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Research Article

On the local antimagic chromatic number of the lexicographic product of graphs

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Abstract

Let $G = (V, E)$ be a connected simple graph. A bijection $f : E \to \{1, 2, \dots, |E|\}$ is said to be a local antimagic labeling of G if $f^+(u) \neq f^+(v)$ holds for any two adjacent vertices u and v of G, where $E(u)$ is the set of edges incident to u and $f^+(u) = \sum_{e \in E(u)} f(e)$. A graph G is called local antimagic if G admits at least one local antimagic labeling. The local antimagic chromatic number, denoted $\chi_{la}(G)$, is the minimum number of induced colors taken over local antimagic labelings of G. Let G and H be two disjoint graphs. The graph $G[H]$ is obtained by the lexicographic product of G and H. In this paper, we obtain sufficient conditions for $\chi_{la}(G[H]) \le \chi_{la}(G)\chi_{la}(H)$. Consequently, we give examples of G and H such that $\chi_{la}(G[H]) = \chi(G)\chi(H)$, where $\chi(G)$ is the chromatic number of G. We conjecture that (i) there are infinitely many graphs G and H such that $\chi_{la}(G[H]) = \chi_{la}(G)\chi_{la}(H) = \chi(G)\chi(H)$, and (ii) for $k \ge 1$, $\chi_{la}(G[H]) = \chi(G)\chi(H)$ if and only if $\chi(G)\chi(H) = 2\chi(H) + \lceil \frac{\chi(H)}{k} \rceil$, where $2k + 1$ is the length of a shortest odd cycle in G.

Keywords: lexicographic product; regular; local antimagic chromatic number.

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1. Introduction

Let $G = (V, E)$ be a connected simple graph of order p and size q. A bijection $f : E \to \{1, 2, \dots, q\}$ is called a *local antimagic labeling* of G if $f^+(u) \neq f^+(v)$ holds for any two adjacent vertices u and v, where $f^+(u) = \sum_{e \in E(u)} f(e)$, and $E(u)$ is the set of edges incident to u. Clearly, a local antimagic labeling induces a proper coloring of G. The function f is called a *local antimagic t-coloring* of G if f induces a proper t-coloring of G, and we say $c(f) = t$. The *local antimagic chromatic number* of G, denoted by $\chi_{la}(G)$, is the minimum number of $c(f)$, where f takes over all local antimagic labelings of G [1]. Interested readers may refer to [6, 7, 11] for results related to local antimagic chromatic numbers of graphs.

Let G and H be two disjoint graphs. The *lexicographic product* $G[H]$ of graphs G and H is a graph such that its vertex set is the cartesian product $V(G) \times V(H)$, and any two vertices (u, u') and (v, v') are adjacent in $G[H]$ if and only if either $uv \in E(G)$ or $u = v$ and $u'v' \in E(H)$. In [10], the first two authors studied the exact value of $\chi_{la}(G[O_n])$, where O_n is a null graph of order $n \ge 2$. Motivated by the above result, we investigate the sharp upper bound of $\chi_{la}(G[H])$ for any two disjoint non-null graphs G and H in this paper. We present the sufficient conditions for

$$\chi_{la}(G[H]) \le \chi_{la}(G)\chi_{la}(H).$$

Further, we conjecture that (i) there are infinitely many graphs G and H with $\chi_{la}(G[H]) = \chi_{la}(G)\chi_{la}(H) = \chi(G)\chi(H)$, where $\chi(G)$ is the chromatic number of G; and (ii) for any positive integer k, $\chi_{la}(G[H]) = \chi(G)\chi(H)$ if and only if $\chi(G)\chi(H) = 2\chi(H) + \lceil \frac{\chi(H)}{k} \rceil$, where $2k + 1$ is the length of the shortest odd cycle in G. We refer to [3] for all undefined notation.

2. Bounds of $\chi_{la}(G[H])$

Before presenting our main results, we introduce some necessary notation and known results which will be used in this section.

Let $[a, b] = \{n \in \mathbb{Z} \mid a \le n \le b\}$ and $S \subseteq \mathbb{Z}$. Let S^- and S^+ be a decreasing sequence and an increasing sequence of S, respectively.

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Lemma 2.1 (see [8, Lemma 2.2]). *For positive integers q and p, let $S_p(a) = [p(a-1)+1, pa]$, $1 \le a \le q$. Then,*

(i) $\{S_p(a) \mid 1 \le a \le q\}$ is a partition of $[1, pq]$;

(ii) when $a < b$, every term of $S_p(a)$ is less than that of $S_p(b)$;

(iii) for each $1 \le i \le p$, the sum of the i-th term of $S_p^+(a)$ and that of $S_p^-(b)$ is independent of i, where $1 \le a, b \le q$;

(iv) for any positive integer k and each $1 \le i \le p$, $\sum_{l=1}^{k} \big(i\text{-th term of } S_p^+(a_l)\big) + \sum_{l=1}^{k} \big(i\text{-th term of } S_p^-(b_l)\big)$ is independent of i, where $1 \le a_l, b_l \le q$.

Note that the proof of Lemma 2.1 in [8] shows that the sum of i-term of $S_p^+(a)$ and that of $S_p^-(b)$ is $p(a+b-1)+1$. According to the definitions of $S_p^+(a)$ and $S_p^-(a)$, we shall write the sequence $S_p^+(a)$ and $S_p^-(a)$ as column vectors in this paper. Now we are ready to present our first main result.

Theorem 2.1. *Suppose H admits a local antimagic t-coloring f that satisfies the following conditions:*

(a) for each vertex, the number of even incident edge labels equals the number of odd incident edge labels under f;

(b) when $f^+(u) = f^+(v)$, $\deg(u) = \deg(v)$;

(c) when $f^+(u) \neq f^+(v)$, $pf^+(u) - \frac{1}{2}\deg(u)(p-1) \neq pf^+(v) - \frac{1}{2}\deg(v)(p-1)$ holds for a fixed integer p.

Then $\chi_{la}(pH) \le t$.

Proof. Let $V(H) = \{x_1, \dots, x_n\}$ and L be the labeling matrix of H according to f (for definition of labeling matrix, please see [5, 12]). Now we define a guide matrix [†] $\mathcal{M}$ by adding a '+' sign to all odd entries and a '−' sign to all even entries in L.

We define p matrices $L_1, \dots, L_p$ as follows. For each $1 \le i \le p$, the (j,k)-entry of L_i is the i-th term of $S_p^+(a)$ (resp. $S_p^-(a)$) if the corresponding (j,k)-entry of $\mathcal{M}$ is $+a$ (resp. $-a$), where $1 \le a \le |E(H)|$.

From the condition (a), for each row of L, the number of odd entries equals that of even entries. Thus, let $a_1, \dots, a_s$ denote the odd numerical entries of the j-th row of L and $b_1, \dots, b_s$ denote the even numerical entries of the j-th row of L, where s is a positive integer. Now,

$$r_j(L_i) = \sum_{l=1}^{s} [i\text{-th term of } S_p^+(a_l)] + \sum_{l=1}^{s} [i\text{-th term of } S_p^-(b_l)].$$

By Lemma 2.1 (iv), $r_j(L_i)$ is constant for a fixed j. Actually, it is

$$p\sum_{l=1}^{s}(a_l+b_l) - ps + s = pr_j(L) - k(p-1) = pf^+(x_j) - \frac{1}{2}\deg(x_j)(p-1).$$

By conditions (a) and (b), the diagonal block matrix

$$\begin{pmatrix} L_1 & \bigstar & \cdots & \bigstar \\ \bigstar & L_2 & \cdots & \bigstar \\ \vdots & \vdots & \ddots & \vdots \\ \bigstar & \bigstar & \cdots & L_p \end{pmatrix}$$

is a local antimagic labeling of pH. Thus

$$\chi_{la}(pH) \le t.$$

□

It is known that $\chi_{la}(K_{1,2n}) = 2n+1$ and $\chi_{la}(mK_{1,2n}) = 2nm+1$ [2, Corollary 3]. Clearly, the upper bound stated in Theorem 2.1 is not sharp. From Theorem 2.1, we obtain the following result immediately.

Corollary 2.1. *If H is an r-regular graph ($r \ge 2$) with a local antimagic t-coloring f satisfying the condition (a) of Theorem 2.1, then $\chi_{la}(pH) \le t$ holds for any positive integer p.*

[†] A *guide matrix* $\mathcal{M}$ is an $(n-r) \times n$ matirx in which (j,k)-th entry is $(\mathcal{S})_{j,k}(\mathcal{M}')_{j,k}$, where $1 \le j \le n-r$, $1 \le k \le n$, $\mathcal{S}$ is an $(n-r) \times n$ matrix obtained from $\mathcal{S}_n$ ($\mathcal{S}_n$ is a 'sign matrix', refer to [8]) by removing the last r rows, and $\mathcal{M}'$ is also an $(n-r) \times n$ matrix in which $(\mathcal{M}')_{j,k} = (\mathcal{M}')_{k,j}$ for $1 \le j < k \le n-r$, the upper part of the off-diagonal entries is the increasing sequence $[1, (n-r)(n+r-1)/2]$ and the entries of the main diagonal is $[(n-r)(n+r-1)/2+1, (n-r)(n+r-1)/2+(n-r)]$ in natural order.

Theorem 2.2. *Let G be a graph of order p admitting a local antimagic $\chi_{la}(G)$-coloring g and H be a graph of order n admitting a local antimagic $\chi_{la}(H)$-coloring h. Suppose h satisfies the following conditions:*

(i) For each vertex, the number of even incident edge labels equals the number of odd incident edge labels under h;

(ii) when $h^+(u) = h^+(v)$, $\deg_H(u) = \deg_H(v)$;

(iii) when $h^+(u) \neq h^+(v)$, $ph^+(u) - \frac{1}{2}\deg_H(u)(p-1) \neq ph^+(v) - \frac{1}{2}\deg_H(v)(p-1)$.

Moreover, g satisfies the following conditions:

(iv) when $g^+(u) = g^+(v)$, $\deg_G(u) = \deg_G(v)$, and

(v) when $g^+(u) \neq g^+(v)$, $g^+(u)n^3 - \frac{(n^3-n)\deg_G(u)}{2} \neq g^+(v)n^3 - \frac{(n^3-n)\deg_G(v)}{2}$.

Then $\chi_{la}(G[H]) \leq \chi_{la}(G)\chi_{la}(H)$.

Proof. Let $q(G)$ and $q(H)$ denote the sizes of G and H respectively. Clearly, $G[H]$ is a graph of order pn and size $pq(H) + q(G)n^2$. Suppose that $\{u_1, \ldots, u_p\}$ and $\{x_1, \ldots, x_n\}$ are the vertex lists of G and H respectively. According to these vertex lists, we define that A_G and A_H are the adjacency matrices of G and H respectively. Thus the adjacency matrix of $G[H]$ can be expressed as

$$A_G \otimes J_n + I_p \otimes A_H,$$

where J_n is an $n \times n$ matrix whose entries are all 1, I_p is an identity matrix of order p, and $A_G \otimes J_n$ is the Kronecker product of A_G and J_n. Note that $A_G \otimes J_n$ is the adjacency matrix of $G[O_n]$ and $I_p \otimes A_H$ is the adjacency matrix of $O_p[H]$, where O_n and O_p are null graphs of orders n and p. Therefore, the diagonal blocks of $A_G \otimes J_n$ are zero matrices and only the diagonal blocks of $I_p \otimes A_H$ are non-zero matrices.

Now we shall label the edges of $O_p[H]$ and $G[O_n]$ separately. According to the definition, $O_p[H] \cong pH$. By Theorem 2.1, pH has a local antimagic $\chi_{la}(H)$-coloring, say ϕ, by using integers in $[1, pq(H)]$ such that for each vertex (u_i, x_j) in $O_p[H]$, $\phi^+(u_i, x_j)$ is independent of i, where $1 \leq i \leq p$. The labeling matrix of ϕ is denoted by $\mathscr{M}_1$.

Next we shall label $G[O_n]$ by integers in $[1, q(G)n^2]$. This labeling was constructed in the proof of [10, Theorem 2.1]. For completeness, we list the outline of the construction.

Let M_g be the labeling matrix of G corresponding to g. Suppose Ω is a magic square of order n. Let $\Omega_i = \Omega + (i-1)n^2 J_n$, where $1 \leq i \leq q(G)$ and ψ_0 be the labeling of $G[O_n]$ such that its labeling matrix $\mathscr{M}$ is defined by replacing each entry of M_G with an $n \times n$ matrix as follows:

(1) replace $*$ by $\bigstar$ which is an $n \times n$ matrix whose entries are $*$;

(2) replace i by Ω_i, if i lies in the upper triangular part of M_g;

(3) replace i by Ω_i^T, if i lies in the lower triangular part of M_g, where Ω_i^T is the transpose of Ω_i.

For each vertex $(u_l, x_j) \in V(G[O_n])$, the row sum of $\mathscr{M}_1$ corresponding to the vertex (u_l, x_j) is

$$\psi_0^+(u_l, x_j) = g^+(u_l)n^3 - \frac{(n^3-n)\deg_G(u_l)}{2},$$

which is independent of j. By condition (i), ψ_0 is a local antimagic labeling of $G[O_n]$. According to condition (v), there are at most $\chi(G)$ distinct row sums of $\mathscr{M}$. Let $\mathscr{M}_2$ be the matrix obtained from $\mathscr{M}$ by adding all numerical entries with $pq(H)$ and ψ be the corresponding labeling. Then, $\psi^+(u_l, x_j) = \psi_0^+(u_l, x_j) + npq(H)$, which is independent of j.

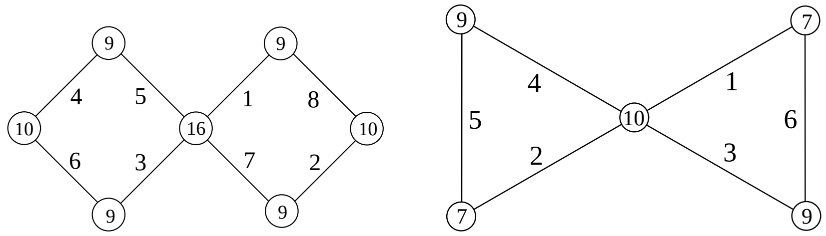
Therefore, $\mathscr{M}_1 + \mathscr{M}_2$ is a labeling matrix that corresponds to a local antimagic labeling of $G[H]$, where $*$ is treated as 0. Hence $\chi_{la}(G[H]) \leq \chi_{la}(G)\chi_{la}(H)$. □

The following is an example of Theorem 2.2.

Example 2.1. *Let G be the one point union of two 4-cycles and H be the one point union of two 3-cycles. Figure 1 shows the local antimagic 3-colorings of G and H.*

Note that $\chi_{la}(G) = \chi_{la}(H) = 3$. It is easy to check that the above local antimagic 3-colorings of G and H satisfy the conditions of Theorem 2.2 respectively. Thus, the labeling matrices of G and H are shown below:

$$M_g = \begin{pmatrix} * & * & * & * & 8 & * & 1 \\ * & * & * & * & 2 & * & 7 \\ * & * & * & * & * & 6 & 3 \\ * & * & * & * & * & 4 & 5 \\ 8 & 2 & * & * & * & * & * \\ * & * & 6 & 4 & * & * & * \\ 1 & 7 & 3 & 5 & * & * & * \end{pmatrix}, \quad M_h = \begin{pmatrix} * & * & 6 & * & 1 \\ * & * & * & 5 & 2 \\ 6 & * & * & * & 3 \\ * & 5 & * & * & 4 \\ 1 & 2 & 3 & 4 & * \end{pmatrix}.$$



(a) Graph G (b) Graph H

Figure 1: Local antimagic 3-colorings of graphs G and H.

Let

$$L_1=\begin{pmatrix} * & * & 42 & * & 1\\ * & * & * & 29 & 14\\ 42 & * & * & * & 15\\ * & 29 & * & * & 28\\ 1 & 14 & 15 & 28 & * \end{pmatrix},\quad L_2=\begin{pmatrix} * & * & 41 & * & 2\\ * & * & * & 30 & 13\\ 41 & * & * & * & 16\\ * & 30 & * & * & 27\\ 2 & 13 & 16 & 27 & * \end{pmatrix},\quad L_3=\begin{pmatrix} * & * & 40 & * & 3\\ * & * & * & 31 & 12\\ 40 & * & * & * & 17\\ * & 31 & * & * & 26\\ 3 & 12 & 17 & 26 & * \end{pmatrix},$$

$$L_4=\begin{pmatrix} * & * & 39 & * & 4\\ * & * & * & 32 & 11\\ 39 & * & * & * & 18\\ * & 32 & * & * & 25\\ 4 & 11 & 18 & 25 & * \end{pmatrix},\quad L_5=\begin{pmatrix} * & * & 38 & * & 5\\ * & * & * & 33 & 10\\ 38 & * & * & * & 19\\ * & 33 & * & * & 24\\ 5 & 10 & 19 & 24 & * \end{pmatrix},\quad L_6=\begin{pmatrix} * & * & 37 & * & 6\\ * & * & * & 34 & 9\\ 37 & * & * & * & 20\\ * & 34 & * & * & 23\\ 6 & 9 & 20 & 23 & * \end{pmatrix},$$

$$L_7=\begin{pmatrix} * & * & 36 & * & 7\\ * & * & * & 35 & 8\\ 36 & * & * & * & 21\\ * & 35 & * & * & 22\\ 7 & 8 & 21 & 22 & * \end{pmatrix}.$$

Obviously, for each $1 \le i \le 7$, the row sums of L_i are 43, 43, 57, 57, 58 respectively. Let Ω be a magic square of order 5 with row sum 65 and $\Omega_i = \Omega + 25(i-1)J_5$, where $1 \le i \le 8$. For each $1 \le i \le 8$, let $\Psi_i = \Omega_i + 42J_5$. Then, the labeling matrix of $G[H]$ is

$$\begin{pmatrix} L_1 & \bigstar & \bigstar & \bigstar & \Psi_8 & \bigstar & \Psi_1\\ \bigstar & L_2 & \bigstar & \bigstar & \Psi_2 & \bigstar & \Psi_7\\ \bigstar & \bigstar & L_3 & \bigstar & \bigstar & \Psi_6 & \Psi_3\\ \bigstar & \bigstar & \bigstar & L_4 & \bigstar & \Psi_4 & \Psi_5\\ \Psi_8^T & \Psi_2^T & \bigstar & \bigstar & L_5 & \bigstar & \bigstar\\ \bigstar & \bigstar & \bigstar & \Psi_6^T & \Psi_4^T & L_6 & \bigstar\\ \Psi_1^T & \Psi_7^T & \Psi_3^T & \Psi_5^T & \bigstar & \bigstar & L_7 \end{pmatrix}$$

By calculating the row sums of the above matrix, we obtain that the distinct row sums are 1468, 1482, 1483, 1593, 1607, 1608, 2643, 2657, 2658. Thus, $\chi_{la}(G[H]) \le 9$.

In [4], N. Čižek and S. Klavžar gave the lower bound of chromatic number of the lexicographic product as follows.

Corollary 2.2 (see [4, Corollary 3])**.** *Let G be a non-bipartite graph. Then for any graph H, $\chi(G[H]) \ge 2\chi(H) + \lceil \frac{\chi(H)}{k} \rceil$, where $k \ge 1$ and $2k+1$ is the length of a shortest odd cycle in G.*

Combining Theorem 2.2 and Corollary 2.2, we obtain the following results.

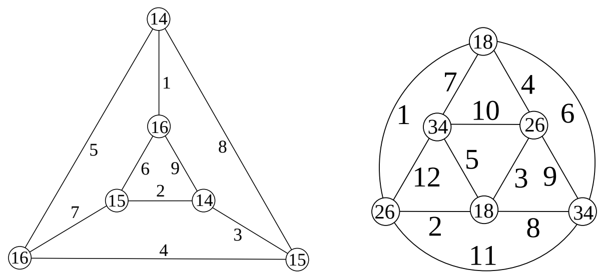
Corollary 2.3. *Suppose G and H are graphs satisfying the conditions listed in Theorem 2.2. If the length of a shortest odd cycle in G is $2k+1$, then $2\chi(H) + \lceil \frac{\chi(H)}{k} \rceil \le \chi(G[H]) \le \chi_{la}(G[H]) \le \chi_{la}(G)\chi_{la}(H)$. In particular, if C_3 is a subgraph of G, then $3\chi(H) \le \chi_{la}(G[H]) \le \chi_{la}(G)\chi_{la}(H)$.*

Proof. $\chi(G[H]) \le \chi_{la}(G[H])$ is trivial. The lower bound follows from Corollary 2.2 and the upper bound follows from Theorem 2.2. □

Corollary 2.4. *Let G and H be regular graphs and H admit a local antimagic $\chi_{la}(H)$-coloring h. Suppose for each vertex of H, the number of even incident edge labels equals the number of odd incident edge labels under h. If the length of a shortest odd cycle in G is $2k+1$, then $2\chi(H) + \lceil \frac{\chi(H)}{k} \rceil \le \chi(G[H]) \le \chi_{la}(G[H]) \le \chi_{la}(G)\chi_{la}(H)$. In particular, if C_3 is a subgraph of G, then $3\chi(H) \le \chi_{la}(G[H]) \le \chi_{la}(G)\chi_{la}(H)$.*

By applying Corollary 2.4, we can obtain $\chi_{la}(G[H])$ for some graphs G and H. An example is shown in Example 2.2.

Example 2.2. *Let $G = C_3 \times K_2$ and H be the octahedral graph. Figure 2 presents their local antimagic 3-colorings which are shown in [9].*



(a) Graph $G = C_3 \times K_2$ (b) Graph H

Figure 2: Local antimagic 3-colorings of graphs G and H.

It is easy to verify that G and H satisfy the conditions of Corollary 2.4, which implies that

$$3\chi(H) \le \chi_{la}(G[H]) \le \chi_{la}(G)\chi_{la}(H).$$

Since $\chi_{la}(G) = \chi_{la}(H) = 3$, $\chi_{la}(G[H]) = 9$.

In [7, Theorem 3.3], the first two authors proved that $\chi_{la}(C_{2m} \vee O_{2n}) = 3$ for $m \ge 2$, $n \ge 1$, where $C_{2m} \vee O_{2n}$ is the join of graphs C_{2m} and O_{2n}. In the following, we give another local antimagic 3-coloring of $C_{2m} \vee O_{2n}$ that satisfies the conditions (i) and (ii) of Theorem 2.2.

Theorem 2.3. *For $m \ge 2$ and $n \ge 1$, there is a local antimagic 3-coloring of $C_{2m} \vee O_{2n}$ satisfying conditions (i) and (ii) of Theorem 2.2.*

Proof. Let $V(C_{2m}) = \{u_i \mid 1 \le i \le 2m\}$ and $V(O_{2n}) = \{v_j \mid 1 \le j \le 2n\}$. We separate $C_{2m} \vee O_{2n}$ into two edge-disjoint graphs, C_{2m} and $O_{2m} \vee O_{2n}$, where $V(O_{2m}) = V(C_{2m})$.

Firstly, define a labeling f for C_{2m}. Let $f : V(C_{2m}) \to [1, 2m]$ such that $f(u_i u_{i+1}) = i$, where $1 \le i \le 2m$ and $u_{2m+1} = u_1$. Thus, $f^+(u_1) = 2m + 1$, $f^+(u_i) = 2i - 1$ for $2 \le i \le 2m$.

Next, we define a labeling g for $O_{2m} \vee O_{2n} \cong K_{2m,2n}$. The labeling matrix of g is $\begin{pmatrix} \bigstar & B \\ B^T & \bigstar \end{pmatrix}$ under the vertex lists $V(O_{2m}) = \{u_1, u_3, \dots, u_{2m-1}, u_2, \dots, u_{2m}\}$ and $V(O_{2n}) = \{v_1, v_2, \dots, v_{2n}\}$. So we only need to fill the integers in $[2m + 1, 2m + 4mn]$ into the matrix B.

Let $\mathcal{M}$ be a guide matrix as follows:

$$\left(\begin{array}{cc|cc|cccccccc} -2 & -3 & +\boxed{2} & -\boxed{2n-1} & +\boxed{3} & -(2n-1) & +\boxed{5} & -(2n-3) & \cdots & +\boxed{2n-3} & -5 \\ -\boxed{2n+1} & -\boxed{2n} & -(2n+1) & +4 & -(2n) & +\boxed{4} & -(2n-2) & +\boxed{6} & \cdots & -6 & +\boxed{2n-2} \end{array}\right).$$

We replace each entry of $\mathcal{M}$ by a column vector according to the rules:

(1) replace $-a$ by $2S^-_m(a) - J_{m,1}$; replace $+a$ by $2S^+_m(a) - J_{m,1}$, where $J_{m,1}$ is an $m \times 1$ matrix with all entries 1;

(2) replace $-\boxed{a}$ by $2S^-_m(a)$; replace $+\boxed{a}$ by $2S^+_m(a)$.

Let $\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ be the resulting matrix, where B_1 and B_2 are $m \times 2n$ matrices. The row sums of B_1 in column matrix is

$$\begin{aligned}
&(2S^-_m(2) - J_{m,1}) + (2S^-_m(3) - J_{m,1}) + 2S^+_m(2) + 2S^-_m(2n-1) + \sum_{i=1}^{n-2} 2S^+_m(2i+1) + \sum_{j=2}^{n-1} [2S^-_m(2j+1) - J_{m,1}] \\
&= 4S^-_m(2n-1) + 2[S^-_m(2) + S^+_m(2)] + 2\sum_{i=1}^{n-2} [S^+_m(2i+1) + S^-_m(2i+1)] - nJ_{m,1} \\
&= 4S^-_m(2n-1) + 2(3m+1)J_{m,1} + 2\sum_{i=1}^{n-2} [m(4i+1)+1]J_{m,1} - nJ_{m,1} \\
&= 4S^-_m(2n-1) + [4mn^2 - 10mn + 10m + n - 2]J_{m,1} = A_1.
\end{aligned}$$

Clearly, the entries of the column matrix A_1 form a decreasing sequence with common difference 4. Now the first column of B_1 is the vector $2S_m^-(2) - J_{m,1}$. We shift each entry of this vector downward to 1 and move the last entry of this vector to the top, i.e., increase the entries by 2 except the $(1,1)$-entry and subtract the $(1,1)$-entry by $2(m-1)$. Let this new matrix be B_1'. Now, the first column of B_1' has entries $2m+1, 4m-1, 4m-3, \dots, 2m+3$ so that the second entry up to the last entry of the first column of B_1' form a decreasing sequence with common difference 2 and the difference between the first entry and second entry is $2-2m$.

Similarly the row sums of B_2 in column matrix is

$$\begin{aligned}
&4S_m^-(2n+1) + 4S_m^-(2n) + 4S_m^+(4) + 2\sum_{i=3}^{n-1}[S_m^+(2i) + S_m^-(2i)] - nJ_{m,1}\\
&= 4S_m^-(2n+1) + 4(m(2n+3)+1)J_{m,1} + [2m(n-3)(2n+3) + 2(n-3)]J_{m,1} - nJ_{m,1}\\
&= 4S_m^-(2n+1) + [4mn^2 + 2mn - 6m + n - 2]J_{m,1} = A_2.
\end{aligned}$$

It is clear that the entries of the column matrix A_2 form a decreasing sequence with common difference 4.

Combining the labelings f and g, we have a labeling ϕ for the whole graph $C_{2m} \vee O_{2n}$. One may check that $\phi^+(u_{2j-1}) = f^+(u_{2j-1}) + r_j(B_1') = 4mn^2 - 2mn + 6m + n + 1$ for each $1 \le j \le m$; and $\phi^+(u_{2i}) = f^+(u_{2i}) + r_i(B_2) = 4mn^2 + 10mn - 2m + n + 1$ for each $1 \le i \le m$. Hence $\phi^+(u_{2i}) > \phi^+(u_{2j-1})$ for $1 \le i, j \le m$.

Clearly, the column sum of $\begin{pmatrix} B_1' \\ B_2 \end{pmatrix}$ is $(4mn + 4m + 1)m$. So $\phi^+(v_l) = (4mn + 4m + 1)m$.

$$\begin{aligned}
\phi^+(u_{2j-1}) - \phi^+(v_l) &= 4mn^2 - 4m^2n - 4m^2 - 2mn + 5m + n + 1\\
&= 4mn(n-m-1) - 4m^2 + 2mn + 5m + n + 1. \qquad (1)
\end{aligned}$$

If $n \ge m+2$, then $\phi^+(u_{2i}) - \phi^+(v_l) > \phi^+(u_{2j-1}) - \phi^+(v_l) \ge 4mn - 4m^2 + 2mn + 5m + n + 1 > 0$.

$$\begin{aligned}
\phi^+(v_l) - \phi^+(u_{2i}) &= 4m^2n - 4mn^2 + 4m^2 - 10mn + 3m - n - 1\\
&= 4mn(m-n-2) + 4m^2 - 2mn + 3m - n - 1. \qquad (2)
\end{aligned}$$

If $m \ge n+2$, then $\phi^+(v_l) - \phi^+(u_{2j-1}) > \phi^+(v_l) - \phi^+(u_{2i}) > 0$.

1) If $n = m+1$, then $\phi^+(u_{2j-1}) - \phi^+(v_l) = -2m^2 + 8m + 2 \neq 0$ (since the discriminant is not a prefect square) and $\phi^+(u_{2i}) - \phi^+(v_l) = 10m^2 + 12m + 2 > 0$

2) If $n = m$, then $\phi^+(u_{2j-1}) - \phi^+(v_l) = -6m^2 + 6m + 1 < 0$ and $\phi^+(u_{2i}) - \phi^+(v_l) = 6m^2 - 2m + 1 > 0$.

3) If $n = m-1$, then $\phi^+(u_{2j-1}) - \phi^+(v_l) = -10m^2 + 12m < 0$, but $\phi^+(u_{2i}) - \phi^+(v_l) = 2m^2 - 8m \neq 0$ when $m \neq 4$. So, for $n = m-1 = 3$, we have to find another labeling for $C_8 \vee O_6$.

   Label the edges of C_8 by 1, 8, 3, 2, 5, 4, 7, 6 in the natural order. Let this labeling be f. So the induced vertex labels of $u_1, u_2, \dots, u_8$ are 7, 9, 11, 5, 7, 9, 11, 13.

   We start from a 8×6 magic rectangle Ω (shown below). Increase each entry by 8 and swap some entries within the same column (indicated in italic). We have

$$\Omega = \begin{pmatrix}
1 & 44 & 9 & 36 & 29 & 28\\
2 & 43 & 10 & 35 & 30 & 27\\
3 & 42 & 11 & 34 & 31 & 26\\
4 & 41 & 12 & 33 & 32 & 25\\
45 & 8 & 37 & 16 & 17 & 24\\
46 & 7 & 38 & 15 & 18 & 23\\
47 & 6 & 39 & 14 & 19 & 22\\
48 & 5 & 40 & 13 & 20 & 21
\end{pmatrix} \longrightarrow
\begin{matrix} 7\\ 11\\ 7\\ 11\\ 9\\ 9\\ 13\\ 5 \end{matrix}
\begin{pmatrix}
9 & 52 & 17 & 44 & 37 & 36\\
10 & 51 & 18 & 43 & 38 & \mathit{31}\\
11 & 50 & 19 & 42 & 39 & 34\\
12 & 49 & 20 & 41 & 40 & \mathit{29}\\
53 & 16 & 45 & 24 & \mathit{27} & 32\\
54 & 15 & 46 & 23 & 26 & \mathit{33}\\
55 & 14 & 47 & 22 & \mathit{25} & 30\\
56 & 13 & 48 & 21 & 28 & \mathit{35}
\end{pmatrix}
\begin{matrix} 202\\ 202\\ 202\\ 202\\ 206\\ 206\\ 206\\ 206 \end{matrix}$$

   This matrix forms a labeling matrix of a labeling g of $K_{8,6}$ under the vertex list $\{u_1, u_3, u_5, u_7, u_2, u_6, u_8, u_4\}$ of C_8, The column in front of the matrix is the corresponding induced vertex labels under f on C_8, and the column behind of the matrix is the induced vertex labels of the labeling ϕ for $C_8 \vee O_6$. Thus $\phi^+(u_{2i-1}) = 202$, $\phi^+(u_{2i}) = 206$ and $\phi^+(v_j) = 260$ for $1 \le i \le 4$ and $1 \le j \le 6$.

Clearly, all labels are used. So ϕ is a local antimagic 3-coloring for $C_{2m} \vee O_{2n}$. Moreover, the number of even incident edge labels equals the number of odd incident edge labels for each vertex. Hence ϕ satisfies conditions (i) and (ii) of Theorem 2.2 □

Corollary 2.5. *If $G = C_3 \times K_2$ and $H = C_{2m} \vee O_{2n}$, $m \geq 2$, $n \geq 1$, then $\chi_{la}(G[H]) = 9$.*

Proof. Keep all notation defined in the proof of Theorem 2.3. Now $\deg_H(u_i) = 2n+2$, $\deg_H(v_l) = 2m$ and $p = 6$. By Theorems 2.2 and 2.3, it suffices to check condition (iii) of Theorem 2.2, i.e., $6[\phi^+(u_1) - \phi^+(v_1)] - 5(n+1-m) \neq 0$ and $6[\phi^+(v_1) - \phi^+(u_2)] - 5(m-n-1) \neq 0$.

By (1), we have

$$\begin{aligned} & 6[4mn(n-m-1) - 4m^2 + 2mn + 5m + n + 1] - 5(n+1-m) \\ &= -24m^2 + 24mn^2 - 24m^2n - 12mn + 35m + n + 1 \\ &= 24mn(n-m) - 24m(m-1) - 12mn + 11m + n + 1. \end{aligned} \tag{3}$$

Clearly (3) is less than zero for $n \geq m$. When $n \geq m+2$, (3) $\geq 36mn - 24m^2 + 35m + n + 1 > 0$. When $n = m+1$, (3) $= 12mn - 24m^2 + 35m + n + 1 = -12m^2 + 48m + 2 \neq 0$ since the discriminant is 2400 which is not a prefect square.

By (2), we have

$$\begin{aligned} & 6[4mn(m-n-2) + 4m^2 - 2mn + 3m - n - 1] - 5(m-n-1) \\ &= 24m^2 + 24m^2n - 24mn^2 - 60mn + 13m - n - 1 \\ &= 24mn(m-n-2) + 12m(m-n) + 12m^2 + 13m - n - 1. \end{aligned} \tag{4}$$

Clearly (4) is greater than 0 for $m \geq n+2$. When $m \leq n$, (4) $\leq -48mn + 12m^2 + 13m - n - 1 = 12m(m-4n+1) + m - n - 1 < 0$. When $m = n+1$, then H is regular so condition (iii) holds. The proof is complete. □

Example 2.3. *Let $V(C_6) = \{u_1, u_3, u_5, u_2, u_4, u_6\}$ and $V(O_8) = \{v_j \mid 1 \leq j \leq 8\}$ be the vertex lists of C_6 and O_8. According to the proof of Theorem 2.2 we label the edges of C_6 by 1 to 6 in natural order. So the induced vertex labels are 7, 3, 5, 7, 9, 11. Then*

$$\begin{array}{c} 7 \\ 5 \\ 9 \\ \hline 3 \\ 7 \\ 11 \end{array} \left(\begin{array}{cccc|cccc} 7 & 17 & 8 & 42 & 14 & 41 & 26 & 29 \\ 11 & 15 & 10 & 40 & 16 & 39 & 28 & 27 \\ 9 & 13 & 12 & 38 & 18 & 37 & 30 & 25 \\ \hline 54 & 48 & 53 & 19 & 47 & 20 & 35 & 32 \\ 52 & 46 & 51 & 21 & 45 & 22 & 33 & 34 \\ 50 & 44 & 49 & 23 & 43 & 24 & 31 & 36 \end{array} \right) \begin{array}{c} 191 \\ 191 \\ 191 \\ \hline 311 \\ 311 \\ 311 \end{array}$$

The column in front of the matrix is the corresponding induced vertex labels under $f$on C_6, and the column behind of the matrix is the induced vertex labels of the labeling ϕ for $C_6 \vee O_8$. One may check that the column sum of the matrix is 183, which is $\phi^+(v_j)$ for all j.

Corollary 2.5 shows that there are infinite numbers of graphs H such that $\chi_{la}(G[H]) = \chi_{la}(G)\chi_{la}(H) = \chi(G)\chi(H)$, where $G = C_3 \times K_2$.

We shall end the article by proposing the following conjectures and problem regarding the local antimagic chromatic number of the lexicographic product of graphs for further study.

Conjecture 2.1. *There exist infinite numbers of graphs G and H, respectively, such that*

$$\chi_{la}(G[H]) = \chi_{la}(G)\chi_{la}(H) = \chi(G)\chi(H).$$

Conjecture 2.2. *For graphs G and H, $\chi_{la}(G[H]) = \chi(G)\chi(H)$ if and only if $\chi(G)\chi(H) = 2\chi(H) + \lceil \frac{\chi(H)}{k} \rceil$, where $k \geq 1$ and $2k+1$ is the length of a shortest odd cycle in G.*

Problem 2.1. *Characterize the graphs that satisfy the upper bounds of Theorems 2.1 and 2.2 respectively.*

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References

[1] S. Arumugam, K. Premalatha, M. Bača, A. Semaničová-Feňovčíková, Local antimagic vertex coloring of a graph, *Graphs Combin.* **33** (2017) 275–285.

[2] M. Bača, A. Semaničová-Feňovčíková, T. M. Wang, Local antimagic chromatic number for copies of graphs, *Mathematics* **9** (2021) #1230.

[3] J. A. Bondy, U. S. R. Murty, *Graph theory with applications*, MacMillan, New York, 1976.

[4] N. Čižek, S. Klavžar, On the chromatic number of the lexicographic product and the Cartesian sum of graphs, *Discrete Math.* **134** (1994)17–24.

[5] G. C. Lau, H. K. Ng, W. C. Shiu, Cartesian magicness of 3-dimensional boards, Malaya J. Mat. **8** (2020) 1175–1185.

[6] G. C. Lau, H. K. Ng, W. C. Shiu, Affirmative solutions on local antimagic chromatic number, *Graphs Combin.* **36** (2020) 1337–1354.

[7] G. C. Lau, H. K. Ng, W. C. Shiu, On local antimagic chromatic number of cycle-related join graphs, *Discuss. Math. Graph Theory* **41** (2021) 133–152.

[8] G. C. Lau, W. C. Shiu, On local antimagic total labeling of amalgamation graphs, (2022) arXiv:2203.06337 [math.CO].

[9] G. C. Lau, W. C. Shiu, On join product and local antimagic chromatic number of regular graphs, *Acta Math. Hungar.*, To appear.

[10] G. C. Lau, W. C. Shiu, On local antimagic chromatic number of lexicographic product graphs, *Acta Math. Hungar.*, To appear.

[11] G. C. Lau, W. C. Shiu, H. K. Ng, On local antimagic chromatic number of graphs with cut-vertices, *Iran. J. Math. Sci. Inform.*, To appear.

[12] W. C. Shiu, P. C. B. Lam, S-M. Lee, Edge-magicness of the composition of a cycle with a null graph, *Congr. Numer.* **132** (1998) 9–18.