\{[xz],[yz],P'\}$. We have

$$\delta(\mathcal{P}(W_6))\ge\delta(T)\ge d_{\mathcal{P}(W_6)}(p,[xz]\cup[yz])=\frac{5}{2}=\frac{\operatorname{diam}(\mathcal{P}(W_6))}{2}\ge\delta(\mathcal{P}(W_6)).$$

Consider $n\ge 7$, then $\operatorname{diam}V(\mathcal{P}(W_n))=5$. Let x,y be the midpoints of $v_{v_1v^{1,6}}v_{v_1v^{1,2}}$ and $v_{v^{3,4}v_4}v_{v_4v^{4,5}}$, respectively. Consider P^* and P' two geodesics joining x and y such that

$$P^*\cap V(\mathcal{P}(W_n))=\{v_{v_1v^{1,6}},v_{v_1v^{0,1}},v_{v_0v^{0,1}},v_{v_0v^{0,4}},v_{v_4v^{0,4}},v_{v_4v^{4,5}}\},$$
$$P'\cap V(\mathcal{P}(W_n))=\{v_{v_1v^{1,2}},v_{v^{1,2}v_2},v_{v_2v^{2,3}},v_{v^{2,3}v_3},v_{v_3v^{3,4}},v_{v^{3,4}v_4}\}.$$

Let z and p be the midpoints of P^* and P', respectively, and consider the geodesic triangle $T=\{[xz],[yz],P'\}$. We have

$$\delta(\mathcal{P}(W_n))\ge\delta(T)\ge d_{\mathcal{P}(W_n)}(p,[xz]\cup[yz])=3=\frac{\operatorname{diam}V(\mathcal{P}(W_n))+1}{2}\ge\delta(\mathcal{P}(W_n)).$$

vi. Consider the Petersen graph P and $\mathcal{P}(P)$ as in Figures 1a and 1b. Note that $\operatorname{diam}V(\mathcal{P}(P))=5$. Let x,y be the midpoints of $v_{v_7v^{7,10}}v_{v_10v^{7,10}}$ and $v_{v_3v^{3,4}}v_{v_4v^{3,4}}$, respectively. Consider P^* and P' two geodesics joining x and y such that $P^*\cap V(\mathcal{P}(P))=\{v_{v_7v^{7,10}},v_{v_2v^{2,7}},v_{v_2v^{2,3}},v_{v_3v^{2,3}},v_{v_3v^{3,4}}\}$ and $P'\cap V(\mathcal{P}(P))=\{v_{v_10v^{7,10}},v_{v_5v^{5,10}},v_{v_5v^{4,5}},v_{v_4v^{4,5}},v_{v_4v^{3,4}}\}$. Let z and p be the midpoints of P^* and P', respectively, and consider the geodesic triangle $T=\{[xz],[yz],P'\}$. We have

$$\delta(\mathcal{P}(P))\ge\delta(T)\ge d_{\mathcal{P}(P)}(p,[xz]\cup[yz])=3=\frac{\operatorname{diam}V(\mathcal{P}(P))+1}{2}\ge\delta(\mathcal{P}(P)).$$

□

Semi total and total operators

If H is a subgraph of G, we always have $d_H(x,y)\ge d_G(x,y)$ for every $x,y\in H$. A subgraph H of G is said *isometric* if $d_H(x,y)=d_G(x,y)$ for every $x,y\in H$. Note that this condition is equivalent to $d_H(u,v)=d_G(u,v)$ for every vertices $u,v\in V(H)$. The following result appeared in [38].

Lemma 2.1. *If H is an isometric subgraph of G, then $\delta(H)\le\delta(G)$.*

Since G is an isometric subgraph of $T(G)$ and $R(G)$, and $\mathcal{L}(G)$ is an isometric subgraph of $T(G)$ and $Q(G)$, we have the following consequence of Lemma 2.1.

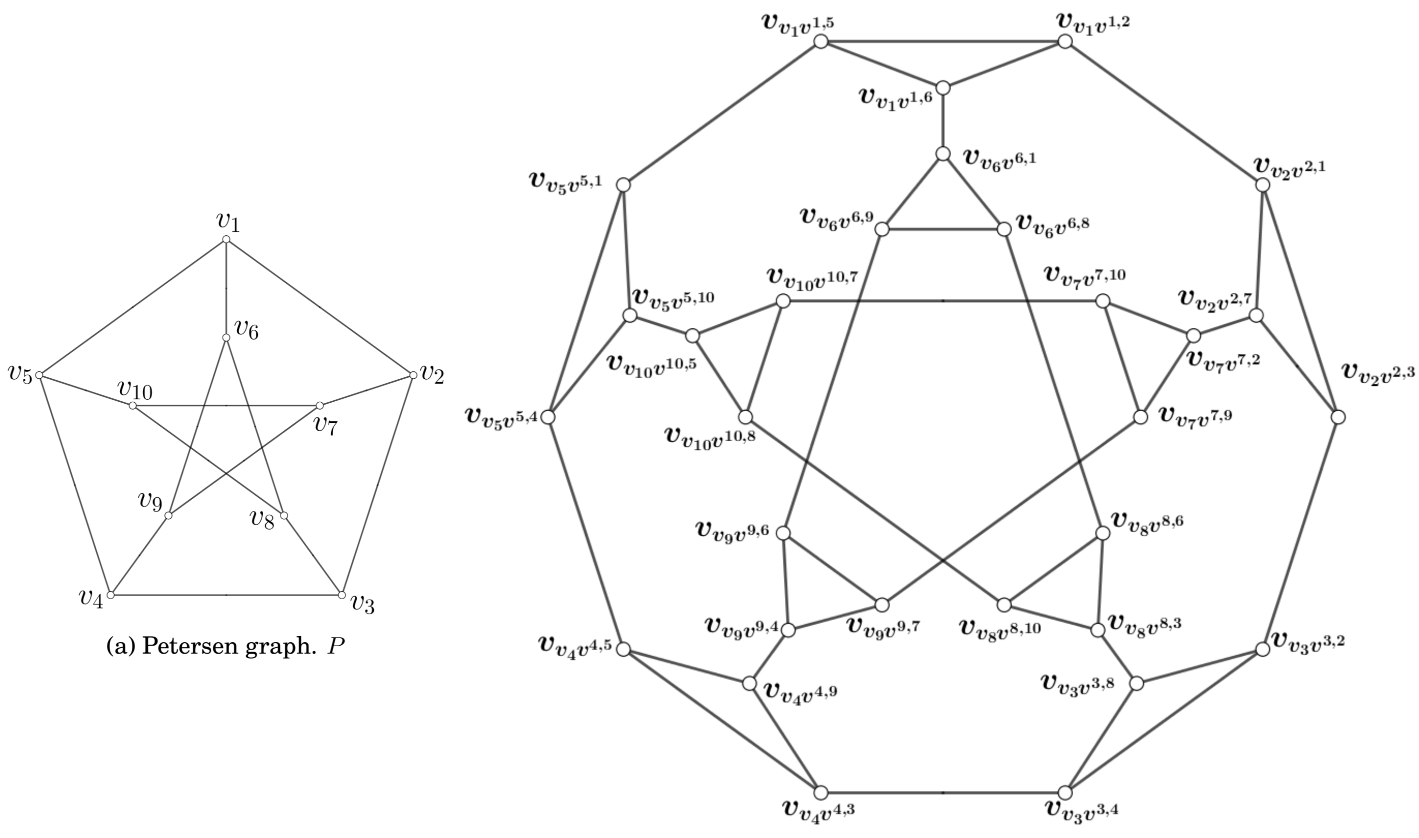
Proposition 2.4. *[29, Corollary 2] For any graph G, we have*

$$\begin{aligned}
\delta(G)&\le\delta(T(G)), & \delta^*(G)&\le\delta^*(T(G)), & \delta^*_v(G)&\le\delta^*_v(T(G)),\\
\delta(G)&\le\delta(R(G)), & \delta^*(G)&\le\delta^*(R(G)), & \delta^*_v(G)&\le\delta^*_v(R(G)),\\
\delta(\mathcal{L}(G))&\le\delta(T(G)), & \delta^*(\mathcal{L}(G))&\le\delta^*(T(G)), & \delta^*_v(\mathcal{L}(G))&\le\delta^*_v(T(G)),\\
\delta(\mathcal{L}(G))&\le\delta(Q(G)), & \delta^*(\mathcal{L}(G))&\le\delta^*(Q(G)), & \delta^*_v(\mathcal{L}(G))&\le\delta^*_v(Q(G)).
\end{aligned}$$

Theorems 2.3, 2.4, 2.5 and Proposition 2.4 have the following consequence.

Corollary 2.5. *[29, Corollary 4] Let G be a graph. Then*

$$\begin{aligned}
\delta(G)&\le\delta(Q(G)),\\
\delta^*_v(G)&\le\delta^*_v(Q(G))+1,\\
\delta^*(G)&\le\delta^*(Q(G))+4.
\end{aligned}$$



(a) Petersen graph. P

(b) $\mathcal{P}(P)$

Figure 1: Petersen graph and its para-line graph.

Given a graph G with multiple edges, we define $B(G)$ as the graph (without multiple edges) obtained from G by replacing each multiple edge by a single edge with the minimum length of the edges corresponding to that multiple edge. From [7] we have the following result.

Lemma 2.2. *If G is a graph with multiple edges, then G is hyperbolic if and only if $B(G)$ is hyperbolic and $J := \sup\{L(\beta) : \beta$ is an edge contained in a multiple edge of G$\}$ is finite. Besides, if $j := \inf\{d(x, y) : x, y$ are joined by a multiple edge of G$\}$, then*

$$\max\Big\{\delta(B(G)), \frac{J+j}{4}\Big\} \le \delta(G) \le \max\Big\{\delta(B(G)) + \frac{J-j}{2}, J\Big\}.$$

Proposition 2.5. *[29, Corollary 5] If G is a graph, then* $\max\Big\{\delta(G), \frac{3}{4}\Big\} \le \delta(R(G)) \le \max\Big\{\delta(G) + \frac{1}{2}, \frac{3}{4}\Big\}$.

Given any graph G which is not a tree, we define its *girth* $g(G)$ as the infimum of the lengths of the cycles in G. From [30] we have the following result.

Theorem 2.9. *Let G be a graph that is not a tree. Then*

$$\delta(G) \ge \frac{g(G)}{4}.$$

Proposition 2.6. *[29, Corollary 7] Let G be a graph which is not a tree. Then $\delta(G) \le \delta(R(G)) \le \delta(G) + \frac{1}{2}$.*

Theorem 2.3 and Proposition 2.6 have the following consequence.

Corollary 2.6. *[29, Corollary 8] Let G be a graph that is not a tree. Then*

$$\delta(R(G)) - \frac{1}{2} \le \delta(\mathcal{L}(G)) \le 5\delta(R(G)) + \frac{5}{2}.$$

Proposition 2.2 and Proposition 2.6 have the following consequence.

Corollary 2.7. *[29, Corollary 9] Let G be a graph that is not a tree. Then*

$$\delta(S(G)) \le 2\delta(R(G)) \le \delta(S(G)) + 1.$$

Theorem 2.10. *[29, Theorem 6] If G is a graph, then*

$$\begin{aligned}
\delta^*(\mathcal{L}(G)) &\le \delta^*(Q(G)) \le \delta_v^*(\mathcal{L}(G)) + 6 \le \delta^*(\mathcal{L}(G)) + 6,\\
\delta_v^*(\mathcal{L}(G)) &\le \delta_v^*(Q(G)) \le \delta_v^*(\mathcal{L}(G)) + 6,\\
\delta^*(\mathcal{L}(G)) &\le \delta^*(T(G)) \le \delta_v^*(\mathcal{L}(G)) + 9 \le \delta^*(\mathcal{L}(G)) + 9,\\
\delta_v^*(\mathcal{L}(G)) &\le \delta_v^*(T(G)) \le \delta_v^*(\mathcal{L}(G)) + 6,\\
\delta^*(G) &\le \delta^*(R(G)) \le \delta_v^*(G) + 6 \le \delta^*(G) + 6,\\
\delta_v^*(G) &\le \delta_v^*(R(G)) \le \delta_v^*(G) + 6,\\
\delta^*(G) &\le \delta^*(T(G)) \le \delta_v^*(G) + 9 \le \delta^*(G) + 9,\\
\delta_v^*(G) &\le \delta_v^*(T(G)) \le \delta_v^*(G) + 6.
\end{aligned}$$

Corollary 2.5 and Theorems 2.4 and 2.10 have the following consequence.

Corollary 2.8. *[29, Corollary 10] Let G be a graph. Then*

$$\begin{aligned}
\delta_v^*(G) - 1 &\le \delta_v^*(Q(G)) \le \delta_v^*(G) + 7,\\
\delta^*(G) - 4 &\le \delta^*(Q(G)) \le \delta_v^*(G) + 7 \le \delta^*(G) + 7.
\end{aligned}$$

The inequalities $\delta(G) \le 3\delta^*(G)$ and $\delta^*(G) \le 2\delta(G)$, Theorem 2.10, Proposition 2.4 and Corollaries 2.5 and 2.8 have the following consequence.

Proposition 2.7. *[29, Corollary 11] If G is a graph, then*

$$\begin{aligned}
\delta(\mathcal{L}(G)) &\le \delta(Q(G)) \le 6\delta(\mathcal{L}(G)) + 18,\\
\delta(\mathcal{L}(G)) &\le \delta(T(G)) \le 6\delta(\mathcal{L}(G)) + 27,\\
\delta(G) &\le \delta(T(G)) \le 6\delta(G) + 27,\\
\delta(G) &\le \delta(Q(G)) \le 6\delta(G) + 21.
\end{aligned}$$

Proposition 2.8. *[29, Theorem 7] If G is a path graph, then*

$$0 = \delta(\mathcal{L}(G)) \le \delta(Q(G)) \le 3/4.$$

The union of the set of the midpoints of the edges of a graph G and the set of vertices, $V(G)$, will be denoted by $J(G)$. Let $\mathbb{T}_1$ be the set of geodesic triangles T in G that are cycles and such that each vertex of T belongs to $J(G)$. Let us define

$$\delta_1(G) := \inf\{\lambda : \text{every geodesic triangle in } \mathbb{T}_1 \text{ is } \lambda\text{-thin}\}.$$

The following results appeared in [6].

Theorem 2.11. *For every graph G we have $\delta_1(G) = \delta(G)$.*

Theorem 2.12. *If G is an hyperbolic graph, then there exists $T \in \mathbb{T}_1$ with $\delta(T) = \delta(G)$.*

The following results improve the inequality $\delta(Q(G)) \le 6\delta(\mathcal{L}(G)) + 18$ in Proposition 2.7.

Theorem 2.13. *[29, Theorem 8] If G is not a path graph, then*

$$\delta(\mathcal{L}(G)) \le \delta(Q(G)) \le \delta(\mathcal{L}(G)) + 1/2.$$

Proposition 2.2, Theorems 2.3 and 2.13, and Corollary 2.4 have the following consequence.

Corollary 2.9. *[29, Corollary 12] Let G be a graph. If G is not a path graph, then*

$$\delta(S(G)) \le 2\delta(Q(G)) \le 5\delta(S(G)) + 6.$$

Central graph

The central graph of G, $C(G)$, has the same set of vertices as $S(G)$, and the set of edges of $C(G)$ is the union of the edge sets of $S(G)$ and $\overline{G}$. We can write

$$C(G) = (V(G) \cup V(E), E\,(S(G)) \cup E(\overline{G})).$$

The central graph is related to the complement graph; in such a direction, it is important to note that hyperbolicity in the complement graph has been studied in [8, 22, 36, 37]. Next, we expose some of its main results. The following result in [8] gives a sharp bound for the hyperbolicity constant of the complement of a graph.

Theorem 2.14. *If G is a graph with $\operatorname{diam} V(G) \geq 3$, then its complement graph $\overline{G}$ satisfies $0 \leq \delta(\overline{G}) \leq 2$.*

The next theorem about regular graphs can be proved using the following result (see [5]).

Lemma 2.3. *Given any graph G, we have $\delta(G) \geq 5/4$ if and only if there exist a cycle g in G with length $L(g) \geq 5$ and a vertex $w \in g$ such that $\deg_g(w) = 2$.*

Theorem 2.15. *[36, Theorem 4.3] Let G be a $(n-3)$-regular graph with $n \geq 5$ vertices. Then $\delta(G) = 1$ if $\overline{G}$ is a union of cycle graphs with three vertices, and $\delta(G) = 5/4$ otherwise.*

Theorem 2.16. *Let G be a regular graph of order n and $\operatorname{diam} V(G) = 3$, then $\delta(\overline{G}) \leq \frac{3}{2}$. Furthermore, if G is a k-regular graph with $k < n-2$, then $\delta(\overline{G}) \geq \frac{3}{4}$.*

Proof. Suppose that $\operatorname{diam} V(G) = 3$ and $\operatorname{diam} V(\overline{G}) \geq 3$, then there exist four different vertices, $x, y, w, z \in V(G)$, such that $d_G(x, y) = 3$ and $d_{\overline{G}}(w, z) = 3$; note that $uw \in E(G)$ or $uz \in E(G)$ for all $u \in V(G)$. Without loss of generality suppose $xw \in E(G)$ and so, $yz \in E(G)$.

Consider the set $S = V(G) - \{x, y, w, z\}$. Let l be the number of edges in $E(G)$ between the vertices of the sets S and $\{x, y\}$, and let m be the number of edges in $E(G)$ between the vertices of the sets S and $\{w, z\}$. It is clear that $m \geq l$, $\deg(w) + \deg(z) = m + 4$, $\deg(x) + \deg(y) = l + 2$ and $\deg(w) + \deg(z) > \deg(x) + \deg(y)$, so G is not regular.

Thus, we have that if G is a regular graph of order n and $\operatorname{diam} V(G) = 3$, then $\operatorname{diam} V(\overline{G}) = 2$. Since $\delta(G) \leq \frac{\operatorname{diam} V(G)+1}{2}$, the result follows.

Since G is a k-regular graph with $k < n-2$, then $\overline{G}$ is $(n-k-1)$-regular graph. Therefore, $\overline{G}$ is not a tree and $\overline{G}$ has a cycle of length at least 3. The result follows because $\delta(G) \geq \frac{g(G)}{4}$. □

We can write

$$C(G) := (V \cup V(E), E(S(G)) \cup E(\overline{G})).$$

Theorem 2.17. *If $G = (V, E)$ is a graph, then $\operatorname{diam} V(C(G)) \leq 4$ and this bound is sharp.*

Proof. Fix two vertices $u, v \in V(C(G))$. We have three cases:

- *Case 1:* $u, v \in V$. If $uv \in E(G)$, then $d_{C(G)}(u, v) = 2$. If $uv \notin E(G)$, then $d_{C(G)}(u, v) = 1$.
- *Case 2:* $u \in V$, $v \in V(E)$. Consider $v_1, v_2 \in V(G)$ such that v_1v, $v_2v \in E(S(G))$. We have the following cases:
  - *Case 2.1:* If $u = v_1$ or $u = v_2$, then $d_{C(G)}(u, v) = 1$.
  - *Case 2.2:* If $uv_1 \notin E(G)$ or $uv_2 \notin E(G)$, then $d_{C(G)}(u, v) = 2$.
  - *Case 2.3:* If uv_1, $uv_2 \in E(G)$ (i.e. u, v_1, v_2 induce a subgraph K_3), then $d_{C(G)}(u, v) = 3$.
- *Case 3:* $u, v \in V(E)$. Consider $u_1, u_2, v_1, v_2 \in V(G)$ such that $u_iu, v_iv \in E(S(G))$ for $i = 1, 2$. By Case 1, we have

$$d_{C(G)}(u, v) = 1 + \min\{d_{C(G)}(u_i, v_j) : 1 \leq i, j \leq 2\} + 1 \leq 1 + 2 + 1 = 4.$$

The bound is attained, for example, on the central graph of the complete graph K_n $(n \geq 4)$ by considering the vertices obtained by the subdivision of two non-adjacent edges in K_n. □

Corollary 2.10. *If G is a graph, then $\operatorname{diam} C(G) \leq 4$.*

Proof. Let $x, y \in C(G)$ be points such that $d_{C(G)}(x, y) = \operatorname{diam} C(G)$. Since the diameter of the graph is reached by considering the distances between vertices and midpoints of edges, let us consider the following cases:

(i) $x, y \in V(C(G))$. Theorem 2.17 gives $d_{C(G)}(x, y) \leq 4$.

(ii) $x \in V(C(G))$, y is the midpoint of $e = v_1v_2 \in E(S(G)) \cup E(\overline{G})$. Without loss of generality we can assume that $v_1 \in V(G)$. Case 2 in Theorem 2.17 gives $d_{C(G)}(x, y) \leq d_{C(G)}(x, v_1) + 1/2 \leq 7/2$.

(iii) x, y are the midpoints of $u_1u_2, v_1v_2 \in E(C(G))$, respectively. Without loss of generality we can assume that $u_1, v_1 \in V(G)$. Case 1 in Theorem 2.17 gives $d_{C(G)}(x, y) \leq d_{C(G)}(x, u_1) + d_{C(G)}(u_1, v_1) + d_{C(G)}(v_1, y) \leq 3$.

□

Note that it is not necessarily the case that $\operatorname{diam} V(C(G)) = \operatorname{diam} C(G)$. For example, if $G = P_2$, then $C(P_2) = C_5$ and $\operatorname{diam} V(C_5) = 2 \neq 5/2 = \operatorname{diam} C(G)$. Theorem 2.6 and Corollary 2.10 give the next result.

Theorem 2.18. *If G is a graph, then $\delta(C(G)) \leq 2$.*

Theorem 2.19. *If G is a graph of order $n \geq 2$, then the following statements hold:*

i. $\operatorname{diam} V(C(G)) = 1$ *if and only if* $E(G) = \emptyset$.

ii. $\operatorname{diam} V(C(G)) = 2$ *if and only if* $G = S_n$.

iii. $\operatorname{diam} V(C(G)) = 4$ *if and only if G has an induced subgraph* K_4.

iv. $\operatorname{diam} V(C(G)) = 3$ *otherwise.*

Proof. i. If $\operatorname{diam} V(C(G)) = 1$, then for each pair of vertices $u, v \in V(G)$, we have that $uv \notin E(G)$ and so, $E(G) = \emptyset$. The converse is immediate.

ii. Assume now that $\operatorname{diam} V(C(G)) = 2$. If G has an induced subgraph C_3 with vertices v_1, v_2, v_3, then $d_{C(G)}(v_1, v^{2,3}) = 3$. If the length of the girth of G is greater than or equal to 4, then there exist two non-adjacent edges u_1u_2 and v_1v_2, and $d_{C(G)}(u^{1,2}, v^{1,2}) = 3$. Therefore, G has no cycles, i.e., G is a tree. Since $\operatorname{diam} V(C(G)) = 2$, then each pair of edges in $E(G)$ has a common vertex in $V(G)$ and so, $G = S_n$. The converse is immediate.

iii. Finally, assume that $\operatorname{diam} V(C(G)) = 4$. Let $u, v \in C(G)$ be such that $d_{C(G)}(u, v) = 4$. Theorem 2.17 gives $u, v \in V(E)$. Let $u_1, u_2, v_1, v_2 \in V(G)$ be such that $u_iu, v_iv \in E(S(G))$ for $i = 1, 2$. Note that $u_1u_2, v_1v_2 \in E(G)$ and if $u_iv_j \notin E(G)$, $i, j \in \{1, 2\}$, then $d_{C(G)}(u, v) = 3$, which contradicts $d_{C(G)}(u, v) = 4$. Thus u_1, u_2, v_1, v_2 induce a complete subgraph K_4. Let us prove now the converse. Let $u_1, u_2, v_1, v_2 \in V(G)$ be the vertices in an induced subgraph $K_4 \subset G$. The argument in the proof of Theorem 2.17 implies $d_{C(G)}(u^{1,2}, v^{1,2}) = 4 = \operatorname{diam} C(G)$. □

Definition 2.2. *Let v_1, v_2, v_3, v_4 be the vertices in the complete graph K_4. We define the subgraphs $H_0 := K_4 \setminus \{v_1v_4, v_2v_4\}$ and $H_1 := K_4 \setminus \{v_2v_4\}$.*

Theorem 2.20. *If G is a graph of order $n \geq 4$, then the following statements hold:*

i. *If* $\operatorname{diam} V(C(G)) = 1$, *then* $\operatorname{diam} C(G) = 2$.

ii. *If* $\operatorname{diam} V(C(G)) = 2$, *then* $\operatorname{diam} C(G) = 3$.

iii. *If* $\operatorname{diam} V(C(G)) = 3$ *and G has an induced subgraph H_0 or H_1, then* $\operatorname{diam} C(G) = 7/2$.

iv. *If* $\operatorname{diam} V(C(G)) = 3$ *and G does not have H_0 or H_1 as induced subgraphs, then* $\operatorname{diam} C(G) = 3$.

v. *If* $\operatorname{diam} V(C(G)) = 4$, *then* $\operatorname{diam} C(G) = 4$.

Proof. i. If $\operatorname{diam} V(C(G)) = 1$, then $C(G) = K_n$ and so, $\operatorname{diam} C(G) = \operatorname{diam} K_n = 2$.

ii. Since $\operatorname{diam} V(C(G)) = 2$, Theorem 2.19 gives $G = S_n$ and there are four vertices $v_0, v_1, v_2, v_3 \in V(G)$ such that $v_0v_i \in E(G)$, for $i = 1, 2, 3$. Let $x, y \in C(S_n)$ be the midpoints of v_1v_2 and $v_0v^{0,3}$, respectively. We have $d_{C(G)}(v_0, \{v_1, v_2\}) = d_{C(G)}(v^{0,3}, \{v_1, v_2\}) = 2$ and so, Theorem 2.6 gives

$$\begin{aligned}\operatorname{diam} C(G) \geq d_{C(G)}(x, y) &= d(x, \{v_1, v_2\}) + \min\{d_{C(G)}(v_0, \{v_1, v_2\}), d_{C(G)}(v^{0,3}, \{v_1, v_2\})\} + d(y, \{v_0, v^{0,3}\}) \\ &= \operatorname{diam} V(C(G)) + 1 \geq \operatorname{diam} C(G).\end{aligned}$$

iii. Since $\operatorname{diam} V(C(G)) = 3$, *item ii* in Proposition 2.10 gives $\operatorname{diam} C(G) \leq 7/2$. Let $v_1, v_2, v_3, v_4 \in V(G)$ be such as in Definition 2.2 and and let $H_i \subset G$, $i \in \{0, 1\}$. Let x be the midpoint of $v_3v^{3,4} \in E(C(G))$, we have

$$7/2 \geq \operatorname{diam} C(G) \geq d_{C(G)}(v^{1,2}, x) = 7/2.$$

We prove now the converse. Since $\operatorname{diam} C(G) = 7/2$, there exist $v \in V(C(G))$ and $e = u_1u_2 \in E(C(G))$ such that if x is the midpoint of e, then $d_{C(G)}(x, v) = 7/2$. Note that $d_{C(G)}(v, u_1) = d_{C(G)}(v, u_2) = 3$. We can suppose that $u_1 \in V(G)$. The argument in the proof of Theorem 2.17 gives $v \in V(E)$ and so, there are vertices $v_1, v_2 \in V(G)$ such that $vv_1, vv_2 \in E(C(G))$, i.e., $v_1v_2 \in E(G)$ and $v = v^{1,2}$; therefore, $d_{C(G)}(v_1, u_1) = d_{C(G)}(v_2, u_1) = 2$ and $u_1v_1, u_1v_2 \in E(G)$. If $u_2 \in V(G)$, then a similar argument gives $u_2v_1, u_2v_2 \in E(G)$ and so, G has an induced subgraph K_4 which contradicts *item iii* of Theorem 2.19. Hence, $u_2 \in V(E)$ and there exists $u_3 \in V(G)$ such that $u_1u_3 \in E(G)$ and $u_2 = u^{1,3}$. Since $d_{C(G)}(v^{1,2}, u_3) = 2$, we have either $v_1u_3 \notin E(G)$ or $v_2u_3 \notin E(G)$ and so, v_1, v_2, u_1, u_3 induce either the subgraph H_0 or H_1.

iv. By the previous item, we have

$$\frac{7}{2} > \operatorname{diam} C(G) \geq \operatorname{diam} V(C(G)) = 3$$

and so, $\operatorname{diam} C(G) = 3$.

v. Corollary 2.10 gives the result. □

Theorem 2.7 gives the following result.

Theorem 2.21. *If G is a graph of order $n \leq 3$, then the following statements hold:*

i. *If $n \leq 2$, then $\delta(C(G)) = 0$.*

ii. *If G is a set of three isolated vertices, then $\delta(C(G)) = 3/4$.*

iii. *If G is an edge and an isolated vertex, then $\delta(C(G)) = 1$.*

iv. *If $G = P_3$, then $\delta(C(P_3)) = 5/4$.*

v. *If $G = K_3$, then $\delta(C(K_3)) = 3/2$.*

Theorem 2.22. *If G is a graph of order $n \geq 4$, then*

$$\delta(C(G)) = \frac{1}{2} \operatorname{diam} C(G).$$

Proof. Since Theorem 2.20 relates $\operatorname{diam} C(G)$ and $\operatorname{diam} V(C(G))$, we have the following cases.

Case i. $\operatorname{diam} V(C(G)) = 1$. By the argument in the proof of Theorem 2.20, we have $C(G) = K_n$ and $\operatorname{diam} C(G) = 2$, and Theorem 2.7 gives $\delta(C(G)) = 1 = \frac{1}{2} \operatorname{diam} C(G)$.

Case ii. $\operatorname{diam} V(C(G)) = 2$. Theorem 2.19 gives $G = S_n$. Let $v_0, v_1, v_2, v_3 \in V(G)$ be such that $v_0 v_i \in E(G)$ for $1 \leq i$. Let $x, y \in C(G)$ be the midpoints of $v_1 v_2$ and $v_0 v^{0,3}$, respectively. Note that $d_{C(G)}(x, y) = 3 = \operatorname{diam} C(G)$. Let P and P' be two geodesics joining x and y such that $P \cap V(C(G)) = \{v_1, v_3, v^{0,3}\}$ and $P' \cap V(C(G)) = \{v_0, v^{0,2}, v_2\}$. Let z and p' be the midpoints of P and P', respectively, and consider the geodesic triangle $T = \{[xz], [yz], P'\}$. We have

$$\frac{3}{2} = \frac{1}{2} \operatorname{diam} C(G) \geq \delta(C(G)) \geq \delta(T) \geq d_{C(G)}(p', [xz] \cup [yz]) = d_{C(G)}(p', \{x, y\}) = \frac{3}{2}.$$

Case iii. $\operatorname{diam} V(C(G)) = 3$. We have two cases.

Case iii-A. G contains either H_0 or H_1 as subgraphs. Suppose that $H_1 \subset G$. Let $x \in C(G)$ be the midpoint of $v_3 v^{3,4}$ and $y = v^{1,2}$. Let P and P' be geodesics joining x and y such that $P \cap V(C(G)) = \{v^{1,2}, v_2, v_4, v^{3,4}\}$ and $P' \cap V(C(G)) = \{v^{1,2}, v_1, v^{1,3}, v_3\}$. Let z and p' be the midpoints of P and P', respectively, and consider the geodesic triangle $T = \{[xz], [yz], P'\}$. We have

$$2 d_{C(G)}(p', [xz] \cup [yz]) = 2 d_{C(G)}(p', \{x, y\}) = d_{C(G)}(p', x) + d_{C(G)}(p', y) = \operatorname{diam} C(G)$$

and so,

$$\frac{1}{2} \operatorname{diam} C(G) \geq \delta(C(G)) \geq \delta(T) \geq d_{C(G)}(p', [xz] \cup [yz]) = \frac{1}{2} \operatorname{diam} C(G).$$

Therefore, $\delta(C(G)) = \frac{1}{2} \operatorname{diam} C(G)$. The proof is similar if $H_2 \subset G$.

Case iii-B. G does not contain H_0 or H_1 as subgraphs. Theorem 2.6 gives $\delta(C(G)) \leq 3/2$. Let us prove the converse inequality. Let $u, v \in V(C(G))$ such that $d_{C(G)}(u, v) = 3$. We have two cases.

Case iii-B-1. $v \in V(G)$ and $u \in V(E)$. Let $u_1, u_2 \in V(G)$ such that $u_1 u, u_2 u \in E(C(G))$. Note that v, u_1, u_2 are the vertices in the subgraph K_3. Since $n \geq 4$, if $w \in V(G)$, then $vw, u_1 w, u_2 w \notin E(G)$. Since $C(K_3) = C_6$ is an isometric subgraph in $C(G)$, Theorem 2.7 and Proposition 2.4 give $\delta(C(G)) \geq \delta(C_6) = 3/2$.

Case iii-B-2. $u, v \in V(E)$. Let v_1, v_2, u_1, $u_2, \in V(G)$ such that $uu_i, vv_i \in E(C(G))$ for $i = 1, 2$. Since $d_{C(G)}(u, v) = 3$, there exist i, j such that $u_i v_j \notin E(G)$. By symmetry, we can assume that $u_2 v_2 \notin E(G)$.

If $u_1 v_1 \in E(G)$, then $u_1 v_2, u_2 v_1 \notin E(G)$. Let P and P' be two geodesics joining u and v such that $P \cap V(C(G)) = \{u, u_1, v_2, v\}$ and $P' \cap V(C(G)) = \{u, u_2, v_1, v\}$. If $u_1 v_1 \notin E(G)$, consider P and P' such that $P \cap V(C(G)) = \{u, u_1, v_1, v\}$ and $P' \cap V(C(G)) = \{u, u_2, v_2, v\}$. Let p and z be the midpoints of P and P', respectively, and consider the geodesic triangle $T = \{[uz] \cup [vz] \cup P\}$. We have

$$\delta(C(G)) \geq \delta(T) \geq d_{C(G)}(p, [uz] \cup [vz]) = \frac{3}{2} = \frac{1}{2} \operatorname{diam} C(G).$$

Case iv. $\operatorname{diam} V(C(G)) = 4$. Let $v_1, v_2, v_3, v_4 \in V(G)$ be the vertices in the induced subgraph $K_4 \subset G$. Let P be the geodesic joining $v^{1,2}$ and $v^{3,4}$ such that $P \cap V(C(G)) = \{v^{1,2}, v_1, v^{1,4}, v_4, v^{3,4}\}$. If we consider the geodesic triangle $T = \{[v^{1,2} v^{2,3}], [v^{2,3} v^{3,4}], P\}$, then

$$\begin{aligned} \frac{1}{2} \operatorname{diam} C(G) &\geq \delta(C(G)) \geq \delta(T) \geq d_{C(G)}\left(v^{1,4}, [v^{1,2} v^{2,3}] \cup [v^{2,3} v^{3,4}]\right) \\ &= d_{C(G)}(v^{1,4}, \{v^{1,2}, v^{3,4}\}) = 2 = \frac{1}{2} \operatorname{diam} C(G). \end{aligned}$$

□

Theorems 2.19, 2.20 and 2.22 have the following consequence.

Theorem 2.23. *The following graphs have the following values of δ:*

- *The path graph P_n satisfies $\delta(C(P_n)) = 3/2$ for every $n \geq 4$.*
- *The cycle graph C_n satisfies $\delta(C(C_n)) = 3/2$ for every $n \geq 3$.*
- *The complete graph K_n satisfies $\delta(C(K_n)) = 2$ for every $n \geq 4$.*
- *The complete bipartite graph $K_{m,n}$ satisfies $\delta(C(K_{m,n})) = 3/2$ for every m, $n \geq 2$.*
- *The wheel graph W_n satisfies $\delta(C(W_n)) = 7/4$ for every $n \geq 5$.*
- *The star graph S_n satisfies $\delta(C(S_n)) = 3/2$ for every $n \geq 4$.*
- *The Petersen graph P satisfies $\delta(C(P)) = 3/2$.*

Acknowledgements

The research of Rosalio Reyes, José M. Rodríguez, José M. Sigarreta was supported by a grant from Agencia Estatal de Investigación (PID2019-106433GBI00/AEI/10.13039/501100011033), Spain. José M. Rodríguez was also supported by the Madrid Government (Comunidad de Madrid-Spain) under the Multiannual Agreement with UC3M in the line of Excellence of University Professors (EPUC3M23), and in the context of the V PRICIT (Regional Programme of Research and Technological Innovation).

References

[1] M. Abu-Ata, F. F. Dragan, Metric tree-like structures in real-life networks: an empirical study, *Networks* **67** (2016) 49–68.

[2] J. Alonso, T. Brady, D. Cooper, T. Delzant, V. Ferlini, M. Lustig, M. Mihalik, M. Shapiro, H. Short, Notes on word hyperbolic groups, In: E. Ghys, A. Haefliger, A. Verjovsky (Eds.), *Group Theory from a Geometrical Viewpoint*, World Scientific, 1992, 3–63.

[3] Y. Baryshnikov, G. H. Tucci, Asymptotic traffic flow in an hyperbolic network I: Definition and properties of the core, *arXiv*:1010.3304 [math.GR], (2010).

[4] B. Basavanagoud, S. M. Hosamani, S. H. Malghan, Edge semitotal - point domination in graphs, *Int. J. Math. Sci. Eng. Appl.* **5** (2011) 53–60.

[5] S. Bermudo, J. M. Rodríguez, O. Rosario, J. M. Sigarreta, Small values of the hyperbolicity constant in graphs, *Discrete Math.* **339** (2016) 3073–3084.

[6] S. Bermudo, J. M. Rodríguez, J. M. Sigarreta, Computing the hyperbolicity constant, *Comput. Math. Appl.* **62** (2011) 4592–4595.

[7] S. Bermudo,J. M. Rodríguez, J. M. Sigarreta, J.-M. Vilaire, Gromov hyperbolic graphs, *Discrete Math.* **313** (2013) 1575–1585.

[8] S. Bermudo, J. M. Rodríguez, J. M. Sigarreta, E. Touris, Hyperbolicity and complement of graphs, *Appl. Math. Lett.* **24** (2011) 1882–1887.

[9] A. R. Bindusree, I. N. Cangul, V. Lokesh, A. S. Cevik, Zagreb polynomials of three graph operators, *Filomat* **30** (2016) 1979–1986.

[10] B. H. Bowditch, Notes on Gromov's hyperbolicity criterion for path-metric spaces, In: E. Ghys, A. Haefliger, A. Verjovsky (Eds.), *Group Theory from a Geometrical Viewpoint*, World Scientific, 1991, 64–167.

[11] G. Brinkmann, J. Koolen, V. Moulton, On the hyperbolicity of chordal graphs, *Ann. Combin.* **5** (2001) 61–69.

[12] W. Carballosa, J. M. Rodríguez, J. M. Sigarreta, New inequalities on the hyperbolicity constant of line graphs, *Ars Combin.* **129** (2016) 367–386.

[13] W. Carballosa, J. M. Rodríguez, J. M. Sigarreta, M. Villeta, On the hyperbolicity constant of line graphs, *Electron. J. Combin.* **18** (2011) #P210.

[14] D. Coudert, G. Ducoffe, On the hyperbolicity of bipartite graphs and intersection graphs, *Discrete Appl. Math.* **214** (2016) 187–195.

[15] M. R. Farahani, W. Gao, M. F. Nadeem, S. Zafar, Z. Zahid, On the para-line graphs of certain nanostructures based on topological indices, *Sci. Bull. Series B: Chem. Mater. Sci.* **79** (2017) 93–104.

[16] M. R. Farahani, M. F. Nadeem, S. Zafar, Z. Zahid, X. Zhang, Study of the para-line graphs of certain polyphenyl chains using topological indices, *Int. J. Biochem. Biotech.* **8** (2017) 2435–2442.

[17] E. Ghys, P. de la Harpe, *Sur les Groupes Hyperboliques d'après Mikhael Gromov*, Birkhäuser Boston Inc., 1990.

[18] M. Gromov, Hyperbolic groups, In: S. M. Gersten (Ed.), *Essays in Group Theory*, Springer, 1987, 75–263.

[19] F. Harary, R. Z. Norman, Some properties of line digraphs, *Rend. Circ. Mat. Palermo.* **9** (1960) 161–168.

[20] Harisha, P. S. W. Ranjini, V. Lokesha, SK indices of graph operator $S(G)$ and $R(G)$ on few nanostructures. *Montes Taurus J. Pure Appl. Math.* **2** (2020) 38–44.

[21] V. Hernández, D. Pestana, J. M. Rodríguez, Bounds on Gromov hyperbolicity constant, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **110** (2016) 321–342.

[22] J. C. Hernández, R. Reyes, J. M. Rodríguez, J. M. Sigarreta, Mathematical properties on the hyperbolicity of interval graphs, *Symmetry* **9** (2017) #255.

[23] E. A. Jonckheere, Contrôle du trafic sur les réseaux à géometrie hyperbolique–Une approche mathématique a la sécurité de l'acheminement de l'information, *J. Europ. Syst. Automat.* **37** (2003) 145–159.

[24] E. A. Jonckheere, M. Lou, J. Hespanha, P. Barooah, Effective resistance of Gromov-hyperbolic graphs: Application to asymptotic sensor network problems, *Proceedings of the 46th IEEE Conference on Decision and Control*, 2007, 1453–1458.

[25] E. A. Jonckheere, P. Lohsoonthorn, Geometry of network security, *Proceedings of the American Control Conference*, 2004, 111–151.

[26] J. Krausz, Démonstration nouvelle d'un théorème de Whitney sur les réseaux, *Mat. Fiz. Lapok.* **50** (1943) 75–85.

[27] R. Krauthgamer, J. R. Lee, Algorithms on negatively curved spaces, *47th Annual IEEE Symposium on Foundations of Computer Science* (FOCS'06), 2006, 119–132.

[28] D. Krioukov, F. Papadopoulos, M. Kitsak, A. Vahdat, M. Boguñá, Hyperbolic geometry of complex networks, *Phys. Rev. E* **82** (2010) #036106.

[29] J. A. Méndez-Bermúdez, R. Reyes, J. M. Rodríguez, J. M. Sigarreta, Hyperbolicity on graph operators, *Symmetry* **10** (2018) #360.

[30] J. Michel, J. M. Rodríguez, J. M. Sigarreta, M. Villeta, Hyperbolicity and parameters of graphs, *Ars Combin.* **100** (2011) 43–63.

[31] F. Montgolfier, M. Soto, L. Viennot, Treewidth and hyperbolicity of the internet, *10th IEEE Int. Symposium on Network Computing and Applications*, 2011, 25–32.

[32] E. Prisner, *Graph Dynamics*, Chapman and Hall/CRC, 1995.

[33] J. Rada, J. M. Rodríguez, J. M. Sigarreta, Geometric properties of chemical graphs, *Int. J. Quantum Chem.* **121** (2021) 1–14.

[34] P. S. Ranjini, V. Lokesha, Smarandache-Zagreb index on three graph operators, *Int. J. Math. Combin.* **3** (2010) 1–10.

[35] M. A. Rashid, S. Ahmad, M. K. Siddiqui, M. Imran, Topological properties of para-line graphs for chemical networks, *Polycycl. Aromat. Compd.* **42** (2022) 260–276.

[36] R. Reyes, J. M. Rodríguez, J. M. Sigarreta, M. Villeta, On the hyperbolicity constant of circular-arc graphs, *Discr. Appl. Math.* **263** (2019) 244–256.

[37] J. M. Rodríguez, J. M. Sigarreta, Bounds on Gromov hyperbolicity constant in graphs, *Proc. Indian Acad. Sci. (Math. Sci.)* **122** (2012) 53–65.

[38] J. M. Rodríguez, J. M. Sigarreta, J.-M. Vilaire, M. Villeta, On the hyperbolicity constant in graphs, *Discrete Math.* **311** (2011) 211–219.

[39] E. Sampathkumar, S. B. Chikkodimath, Semi-total graphs of a graph I, *Karnatak Univ. J.* **18** (1973) 273–280.

[40] Y. Shang, Lack of Gromov-hyperbolicity in colored random networks, *Pan-Am. Math. J.* **21** (2011) 27–36.
[41] Y. Shang, Lack of Gromov-hyperbolicity in small-world networks, *Cent. Eur. J. Math.* **10** (2012) 1152–1158.
[42] Y. Shang, Non-hyperbolicity of random graphs with given expected degrees, *Stoch. Models* **29** (2013) 451–462.
[43] M. Soto, *Quelques Propriétés Topologiques des Graphes et Applications a Internet et Aux Réseaux*, Ph.D. Thesis, Université Paris Diderot, Paris, 2011.
[44] E. Tourís, Graphs and Gromov hyperbolicity of non-constant negatively curved surfaces, *J. Math. Anal. Appl.* **380** (2011) 865–881.
[45] K. Verbeek, S. Suri, Metric embeddings, hyperbolic space and social networks, *Comput. Geom.* **59** (2016) 1–12.
[46] F. Werner, Graph-theoretic problems and their new applications, *Mathematics* **8** (2020) #445.
[47] Y. Wu, C. Zhang. Chordality and hyperbolicity of a graph, *Electron. J. Combin.* **18** (2011) #P43.
[48] W. Yan, B.-Y. Yang, Y.-N. Yeh, The behavior of Wiener indices and polynomials of graphs under five graph decorations, *Appl. Math. Lett.* **20** (2007) 290–295.