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Research Article

Saturation of multidimensional 0-1 matrices

Shen-Fu Tsai*

Department of Mathematics, National Central University, Taiwan

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Abstract

A 0-1 matrix M is *saturating* for a 0-1 matrix P if M does not contain a submatrix that can be turned into P by flipping any number of its 1-entries to 0-entries, and flipping any 0-entry of M to 1-entry introduces a copy of P. Matrix M is *semisaturating* for P if flipping any 0-entry of M to 1-entry introduces a *new* copy of P, regardless of whether M originally contains P or not. The functions $ex(n;P)$ and $sat(n;P)$ are the maximum and minimum possible number of 1-entries a $n\times n$ 0-1 matrix saturating for P can have, respectively. The function $ssat(n;P)$ is the minimum possible number of 1-entries an $n\times n$ 0-1 matrix semisaturating for P can have. The function $ex(n;P)$ has been studied for decades, while investigation on $sat(n;P)$ and $ssat(n;P)$ was initiated recently. In this paper, a nontrivial generalization of results regarding these functions to multidimensional 0-1 matrices is made. In particular, the exact values of $ex(n;P,d)$ and $sat(n;P,d)$ are found when P is a d-dimensional identity matrix. Finally, a necessary and sufficient condition for a multidimensional 0-1 matrix to have a bounded semisaturation function is given.

Keywords: 0-1 matrix; forbidden pattern; excluded submatrix; multidimensional matrix; saturation.

2020 Mathematics Subject Classification: 05D99.

1. Introduction

Extremal combinatorics on pattern avoidance is a central topic in graph theory and combinatorics. In this broad research area, the key question usually being asked is: how dense an object could be such that it *avoids* a *forbidden* or *excluded* object. In this paper, the object of interest are *multidimensional 0-1 matrices*. A matrix is called a 0-1 matrix if all its entries are either 0 or 1. We say that a 0-1 matrix A *contains* another 0-1 matrix P if A has a submatrix that can be transformed to P by flipping any number of 1-entries to 0-entries. Otherwise, A *avoids* P. With this context, the density of concern is the number of 1-entries of a 0-1 matrix, which is sometimes called its *weight*. Following these, the key question can be formulated as seeking the asymptotic behavior of function $ex(n_1,n_2,\ldots,n_d;P)$, defined as the maximum weight of a d-dimensional $n_1\times n_2\times\ldots\times n_d$ 0-1 matrix that avoids another d-dimensional 0-1 matrix P. This problem can be seen as finding the maximum possible weight of a d-dimensional $n_1\times n_2\times\ldots\times n_d$ 0-1 matrix A that is *saturating* for 0-1 matrix P, i.e., A does not contain P and flipping any 0-entry of A to 1-entry introduces a copy of P in A. It is then natural to also ask for the *minimum* possible weight of a d-dimensional $n_1\times n_2\times\ldots n_d$ 0-1 matrix saturating for P, denoted $sat(n_1,n_2,\ldots,n_d;P)$. A variation of saturation is *semisaturation*: A is semisaturating for P if flipping any 0-entry of A to 1-entry creates a *new* copy of P in A. The corresponding extremal function is denoted $ssat(n_1,n_2,\ldots,n_d;P)$: the minimum possible weight of a d-dimensional $n_1\times n_2\times\ldots\times n_d$ 0-1 matrix semisaturating for P. When $n_1=n_2=\ldots=n_d$, we use the simplified notations $ex(n;P,d)$, $sat(n;P,d)$, and $ssat(n;P,d)$. By definition

$$ssat(n_1,n_2,\ldots,n_d;P)\le sat(n_1,n_2,\ldots,n_d;P)\le ex(n_1,n_2,\ldots,n_d;P).$$

In this paper we present two major results. First, for P as a d-dimensional identity matrix we give the exact value of functions $ex(n_1,n_2,\ldots,n_d;P)$ and $sat(n_1,n_2,\ldots,n_d;P)$, which are shown to be identical. This together with the implied structure of d-dimensional 0-1 matrices saturating for P generalize the two-dimensional result by Brualdi and Cao [3] and Tsai's discovery that every maximal antichain in a strict chain product poset is also maximum [17]. Second, as a partial extension to Fulek and Keszegh's constant versus linear dichotomy for semisaturation function of two-dimensional 0-1 matrices [5], we give the necessary and sufficient condition for a d-dimensional 0-1 matrix to have bounded semisaturation function.

In Section 2 we review previous works relevant to our study. Section 3 contains terminologies used throughout the paper. Our results and proofs are given in Section 4.

*E-mail address: parity@math.ncu.edu.tw

2. Related works

The extremal theory of 0-1 matrix started around 1990 in studies of computational and discrete geometry problems. Mitchell presented an algorithm for finding the shortest L_1 path between two points in a rectilinear grid with obstacles [14]. Its time complexity was bounded above via certain matrices' extremal function $ex(n;P)$ given by Bienstock and Győri [2]. In 1959, Erdős and Moser asked for the maximum number of unit distances among the vertices of a convex n-gon [4]. In 1990, Füredi gave the first upper bound $O(n\log_2 n)$ that is tighter than $n^{1+\epsilon}$ via the extremal theory [6]. Pach and Sharir used extremal functions $ex(n;P)$ to bound the number of pairs of non-intersecting and vertically visible line segments [15]. One of the latest applications is the resolution of Stanley-Wilf Conjecture in enumerative combinatorics [11, 13] in 2004 as Marcus and Tardos showed that every two-dimensional permutation matrix's extremal function $ex(n;P)$ is linear [13]. After applying to geometry-related problems mentioned previously, Füredi and Hajnal [7] and Tardos [16] asymptotically decide the extremal functions $ex(n;P)$ for every 0-1 matrix with no more than four 1-entries.

The extremal theory also extends from two-dimensional to multidimensional 0-1 matrices. Extending Marcus and Tardos' result above on two-dimensional permutation matrices [13], Klazar and Marcus proved that the extremal function $ex(n;P)$ of a d-dimensional $k\times\ldots\times k$ permutation matrix is $O(n^{d-1})$ [12]. Geneson and Tian gave nontrivial bounds on the extremal function of *block permutation matrices*, i.e., Kronecker products of permutation matrices and *block matrices* with no 0-entry [10], extending Geneson's result on two-dimensional tuple permutation matrices [9]. In another direction, they substantially improved the limit inferior and limit superior of the sequence $\frac{ex(n;P)}{n^{d-1}}$ for tuple permutation matrices and permutation matrices.

Recently, Brualdi and Cao initiated the study of the saturation problem for two-dimensional 0-1 matrices [3]. They proved that every maximal matrix avoiding the identity matrix I_k has the same weight. Fulek and Keszegh found a general upper bound on the saturation function in terms of the dimensions of P, and showed that the saturation function is either bounded or linear [5]. Geneson found that almost all permutation matrices have bounded saturation function [8], followed by Berendsohn's full characterization of permutation matrices with bounded saturation function [1]. In addition to the above, we are also motivated by Geneson's result for d-dimensional $r\times s\times 1\times\ldots\times 1$ 0-1 matrices, and his question about the saturation function of d-dimensional permutation matrices [8].

3. Notations

For a positive integer d, let $[d]$ denote $\{1,2,\ldots,d\}$. We denote a d-dimensional $n_1\times n_2\times\ldots\times n_d$ matrix by $A=(a_{x_1,\ldots,x_d})$, where $x_i\in[n_i]$ for each $i\in[d]$. A *k-dimensional cross section* L of a d-dimensional $n_1\times n_2\times\ldots\times n_d$ matrix A is the set of all entries of A whose coordinates of a fixed set C_L of $d-k$ dimensions are fixed. A cross section L of matrix A is a *face* if for every $i\in C_L$, the value of the i^{th} coordinate is fixed to some $p_i\in\{1,n_i\}$. An *i-row* of matrix A is a cross section L with $C_L=[d]-\{i\}$.

Let z and o be an 0-entry and an 1-entry of 0-1 matrices M and P, respectively. If flipping z to 1-entry introduces a new copy of P in M in which the new 1-entry matches o, then we say z *potentially matches* o.

Given a d-dimensional matrix A, entries $a_{x_1,\ldots,x_d}$ and $a_{y_1,\ldots,y_d}$ belong to the same *diagonal* if

$$x_1-y_1=x_2-y_2=\ldots=x_d-y_d.$$

Diagonal is same as *shape* in [17]. An $n_1\times n_2\times\ldots n_d$ matrix has $\prod_{i\in[d]}n_i-\prod_{i\in[d]}(n_i-1)$ diagonals. An entry $a_{x_1,\ldots,x_d}$ is *below* or *above* another entry $a_{y_1,\ldots,y_d}$ if $x_i>y_i$ for every $i\in[d]$ or $x_i<y_i$ for every $i\in[d]$, respectively. An entry $a_{x_1,\ldots,x_d}$ is *semibelow* or *semiabove* another entry $a_{y_1,\ldots,y_d}$ if $x_i\geq y_i$ for every $i\in[d]$ or $x_i\leq y_i$ for every $i\in[d]$, respectively. Since two-dimensional matrices are imagined as smaller indices being left and top if regarded in geometric space, our definitions of (semi)above or (semi)below are intuitive. Two entries are *comparable* if one of them is above the other. A d-dimensional *staircase* generalized from [5] and *zigzag path* in [3] is a set of pairwise incomparable entries in A. When the set is maximal, it is denoted *complete* staircase. Note that staircase in [5] means complete staircase here. From [17], every complete staircase has size $\prod_{i\in[d]}n_i-\prod_{i\in[d]}(n_i-1)$ and intersects with every diagonal at exactly one entry, using the simple to prove fact that a complete staircase contains exactly one 1-entry in every diagonal. An entry e_1 is *above* or *below* a complete staircase S if S has an entry e_2 such that e_1 is above or below e_2, respectively. Every entry in A is either above, below, or part of any complete staircase S of A. A *shell* is the unique complete staircase containing the corner entry $a_{n_1,\ldots,n_d}$. Similar to 0-1 matrices, the weight of a staircase is the number of 1-entries it has.

4. New results

The following extension from Lemma 3.3 of [5] is handy in dealing with d-dimensional identity matrices.

Lemma 4.1. *Let P be a d-dimensional $l_1 \times \ldots \times l_d$ 0-1 matrix with $l_i > 1$ for each $i \in [d]$. The only 1-entry in P's shell is the corner $p_{l_1,\ldots,l_d}$. Then in any d-dimensional 0-1 matrix M saturating for P there is a d-dimensional complete staircase S that is all 1, and all entries below S are 0-entries that potentially match $p_{l_1,\ldots,l_d}$.*

Proof. We claim that the set of bottommost 1-entries from all diagonals of M forms a d-dimensional complete staircase. First we show that if $z_1, z_2, \ldots, z_k$ are consecutive 0-entries of a diagonal D of M where z_1 is the bottom entry of D, then they all potentially match $p_{l_1,\ldots,l_d}$. This will imply that every diagonal of M contains at least one 1-entry, because the top entry of any diagonal does not potentially match $p_{l_1,\ldots,l_d}$. Entry z_1 is not above any other entry of M, so it must potentially match $p_{l_1,\ldots,l_d}$. Suppose there is a positive integer $i \in [k-1]$ such that z_i potentially matches $p_{l_1,\ldots,l_d}$ and z_{i+1} does not. Let P' be obtained from P by removing its shell. In M there must be a copy A of P' that is above z_i. There must also be an 1-entry a of M below z_{i+1}. The copy A and the 1-entry a form a copy of P, contradicting the assumption that M avoids P.

We are left to prove that no bottommost 1-entry o_1 of a diagonal D_1 is above the bottommost 1-entry o_2 of any other diagonal D_2. If o_1 is above o_2, then let z_1 be the 0-entry immediately below o_1 within D_1. Entry z_1 potentially matches $p_{l_1,\ldots,l_d}$ and is semiabove o_2, thus there is a copy of P in M where o_2 matches $p_{l_1,\ldots,l_d}$. □

Let A and B be d-dimensional $l_1 \times l_2 \times \ldots \times l_d$ and $k_1 \times k_2 \times \ldots \times k_d$ 0-1 matrices, respectively. The *diagonal concatenation* M of A and B is a d-dimensional $(l_1+k_1) \times (l_2+k_2) \times \ldots \times (l_d+k_d)$ 0-1 matrix, such that $m_{x_1,x_2,\ldots,x_d} = a_{x_1,x_2,\ldots,x_d}$ if $x_i \le l_i$ for every $i \in [d]$, $m_{x_1,x_2,\ldots,x_d} = b_{x_1-l_1,x_2-l_2,\ldots,x_d-l_d}$ if $l_i + 1 \le x_i$ for every $i \in [d]$, and $m_{x_1,\ldots,x_d} = 0$ otherwise.

The Lemma below generalizes Theorem 1.9 of [5].

Lemma 4.2. *Let A be a d-dimensional $l_1 \times l_2 \times \ldots \times l_d$ 0-1 matrix, I be a d-dimensional $1 \times \ldots \times 1$ identity 0-1 matrix, P' be the diagonal concatenation of A and I, and P be the diagonal concatenation of P' and I. Then*

$$sat(n_1, n_2, \ldots, n_d; P) = sat(n_1-1, n_2-1, \ldots, n_d-1; P') + \prod_{i\in[d]} n_i - \prod_{i\in[d]} (n_i - 1)$$
$$ex(n_1, n_2, \ldots, n_d; P) = ex(n_1-1, n_2-1, \ldots, n_d-1; P') + \prod_{i\in[d]} n_i - \prod_{i\in[d]} (n_i - 1)$$

where $l_i + 2 \le n_i$ for each $i \in [d]$.

Proof. Given a d-dimensional $n_1 \times n_2 \times \ldots n_d$ 0-1 matrix M, let S denote the shell of M and let M' denote the $(n_1-1) \times (n_2-1) \times \ldots (n_d-1)$ submatrix of M that does not contain any entry of S. We show that if S is all 1 and M' is saturating for P', then M is saturating for P. This implies that

$$sat(n_1, n_2, \ldots, n_d; P) \le sat(n_1-1, n_2-1, \ldots, n_d-1; P') + \prod_{i\in[d]} n_i - \prod_{i\in[d]} (n_i - 1)$$

and

$$ex(n_1, n_2, \ldots, n_d; P) \ge ex(n_1-1, n_2-1, \ldots, n_d-1; P') + \prod_{i\in[d]} n_i - \prod_{i\in[d]} (n_i - 1).$$

Indeed, M clearly avoids P. All 0-entries of M lie in M', so flipping any of them creates a copy P'' of P' in M'. The copy P'' and S filled with 1-entries form a copy of P, so M is saturating for P.

We are left to prove that

$$sat(n_1, n_2, \ldots, n_d; P) \ge sat(n_1-1, n_2-1, \ldots, n_d-1; P') + \prod_{i\in[d]} n_i - \prod_{i\in[d]} (n_i - 1)$$

and

$$ex(n_1, n_2, \ldots, n_d; P) \le ex(n_1-1, n_2-1, \ldots, n_d-1; P') + \prod_{i\in[d]} n_i - \prod_{i\in[d]} (n_i - 1).$$

Given a matrix N saturating for P, apply Lemma 4.1 to get a d-dimensional complete staircase T of N that is all 1 and all entries below T are 0-entries that potentially match $p_{l_1,\ldots,l_d}$. The complete staircase T partitions entries of N into N_1, N_2, and T such that every entry of N_1 and N_2 is above and below T, respectively. Obtain a d-dimensional matrix N' from N by deleting T followed by shifting every entry of N_2 upwards along the diagonal it belongs to. It suffices to show that N' is saturating for P'. Matrix N' clearly avoids P'. It could be seen that every 0-entry of N_2 potentially matches the corner $p'_{l_1-1,\ldots,l_d-1}$ of P'. Moreover every 0-entry of N_1 potentially matches some 1-entry of P', because in N together with T flipping any of them to 1-entry introduces a copy of P. □

By successive application of Lemma 4.1 and Lemma 4.2 to the d-dimensional $(k+1)\times(k+1)\times\ldots\times(k+1)$ identity matrix, we obtain the following generalization of Brualdi and Cao's result on 0-1 matrices saturating for identity matrices [3]. It also extends Tsai's discovery that every maximal antichain in a strict chain product poset is also maximum [17], and implies that the greedy algorithm to generate 0-1 matrices saturating for identity matrix [3] also works for multidimensional matrices.

Theorem 4.1. *Let P be a d-dimensional $(k+1)\times(k+1)\times\ldots\times(k+1)$ identity matrix. Suppose $k\le n_i$ for each $i\in[d]$ and $k\le n$. Then*

$$sat(n_1,n_2,\ldots,n_d;P)=ex(n_1,n_2,\ldots,n_d;P)=\prod_{i\in[d]}n_i-\prod_{i\in[d]}(n_i-k)$$
$$sat(n;P,d)=ex(n;P,d)=n^d-(n-k)^d=\Theta(n^{d-1}).$$

Moreover, the 1-entries of every d-dimensional $n_1\times n_2\times\ldots\times n_d$ 0-1 matrix saturating for P can be decomposed into k d-dimensional staircases with weights $\prod_{i\in[d]}n_i-\prod_{i\in[d]}(n_i-1)$, $\prod_{i\in[d]}(n_i-1)-\prod_{i\in[d]}(n_i-2)$, ..., $\prod_{i\in[d]}(n_i-k+1)-\prod_{i\in[d]}(n_i-k)$, respectively.

All remaining results are extended from Fulek and Keszegh's work on two-dimensional 0-1 matrices [5]. For example, the following extends the upper bound of $O(n)$ on saturation function of two-dimensional 0-1 matrices to the upper bound of $O(n^{d-1})$ on saturation function of d-dimensional 0-1 matrices.

Lemma 4.3. *For any d-dimensional 0-1 matrix P, $sat(n;P,d)=O(n^{d-1})$.*

Proof. If P does not contain any 1-entry, then $sat(n;P,d)=0$. Otherwise let the dimensions of P be $l_1\times l_2\times\ldots\times l_d$, and let $p_{x_1,x_2,\ldots,x_d}$ be an 1-entry. Construct a d-dimensional $n\times n\times\ldots\times n$ 0-1 matrix M saturating for P as follows. For any entry $m_{x'_1,x'_2,\ldots,x'_d}$, set it to 0 if and only if $x_i\le x'_i\le n-(l_i-x_i)$ for each $i\in[d]$. Clearly M avoids P and flipping any 0-entry of M in 1-entry makes it contain P. The weight of M is $n^d-(n-l_1+1)(n-l_2+1)\ldots(n-l_d+1)=O(n^{d-1})$. □

The result below is naturally extended from two-dimensional 0-1 matrices.

Lemma 4.4. *Let P be the diagonal concatenation of non-zero d-dimensional 0-1 matrices A and B. Then $sat(n;P,d)=\Theta(n^{d-1})$.*

Proof. We show by contradiction that every i-row of a d-dimensional 0-1 matrix M saturating for P contains at least one 1-entry where $i\in[d]$. Suppose M has an i-row r which is all 0. Flipping the first or last entry of r to 1 creates a copy of P where the new 1-entry matches an 1-entry of A or B, respectively. Thus there exist two adjacent entries z_1 and z_2 of r such that flipping z_1 to 1-entry creates a copy P' of P where z_1 matches an 1-entry of A, and flipping z_2 to 1-entry creates a copy P'' of P where z_2 matches an 1-entry of B. The copy of B in P' and the copy of A in P'' form a copy of P, i.e., M contains P. □

Fulek and Keszegh showed in [5] that for a two-dimensional 0-1 matrix to have bounded semisaturation function, it has to have an 1-entry that is the only 1-entry in its row and column. Below we present a necessary condition for a d-dimensional 0-1 matrix to have $o(n^{d-d'})$ semisaturation function for any $d'\in[d-1]$.

Lemma 4.5. *Let $d'<d$. If a non-zero d-dimensional 0-1 matrix P does not have any 1-entry which is the only 1-entry in every d'-dimensional cross section of P it belongs to, then $ssat(n;P,d)=\Omega(n^{d-d'})$.*

Proof. Suppose M is semisaturating for P. Say a 0-entry and an 1-entry of M are *connected* if they are in the same d'-dimensional cross section of M. Each 0-entry is connected with at least one 1-entry, and each 1-entry is connected with at most $\binom{d}{d'}(n^{d'}-1)$ 0-entries. So the weight of M is at least

$$\frac{n^d}{1+\binom{d}{d'}(n^{d'}-1)}=\Theta(n^{d-d'}).$$

□

With the above result, we characterize d-dimensional 0-1 matrices with bounded semisaturation function.

Theorem 4.2. *Given a d-dimensional pattern P, $ssat(n;P,d)=O(1)$ if and only all the following properties hold for P: (i) For any $d'\in[d-1]$, every d'-dimensional face f of P contains an 1-entry o that is the only 1-entry in every $(d-1)$-dimensional cross section that is orthogonal to f and intersects f at o. (ii) P contains an 1-entry that is the only 1-entry in every $(d-1)$-dimensional cross section it belongs to.*

Proof. Let M be a d-dimensional 0-1 matrix saturating for P. Suppose P does not have property (i), i.e., for some $d' \in [d-1]$ P has a d'-dimensional face f that does not contain any 1-entry o that is the only 1-entry in every $(d-1)$-dimensional cross section that is orthogonal to f and intersects f at o. Let the counterpart of f in M be f'. If f is all 0, then f' must be all 1. Otherwise, we say a 0-entry in f' is connected to an 1-entry in $M-f'$ if they are in the same $(d-1)$-dimensional cross section. Each 0-entry of f' is connected to at least one 1-entry. Each 1-entry of $M-f'$ is in d' $(d-1)$-dimensional cross sections that is orthogonal to f', and each of them contains at most $n^{d'-1}$ 0-entries of f'. Thus each 1-entry of $M-f'$ is connected to at most $d'(n^{d'-1})$ 0-entries in f'. Suppose f' has α 0-entries, then M has at least $(n^{d'}-\alpha)+\frac{\alpha}{d'(n^{d'-1})} \geq \frac{n^{d'}}{d'(n^{d'-1})} = \Theta(n)$ 1-entries. Otherwise suppose P does not have property (ii), i.e., it does not contain any 1-entry that is the only 1-entry in every $(d-1)$-dimensional cross section it belongs to. By Lemma 4.5 $ssat(n;P,d)=\Omega(n)$.

We are left to prove that $ssat(n;P,d)=O(1)$ if P has all the properties. Suppose P's dimensions are $l_1 \times l_2 \times \ldots \times l_d$. We construct an $n \times n \times \ldots \times n$ 0-1 matrix M' semisaturating for P with $O(1)$ 1-entries as follows. Entry $m'_{l'_1,l'_2,\ldots,l'_d}$ is 1 if and only if for each $i \in [d]$, $l'_i < l_i$ or $n+1-l_i < l'_i$. Let an 1-entry of P with property (ii) be $p_{o_1,o_2,\ldots,o_d}$. For a given 0-entry $m'_{z_1,z_2,\ldots,z_d}$, suppose flipping it to 1-entry produces another matrix M''. If $z_i \in [l_i, n+1-l_i]$ for each $i \in [d]$ then for each dimension i we slice M'' by indices $\{1,2,\ldots,o_i-1,z_i,n-l_i+o_i+1,n-l_i+o_i+2,\ldots,n\}$, and the resulting $l_1 \times \ldots \times l_d$ submatrix contains P.

If $z_i \notin [l_i, n+1-l_i]$ for some $i \in [d]$, there exists a partitioning $[d]=X \cup Y \cup Z$ such that $X \cup Y \neq \emptyset$, $Z \neq \emptyset$, and $\forall i \in [d]$

$$\begin{cases} z_i < l_i, & \text{if } i \in X \\ n+1-l_i < z_i, & \text{if } i \in Y \\ z_i \in [l_i, n+1-l_i], & \text{otherwise} \end{cases}$$

Let f be the face of P whose i^{th} coordinate is fixed to 1 or l_i if $i \in X$ or $i \in Y$, respectively, and for each $i \in Z$ let the i^{th} coordinate of the 1-entry contained in f with property (i) be l''_i. We then slice M'' by indices I_i for each $i \in [d]$ to obtain a $l_1 \times \ldots l_d$ submatrix containing P: for each $i \in X$, $I_i = \{z_i, n-l_i+2, n-l_i+3, \ldots, n\}$; for each $i \in Y$, $I_i = \{1,2,\ldots,l_i-1,z_i\}$; otherwise $I_i = \{1,2,\ldots,l''_i-1,z_i,n-l_i+l''_i+1,\ldots,n\}$. □

It would be interesting to investigate what the dichotomy of constant versus linear translates to when moving from (semi)saturation function of two-dimensional 0-1 matrices to (semi)saturation function of higher dimensional 0-1 matrices.

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