Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **11** (2023) 7–13
DOI: 10.47443/dml.2022.050

Research Article

Applications of Zeilberger's algorithm to Ramanujan-inspired series involving harmonic-type numbers

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(Received: 9 April 2022. Received in revised form: 24 June 2022. Accepted: 30 August 2022. Published online: 10 September 2022.)

Abstract

A "harmonic variant" of Zeilberger's algorithm is utilized to improve upon the results introduced by Wang and Chu [*Ramanujan J.* **52** (2020) 641–668]. Wang and Chu's coefficient-extraction methodologies yielded evaluations for Ramanujan-like series involving summand factors of the form $H_n^3+3H_nH_n^{(2)}+2H_n^{(3)}$, where H_n denotes a harmonic number and $H_n^{(x)}$ is a generalized harmonic number. However, it is unclear as to how Wang and Chu's techniques could be applied to improve upon such results by separately evaluating the series obtained upon the expansion of the summands according to the terms of the factor $H_n^3+3H_nH_n^{(2)}+2H_n^{(3)}$. In this note, we succeed in applying Zeilberger's algorithm toward this problem, providing explicit evaluations for the series with a factor of the form $H_n^{(3)}$ obtained from the aforementioned expansion. Our approach toward generalizing Zeilberger's algorithm to non-hypergeometric expressions may be applied much more broadly. The series obtained by replacing $H_n^{(3)}$ with $H_n^{(2)}$ were highlighted as especially beautiful motivating examples in Wang and Chu's article. These $H_n^{(2)}$-series motivate our main results, which are natural higher-order extensions of these $H_n^{(2)}$-series.

Keywords: creative telescoping; Zeilberger's algorithm; harmonic-type number; difference equation; Ramanujan-like series.

2020 Mathematics Subject Classification: 33F10, 39A10.

1. Introduction

This article is mainly devoted to the application of a variant of Zeilberger's algorithm [14, §6] based on non-hypergeometric sums, in order to improve upon the following symbolic evaluations introduced by Wang and Chu [16]:

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2\left(H_n^3+3H_nH_n^{(2)}+2H_n^{(3)}\right)}{16^n(2n-1)}=\frac{8}{\pi}\left(\pi^2-2\pi^2\ln 2+16\ln^3 2-24\ln^2+24\ln 2+6\zeta(3)-12\right) \tag{1}$$

and

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2\left(H_n^3+3H_nH_n^{(2)}+2H_n^{(3)}\right)}{16^n(2n-1)^2}=\frac{8}{\pi}\left(4\pi^2\ln 2-96\ln 2+72\ln^2 2-32\ln^3 2-3\pi^2-12\zeta(3)+60\right), \tag{2}$$

letting $H_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$ to denote the n^{th} harmonic number and writing $H_n^{(x)}=1+\frac{1}{2^x}+\cdots+\frac{1}{n^x}$ to denote generalized harmonic numbers, and recalling the Riemann zeta function $\zeta(x)=1+\frac{1}{2^x}+\frac{1}{3^x}+\cdots$. Wang and Chu proved the closed-form formulas in (1) and (2) by extracting the coefficient of x^3 across both sides of the special case

$${}_2F_1\left[\begin{matrix}\frac{1}{2}-\mu,\frac{1}{2}-\nu\\ 1-x\end{matrix}\,\middle|\,1\right]=\Gamma\left[\begin{matrix}1-x,\mu+\nu-x\\ \frac{1}{2}+\mu-x,\frac{1}{2}+\nu-x\end{matrix}\right] \tag{3}$$

of the Gauss summation theorem [2, §1.3], letting generalized hypergeometric series [2, §2.5] be denoted as

$${}_{p+1}F_p\left[\begin{matrix}a_0,a_1,\ldots,a_{p+1}\\ b_1,b_2,\ldots,b_p\end{matrix}\,\middle|\,z\right]=\sum_{n=0}^{\infty}\frac{(a_0)_n(a_1)_n\cdots(a_{p+1})_n}{n!\,(b_1)_n(b_2)_n\cdots(b_p)_n}z^n,$$

and writing

$$\Gamma(x):=\int_0^{\infty}u^{x-1}e^{-u}\,du$$

for $\Re(x)>0$, with

$$\Gamma\left[\begin{matrix}\alpha,\beta,\ldots,\gamma\\ A,B,\ldots,C\end{matrix}\right]=\frac{\Gamma(\alpha)\Gamma(\beta)\cdots\Gamma(\gamma)}{\Gamma(A)\Gamma(B)\cdots\Gamma(C)},$$

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and where the Pochhammer symbol $(x)_n$ satisfies $(x)_n = \frac{\Gamma(n+x)}{\Gamma(x)}$. It is far from clear as to how the techniques from [16] may be applied to improve upon the Ramanujan-inspired formulas in (1) and (2), by providing separate evaluations for series obtained by expanding the summands of (1) and (2) according to the summand factor $H_n^3 + 3H_n H_n^{(2)} + 2H_n^{(3)}$. We succeed in applying Zeilberger's algorithm toward this problem, by proving the following formulas, and related results, letting $G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$ to denote Catalan's constant:

$$\sum_{k=0}^{\infty} \left(\frac{1}{16}\right)^k \binom{2k}{k}^2 \frac{H_k^{(3)}}{2k-1} = -\frac{2\zeta(3)}{\pi} + \frac{32G}{\pi} - \frac{16}{\pi} - 16\log(2) + 8, \tag{4}$$

$$\sum_{k=0}^{\infty} \left(\frac{1}{16}\right)^k \binom{2k}{k}^2 \frac{H_k^{(3)}}{(2k-1)^2} = \frac{4\zeta(3)}{\pi} - \frac{96G}{\pi} + \frac{80}{\pi} + 48\log(2) - 32. \tag{5}$$

The identities

$$\sum_{k=0}^{\infty} \left(\frac{1}{16}\right)^k \binom{2k}{k}^2 \frac{H_k^{(2)}}{2k-1} = 4 - \frac{\pi}{3} - \frac{8}{\pi} \tag{6}$$

and

$$\sum_{k=0}^{\infty} \left(\frac{1}{16}\right)^k \binom{2k}{k}^2 \frac{H_k^{(2)}}{(2k-1)^2} = \frac{2\pi}{3} - 12 + \frac{32}{\pi} \tag{7}$$

are highlighted as especially beautiful [16] motivating examples in [16], which greatly adds to the interest in our new formulas on display in (4) and (5): We see that the series in (4) and (5) are precisely the series obtained by replacing summand factor $H_k^{(2)}$ with $H_k^{(3)}$ in (6) and (7), respectively. So, (4) and (5) are natural higher-order extensions of the Wang–Chu formulas in (6) and (7) that are highlighted as especially beautiful applications of the techniques from [16]. Since these past techniques introduced in [16] do not seem to apply to (4) and (5), this, in conjunction with the foregoing considerations, emphasizes the remarkable nature about the main results and techniques that we introduce.

Although the main results in this article are our formulas in (4) and (5) along with our proofs for these formulas, the way we extend Zeilberger's algorithm to non-hypergeometric sums, as in Section 2 below, is of interest in its own right. It seems that this "harmonic Zeilberger" approach may be applied quite broadly to further improve upon (1) and (2) and to generalize (4) and (5); for the sake of brevity, we leave a full exploration along these lines for a separate project.

Evaluations as in (1) and (2) are inspired by Ramanujan's series for $\frac{1}{\pi}$ (cf. [4]). Apart from the famous 17 series for $\frac{1}{\pi}$ due to Ramanujan [15] (cf. [3, pp. 352–354]) and Ramanujan's ${}_4F_3(-1)$-series for $\frac{1}{\pi}$ included in his first letter to Hardy, the Ramanujan-inspired series as in [4, 16] are such natural extensions, as recently explored in [7], of Ramanujan's sums of the form

$$S(r) = \sum_{k=0}^{\infty} \left(\frac{1}{16}\right)^k \binom{2k}{k}^2 \frac{1}{k+r};$$

see [1]. Our application of Zeilberger's algorithm to prove the Ramanujan-inspired series in (4) and (5) is very much inspired by Zeilberger's famous proof [8] via the Wilf–Zeilberger method [14] of Ramanujan's formula

$$\frac{2}{\pi} = \sum_{k=0}^{\infty} \left(-\frac{1}{64}\right)^k \binom{2k}{k}^3 (4k+1).$$

In our recent publication [6], we had solved some open problems given by Wang and Chu in the 2022 article [17], which concerned series involving harmonic-type numbers such as the odd harmonic numbers $O_k = 1 + \frac{1}{3} + \cdots + \frac{1}{2k-1}$. We had applied the WZ method in [6] to derive a ${}_3F_2(1)$-identity, and the application of differential operators to this identity resulted in a linear relation among series considered in [17]. In contrast, instead of applying parameter derivatives to obtain harmonic or harmonic-type numbers from hypergeometric formulas, we instead use telescoping arguments for expressions involving $H_k^{(3)}$.

As below, we are to often use the following hypergeometric evaluation, which has been proved in a variety of different ways in [5, 9, 13]:

$${}_4F_3\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}, 1, 1\\ 2, 2, 2\end{matrix} \,\middle|\, 1\right] = \sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2}{16^k (k+1)^3} = 16\log 2 + \frac{48}{\pi} - \frac{32G}{\pi} - 16. \tag{8}$$

Following [5], one way of going about proving the evaluation in (8) relies on the following moment formula for the sequence of Catalan numbers:

$$\frac{\binom{2n}{n}}{n+1} = \frac{1}{2\pi} \int_0^4 x^n \sqrt{\frac{4-x}{x}}\, dx.$$

Again, following [5], we can use this moment formula to show that the ${}_4F_3(1)$-series in (8) is equal to

$$\frac{1}{2\pi}\int_0^4 -8\sqrt{4-x}\left(\frac{1}{x}\right)^{3/2}\left(-2+\sqrt{4-x}+2\ln(2)-2\ln\left(1+\sqrt{1-\frac{x}{4}}\right)\right)dx.$$

A dilogarithmic form for the corresponding indefinite integral is provided in [5], and this antiderivative evaluation may be used in a direct way to obtain the symbolic form in (8).

In order to apply Zeilberger's algorithm, we employ its implementation in Maple. In this regard, we input

```
with(SumTools[Hypergeometric]):
```

and then specify a hypergeometric expression `T` to which we want to apply Zeilberger's algorithm, and then input the following.

```
Zpair := Zeilberger(T, n, k, En):
```

The hypergeometric function G obtained by Zeilberger's algorithm may then be obtained by inputting the following.

```
G := Zpair[2]
```

The polynomial identity satisfied by `T` may be computed with the following.

```
L := Zpair[1]
```

It is useful, with regard to our formulation of the proof of Theorem 2.1, to provide references for the classical series shown in (9) and (10) below, following the exposition in [12]. The closed form for what is referred to as *Forsyth's series*

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2}{16^k(2k-1)^2}=\frac{4}{\pi} \tag{9}$$

dates back to 1883 (see [10]), and Glaisher's formula

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2}{16^k(k+1)}=\frac{4}{\pi} \tag{10}$$

was introduced in 1905 [11] (cf. [12]).

2. Main results

Theorem 2.1. *The symbolic evaluations shown in* (4) *and* (5) *hold true.*

Proof. Set $F(n,k)=\binom{n}{k}^2$. Using Zeilberger's algorithm, this gives us the companion function

$$G(n,k)=\frac{2k^2\left(k-\frac{3n}{2}-\frac{3}{2}\right)\binom{n}{k}^2}{(k-n-1)^2},$$

and we find that the pair (F,G) satisfies the following difference equation:

$$(n+1)F(n+1,k)+(-4n-2)F(n,k)=G(n,k+1)-G(n,k).$$

We multiply both sides of the above equality by $H_k^{(3)}$:

$$\left((n+1)F(n+1,k)+(-4n-2)F(n,k)\right)H_k^{(3)}=G(n,k+1)H_k^{(3)}-G(n,k)H_k^{(3)}.$$

According to the relation $H_{k+1}^{(3)}=H_k^{(3)}+\frac{1}{(k+1)^3}$, we obtain that

$$\left((n+1)F(n+1,k)+(-4n-2)F(n,k)\right)H_k^{(3)}+G(n,k+1)\frac{1}{(k+1)^3}=G(n,k+1)H_{k+1}^{(3)}-G(n,k)H_k^{(3)},$$

so that the right-hand side telescopes upon the application of summation operators over indices $k\in\mathbb{N}_0$. Now, set $n=\frac{1}{2}$ in the above displayed equality, and apply $\sum_{k=0}^{\infty}\cdot$ to both sides of the resultant equality, noting that

$$G(1/2,k)=\frac{2(4k-9)k^2}{(2k-3)^2}\binom{1/2}{k}^2.$$

By the telescoping of the series $\sum_{k=0}^{\infty}(G(n,k+1)H_{k+1}^{(3)}-G(n,k)H_k^{(3)})$, this is easily seen to vanish. So, we have applied Zeilberger's algorithm to prove the following:

$$\sum_{k=0}^{\infty}\left(-4\binom{\frac{1}{2}}{k}^2+\frac{3\binom{\frac{3}{2}}{k}^2}{2}\right)H_k^{(3)}=-\sum_{k=0}^{\infty}\frac{G\left(\frac{1}{2},k+1\right)}{(k+1)^3}$$

$$=-\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{1}{16}\right)^k\binom{2k}{k}^2\frac{4k-5}{(2k-1)^2(k+1)^3}$$

$$=-\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{1}{16}\right)^k\binom{2k}{k}^2\left(\frac{40}{27(2k-1)}-\frac{8}{9(2k-1)^2}-\frac{20}{27(k+1)}-\frac{8}{9(k+1)^2}-\frac{1}{(k+1)^3}\right).$$

So, by expanding the above summand, we obtain a $\mathbb{Q}$-linear combination of the ${}_4F_3$-series in (8), Ramanujan's S-function evaluated at 1 (see [1,7,12]), Ramanujan's S-function evaluated at $-\frac{1}{2}$ (see [1,7,12]), Forsyth's series as shown in (9), and the limit of either side of the following partial sum identity:

$$\sum_{k=0}^{n-1}\left(\frac{1}{16}\right)^k\frac{\binom{2k}{k}^2}{(k+1)^2}=4\frac{(4n+1)\binom{2n}{n}^2}{16^n}-4. \tag{11}$$

Simplifying the linear combination of the corresponding closed forms for all of these series, this gives us the following:

$$\sum_{k=0}^{\infty}\left(\frac{1}{16}\right)^k\binom{2k}{k}^2\left(\frac{27}{8(2k-1)}-\frac{5}{8(2k-1)^2}-\frac{27}{8(2k-3)}+\frac{27}{8(2k-3)^2}\right)H_k^{(3)}=-\frac{16G}{\pi}-\frac{88}{9}+\frac{968}{27\pi}+8\log(2).$$

Now, we set $F(n,k)=(2k-1)\binom{n}{k}^2$, and we again apply Zeilberger's algorithm, which provides the following companion to F:

$$G(n,k)=\frac{k^2\left(k^2(4n+2)-2k(n+2)(3n+1)+3(n+1)(3n+1)\right)\binom{n}{k}^2}{(n-k+1)^2}.$$

Zeilberger's algorithm gives us the discrete difference equation shown below:

$$\left(n^2-1\right)F(n+1,k)+\left(-4n^2-2n\right)F(n,k)=G(n,k+1)-G(n,k).$$

Again, we multiply both sides by $H_k^{(3)}$, and then use the recurrence $H_{k+1}^{(3)}=H_k^{(3)}+\frac{1}{(k+1)^3}$ so as to once again obtain an expression of the form

$$G(n,k+1)H_{k+1}^{(3)}-G(n,k)H_k^{(3)}, \tag{12}$$

so that this expression again telescopes upon the application of $\sum_{k=0}^{\infty}\cdot$. So, setting $n=\frac{1}{2}$, and then following through with the telescoping argument indicated in the preceding sentence, we obtain that:

$$\sum_{k=0}^{\infty}\left(-2(2k-1)\binom{\frac{1}{2}}{k}^2-\frac{3}{4}(2k-1)\binom{\frac{3}{2}}{k}^2\right)H_k^{(3)}=-\sum_{k=0}^{\infty}\frac{G\left(\frac{1}{2},k+1\right)}{(k+1)^3}.$$

The right-hand side may be rewritten as:

$$-\frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{1}{16}\right)^k\binom{2k}{k}^2\left(-\frac{56}{27(2k-1)}+\frac{16}{9(2k-1)^2}+\frac{28}{27(k+1)}+\frac{10}{9(k+1)^2}+\frac{5}{(k+1)^3}\right).$$

Again, we obtain a linear combination of the same ${}_4F_3(1)$-series as before, Forsyth's series, $S(1)$, $S\left(-\frac{1}{2}\right)$, and the sum of the squares of normalized Catalan numbers. So, we may obtain that

$$\sum_{k=0}^{\infty}\left(\frac{1}{16}\right)^k\binom{2k}{k}^2\left(-\frac{59}{16(2k-1)}+\frac{27}{16(2k-3)}-\frac{27}{8(2k-3)^2}\right)H_k^{(3)}=\frac{40G}{\pi}+\frac{190}{9}-\frac{1844}{27\pi}-20\log(2).$$

Now, we set $F(n,k)=(2k-1)^2\binom{n}{k}^2$, and we again apply Zeilberger's algorithm. This gives us the following companion function $G(n,k)$ given by the following Maple output:

```
(k^3-1/2*(3*n^2+8*n-1)/n*k^2+1/4*(36*n^5+78*n^4+60*n^3+42*n^2+9*n-4)/n/(2*n^3+2*n^2+2*n+1)*k-
1/8*(54*n^5+90*n^4+60*n^3+29*n^2+n-4)/n/(2*n^3+2*n^2+2*n+1))*k^2/(-n+k-1)^2*binomial(n,k)^2*(16*n^3+
16*n^2+16*n+8)
```

Zeilberger's algorithm gives us that the following difference equation holds true:

$$\left(2n^4-2n^3+3n-1\right)F(n+1,k)+\left(-8n^4-4n^3-4n^2+2\right)F(n,k)=G(n,k+1)-G(n,k).$$

Once again, we multiply both sides by $H_k^{(3)}$, and, as before, manipulate the resultant equality so as to obtain an expression as in (12) that telescopes as we sum over $k\in\mathbb{N}_0$. So, by applying $\sum_{k=0}^{m}\cdot$, we find that

$$\sum_{k=0}^{m}\left(\left(2n^4-2n^3+3n-1\right)F(n+1,k)+\left(-8n^4-4n^3-4n^2+2\right)F(n,k)\right)H_k^{(3)}+\sum_{k=0}^{m}\frac{G(n,k+1)}{(k+1)^3} \tag{13}$$

is equal to $G(n,m+1)H_{m+1}^{(3)}-G(n,k)H_0^{(3)}$, letting $m\in\mathbb{N}_0$. By setting $n=\frac{1}{2}$, we can show that the limit of $G(n,m+1)H_{m+1}^{(3)}-G(n,k)H_0^{(3)}$ as $m\to\infty$ is $\frac{11\zeta(3)}{2\pi}$, if we restrict m to integer values. More specifically, by writing

$$G\left(\frac{1}{2},m+1\right)H_{m+1}^{(3)}-G\left(\frac{1}{2},k\right)H_0^{(3)}=\frac{\pi(4m(11m(4m-3)-2)+3)H_{m+1}^{(3)}}{32\Gamma\left(\frac{3}{2}-m\right)^2\Gamma(m+1)^2},$$

since $\lim\limits_{m\to\infty}H_{m+1}^{(3)}=\zeta(3)$, it remains to evaluate

$$\lim_{m\to\infty}\frac{(3+4m(11m(4m-3)-2))\pi\zeta(3)}{32\Gamma\left(\frac{3}{2}-m\right)^2\Gamma(m+1)^2},$$

again with m restricted to $\mathbb{N}$. For $\ell\in\mathbb{N}_0$, we have that

$$\Gamma\left(\frac{1}{2}-\ell\right)=\frac{(-4)^\ell\sqrt{\pi}}{\binom{2\ell}{\ell}\ell!},$$

so it remains to evaluate

$$\lim_{m\to\infty}\frac{2^{-4m-1}(4m(11m(4m-3)-2)+3)\zeta(3)\binom{2(m-1)}{m-1}^2}{m^2},$$

which is easily seen to reduce to $\frac{11\zeta(3)}{2\pi}$ using Stirling's approximation, giving us the value of (13) as $m\to\infty$. As for the series

$$\sum_{k=0}^{\infty}\frac{G\left(\frac{1}{2},k+1\right)}{(k+1)^3}$$

required for our evaluation, we find that this is equivalent to

$$\frac{11}{2}\sum_{k=0}^{\infty}\left(\frac{1}{16}\right)^k\binom{2k}{k}^2\left(\frac{16}{297(2k-1)}-\frac{8}{99(2k-1)^2}-\frac{8}{297(k+1)}+\frac{97}{99(k+1)^2}-\frac{3}{4(k+1)^3}\right).$$

Once again, we obtain a linear combination of the ${}_4F_3(1)$-series in (8), the Forsyth series, $S(1)$, $S\left(\frac{1}{2}\right)$, and (11). This gives us that

$$\sum_{k=0}^{\infty}\left(\frac{1}{16}\right)^k\binom{2k}{k}^2\frac{H_k^{(3)}}{(2k-3)^2}=-\frac{352G}{9\pi}+\frac{44\zeta(3)}{27\pi}-\frac{3200}{243}+\frac{24784}{729\pi}+\frac{176\log(2)}{9}.$$

Now, we set $F(n,k)=(2k+1)^2\binom{n}{k}^2$, and we again apply Zeilberger's algorithm. The G-function in this case is as in the Maple output shown below.

```
(k^3-1/2*(3*n^2+4*n+1)/n*k^2+1/4*(12*n^5+70*n^4+156*n^3+170*n^2+93*n+20)/n/(2*n^3+10*n^2+14*n+5)*k-
1/8*(6*n^5+34*n^4+88*n^3+121*n^2+81*n+20)/n/(2*n^3+10*n^2+14*n+5))*k^2/(-n+k-
1)^2*binomial(n,k)^2*(16*n^3+80*n^2+112*n+40)
```

Zeilberger's algorithm gives us that the following difference equation holds:

$$\left(2n^4+6n^3+4n^2-n-1\right)F(n+1,k)+\left(-8n^4-36n^3-36n^2+8n+10\right)F(n,k)=G(n,k+1)-G(n,k).$$

We again employ our harmonic Zeilberger technique, in much the same way as before, giving us that

$$\sum_{k=0}^{m}\left(\left(2n^4+6n^3+4n^2-n-1\right)F(n+1,k)+\left(-8n^4-36n^3-36n^2+8n+10\right)F(n,k)\right)H_k^{(3)}+\sum_{k=0}^{m}\frac{G(n,k+1)}{(k+1)^3}$$

with $n=\frac{1}{2}$ and $m\to\infty$ reduces to $\frac{59\zeta(3)}{2\pi}$. This, together with

$$\sum_{k=0}^{\infty}\frac{G\left(\frac{1}{2},k+1\right)}{(k+1)^3}$$

being equal to

$$\frac{59}{2}\sum_{k=0}^{\infty}\left(\frac{1}{16}\right)^k\binom{2k}{k}^2\left(\frac{112}{1593(2k-1)}-\frac{32}{531(2k-1)^2}-\frac{56}{1593(k+1)}+\frac{511}{531(k+1)^2}-\frac{185}{236(k+1)^3}\right),$$

can be used to show that

$$\sum_{k=0}^{\infty}\left(\frac{1}{16}\right)^k\binom{2k}{k}^2\left(\frac{27}{4(2k-1)}+\frac{27}{8(2k-1)^2}-\frac{27}{4(2k-3)}+\frac{27}{2(2k-3)^2}\right)H_k^{(3)}=-\frac{740G}{\pi}+\frac{59\zeta(3)}{2\pi}-\frac{2308}{9}+\frac{18122}{27\pi}+370\log(2).$$

We define s_1–s_4 as below:

$$s_1=\sum_{k=0}^{\infty}\left(\frac{1}{16}\right)^k\binom{2k}{k}^2\frac{H_k^{(3)}}{(2k-3)^2},$$

$$s_2=\sum_{k=0}^{\infty}\left(\frac{1}{16}\right)^k\binom{2k}{k}^2\frac{H_k^{(3)}}{2k-3},$$

$$s_3=\sum_{k=0}^{\infty}\left(\frac{1}{16}\right)^k\binom{2k}{k}^2\frac{H_k^{(3)}}{(2k-1)^2},$$

$$s_4=\sum_{k=0}^{\infty}\left(\frac{1}{16}\right)^k\binom{2k}{k}^2\frac{H_k^{(3)}}{2k-1}.$$

So, using Zeilberger's algorithm, we have shown that

$$\begin{pmatrix}\frac{27}{8}&-\frac{27}{8}&-\frac{5}{8}&\frac{27}{8}\\-\frac{27}{8}&\frac{27}{16}&0&-\frac{59}{16}\\1&0&0&0\\\frac{27}{2}&-\frac{27}{4}&\frac{27}{8}&\frac{27}{4}\end{pmatrix}\cdot\begin{pmatrix}s_1\\s_2\\s_3\\s_4\end{pmatrix}=\begin{pmatrix}-\frac{16G}{\pi}-\frac{88}{9}+\frac{968}{27\pi}+\frac{8\log(512)}{9}\\\frac{40G}{\pi}+\frac{190}{9}-\frac{1844}{27\pi}-20\log(2)\\-\frac{352G}{9\pi}+\frac{44\zeta(3)}{27\pi}-\frac{3200}{243}+\frac{24784}{729\pi}+\frac{176\log(2)}{9}\\-\frac{740G}{\pi}+\frac{59\zeta(3)}{2\pi}-\frac{2308}{9}+\frac{18122}{27\pi}+370\log(2)\end{pmatrix}.$$

The 4×4 matrix shown above is invertible. So, we find that:

$$\begin{pmatrix}s_1\\s_2\\s_3\\s_4\end{pmatrix}=\begin{pmatrix}0&0&1&0\\-\frac{59}{148}&-\frac{1}{2}&\frac{775}{1184}&-\frac{295}{3996}\\-\frac{16}{37}&0&-\frac{54}{37}&\frac{8}{37}\\-\frac{27}{148}&-\frac{1}{2}&-\frac{729}{1184}&-\frac{5}{148}\end{pmatrix}\cdot\begin{pmatrix}-\frac{16G}{\pi}-\frac{88}{9}+\frac{968}{27\pi}+\frac{8\log(512)}{9}\\\frac{40G}{\pi}+\frac{190}{9}-\frac{1844}{27\pi}-20\log(2)\\-\frac{352G}{9\pi}+\frac{44\zeta(3)}{27\pi}-\frac{3200}{243}+\frac{24784}{729\pi}+\frac{176\log(2)}{9}\\-\frac{740G}{\pi}+\frac{59\zeta(3)}{2\pi}-\frac{2308}{9}+\frac{18122}{27\pi}+370\log(2)\end{pmatrix}.$$

Computing the above matrix product, this gives us our desired symbolic forms for s_3 and s_4. □

3. A harmonic Zeilberger method

It seems that our harmonic Zeilberger approach applied above may also be applied to many variants and extensions of series as in (4) and (5). For example, we encourage the application of our methods to the series obtained by replacing $H_k^{(3)}$ with $H_k^{(4)}$, by replacing $H_k^{(3)}$ with H_k^3, by replacing $\left(\frac{1}{16}\right)^k\binom{2k}{k}^2$ with higher powers of normalized central binomial coefficients, etc. For the sake of brevity, we leave this for a future project. We broadly describe our *harmonic Zeilberger* method as follows, letting $F(n,k)$ be hypergeometric, and such that, informally, inputting $n=\frac{1}{2}$ or some other specified value yields an expression "resembling" the summand of a series that is to be evaluated.

Step 1: Apply Zeilberger's algorithm to obtain a difference equation of the form

$$\sum_{i=0}^{m}p_i(n)F(n+i,k)=G(n,k+1)-G(n,k), \tag{14}$$

for polynomials p.

Step 2: Let $(\Delta(k):k\in\mathbb{N}_0)$ denote a sequence of harmonic-type numbers such that $\Delta(k+1)-\Delta(k)$ is a rational expression $\frac{1}{r(k+1)}$. Multiply both sides of (14) by $\Delta(k)$, and then manipulate this equality to obtain that:

$$\sum_{i=0}^{m}p_i(n)F(n+i,k)\Delta(k)-G(n,k+1)\frac{1}{r(k+1)}=G(n,k+1)\Delta(k+1)-G(n,k)\Delta(k).$$

Step 3: Sum both sides over $k \in \mathbb{N}_0$, so that the right-hand side telescopes. Evaluate the limit of this telescoping series symbolically. Apply partial fraction decomposition to evaluate, if possible, $\sum_{k=0}^{\infty} G(n, k+1) \frac{1}{r(k+1)}$.

Step 4: Simplify the summand of $\sum_{i=0}^{m} p_i(n) F(n+i, k) \Delta(k)$ and apply partial fraction decomposition so as to obtain a linear combination of series involving $\Delta(k)$ as a summand factor.

Step 5: Repeat the above steps with different choices for F, so as to ideally obtain an invertible system of linear equations, given by the linear combinations obtained from **Step 4**.

We have successfully applied this kind of setup, as in the proof of our main result. Subsequent to the submission of this article, the author had contacted Ce Xu in regard to the formula (4) introduced in this article. Afterwards, Xu and Jianqiang Zhao formulated a remarkably different proof of the formula shown in (4), using an integration-based approach concerning the theory of colored multiple zeta values [18]. We greatly encourage the application of the above procedure, as given by **Steps 1–5**, in conjunction with the CMZV-based techniques from [18].

Acknowledgement

The author hereby expresses his sincere thanks to an anonymous reviewer for insightful comments that improved the exposition.

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