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Research Article

Enumeration of r-smooth words over a finite alphabet

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Abstract

In this paper, we enumerate a restricted family of k-ary words called r-smooth words. The restriction is defined through the distance between adjacent changes in the word. Using automata, we enumerate this family of words. Additionally, we give explicit combinatorial expressions to enumerate the words and asymptotic expansions related to the Fibonacci sequence.

Keywords: k-ary word; automata; generating function.

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1. Introduction

The enumeration of words with a given combinatorial property has been studied extensively from various perspectives in enumerative combinatorics and theoretical computer science. For example, the study of occurrences of a given pattern in words or other structures, such as permutations, is an active area of research (cf. [4, 9]). In this paper, we consider a new family of restricted words defined through the distance between adjacent changes in the word. This notion is closely related to one introduced by Knopfmacher et al. [6]. In particular, they considered words such that there are no two adjacent letters with difference greater than 1. This notion was also introduced for compositions [5], integer partitions [7], and polyominoes [8].

For a positive integer k, we denote the set $\{1, 2, \dots, k\}$ by $[k]$. A *k-ary word* of length n is an element of $[k]^n$. Let $w = w_1 w_2 \cdots w_n$ be a k-ary word. We say that w has a *change* in the position i if $w_i \neq w_{i+1}$. We denote by $\mathcal{P}(w)$ the sequence of the position changes that occur in w. For example, $\mathcal{P}(331154222) = (2, 4, 5, 6)$. Let r be a positive integer, a k-ary word $w = w_1 w_2 \cdots w_n$ is *r-smooth* if either w does not change, there is exactly one change, or $p_{i+1} - p_i \leq r$ for all $p_i \in \mathcal{P}(w)$ and

$$1 \leq i \leq |\mathcal{P}(w)| - 1.$$

For example, the word $w = 331154222$ is 2-smooth. Let $\mathcal{S}_{k,n}^{(r)}$ denote the set of r-smooth k-ary words of length n and

$$\mathcal{S}_k^{(r)} := \bigcup_{n \geq 0} \mathcal{S}_{k,n}^{(r)}.$$

The main purpose of this paper is to enumerate the r-smooth words according to the length of the word and the number of changes. We use automata theory, generating functions, bijective arguments, and asymptotic analysis to describe our results.

2. Counting r-smooth words

Given a r-smooth k-ary word w, we denote the length and number of changes of w by $|w|$ and $c(w)$, respectively. We define the bivariate generating function

$$G_{k,r}(x, q) := \sum_{w \in \mathcal{S}_k^{(r)}} x^{|w|} q^{c(w)}.$$

Notice that the coefficient of $x^n q^i$ in $G_{k,r}(x, q)$ is the number of r-smooth k-ary words of length n with exactly i changes.

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Automata theory approach

In order to find an explicit formula for the generating function $G_{k,r}(x,q)$, we define the following finite automaton. Define a finite automaton

$$\mathcal{A}_{k,r} = (V_r \cup \{\alpha_r\}, [k], \delta, \epsilon, V_r)$$

by

- the states are the words in $V_r \cup \{\alpha_r\} = \{1, 12, \ldots, 12^r\} \cup \{12^{r+1}\}$;
- $[k]$ is the input alphabet;
- $\delta : V_r \times [k] \to V_r \cup \{\alpha_r\}$ is the transition function defined by $\delta(12^i, a) = 12$ with $a \in \{1, 3, 4, \ldots, k\}$, $\delta(\epsilon, a) = 1$ for all $a \in [k]$, $\delta(12^i, 2) = 12^{i+1}$ for all $i = 0, 1, \ldots, r$, $\delta(12^{r+1}, 2) = 12^{r+1}$, and $\delta(12^{r+1}, a) = \alpha_r$ for all $a \in [k] \backslash \{2\}$;
- ϵ is the initial state, where ϵ is the empty word;
- all states in V_r are final states.

We will identify $\mathcal{A}_{k,r}$ with a (labelled) directed graph $G_{k,r}$ with vertices in V_r such that there is a (labelled) edge $\tau \to^a \tau'$ between τ and τ', if $\tau' = \delta(\tau, a)$.

Theorem 2.1. *The number paths of length n starting at vertex ϵ in the directed graph $G_{k,r}$ equals the number of words in $\mathcal{S}_k^{(r)}$ with n letters.*

Proof. In order to prove the theorem, we have to show that δ is a bijection between some classes (we define them next) of the words in $\mathcal{S}_k^{(r)}$.

Define $\mathcal{S}_k^{(r)}(\tau, n)$ to be the set of all words in $\mathcal{S}_k^{(r)}$ with prefix τ and n letters. Let $\sigma = \sigma_1 \cdots \sigma_n \in \mathcal{S}_k^{(r)}$, we read σ from left to right. First, we read σ_1. By exchanging the letters 1 and σ_1 in σ, we obtain that $|\mathcal{S}_k^{(r)}(\sigma_1, n)| = |\mathcal{S}_k^{(r)}(1, n)|$, hence, we have that $\delta(\epsilon, a) = 1$ for all $a \in [k]$.

Now, we assumed we read the first $j + 1 \geq 1$ letters of $\sigma = 12^j \sigma_{j+2} \cdots \sigma_n \in \mathcal{S}_k^{(r)}(12^j, n)$, where $j = 0, 1, \ldots, r$. Here we distinguish between two cases:

- Let $\sigma_{j+2} \in [k] \backslash \{2\}$. Then the distance between first two changes of $j + 1 - 1 = j \leq r$, so by removing the first j letters of σ and exchanging the letters $2, \sigma_{j+2}$ to $1, 2$, we obtain that $|\mathcal{S}_k^{(r)}(12^j \sigma_{j+2}, n)| = |\mathcal{S}_k^{(r)}(12, n - j)|$. Hence, we have $\delta(12^j, a) = 12$ for all $a \in [k] \backslash \{2\}$ and $j \leq r$.
- Let $\sigma_{j+2} = 2$. If $j \leq r$, then clearly, $|\mathcal{S}_k^{(r)}(12^j \sigma_{j+2}, n)| = |\mathcal{S}_k^{(r)}(12^{j+1}, n)|$, thus $\delta(12^j, 2) = 12^{j+1}$. Otherwise, when $j = r + 1$, we see that the only word in $\mathcal{S}_k^{(r)}(12^{r+1}, n)$ is 12^{n-1}. Hence, $\delta(12^{r+1}, 2) = 12^{r+1}$ and $\delta(12^{r+1}, a) = \alpha_r$ for all $a \in [k] \backslash \{2\}$.

This completes the proof. □

Example 2.1. *The directed graphs $G_{3,2}$ is given in Figure 1.*

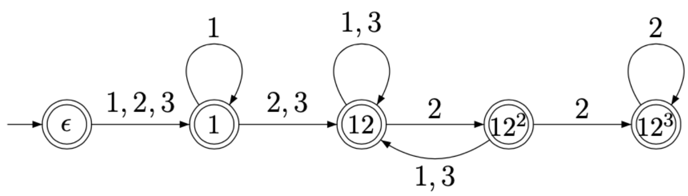

Figure 1: The directed graph $G_{3,2}$.

Let $\tilde{G}_{k,r}$ be the same directed graph $G_{k,r}$ where we weighted each edge $12^j \to^a 12^{j+1}$ by q whenever $a \neq 2$. So, as consequence of Theorem 2.1, we can state the following result.

Theorem 2.2. *The sum of weighted paths of length n starting at vertex ϵ in the weighted directed graph $\tilde{G}_{k,r}$ equals the generating function for the number of words in $\mathcal{S}_k^{(r)}$ with n letters according to the number of changes.*

Let $M_{k,r}$ be the $(r+3)\times(r+3)$ adjacency matrix of the weighted directed graph $\tilde{G}_{k,r}$. Thus,

$$M_{k,r}=\begin{pmatrix} 0 & k & 0 & 0 & \cdots & 0 & 0\\ 0 & 1 & (k-1)q & 0 & \cdots & 0 & 0\\ 0 & 0 & (k-1)q & 1 & \cdots & 0 & 0\\ & & \vdots & & \ddots & & \\ 0 & 0 & (k-1)q & & & 1 & 0\\ 0 & 0 & 0 & & & 0 & 1\end{pmatrix}$$

Let $e_r=(1,0,\dots,0)$ be a row vector and $f_r=(1,1,\dots,1)$ be a column vector of $r+3$ coordinates. Hence, by Theorem 2.1, we have

$$G_{k,r}(x,q)=e_r\left(\sum_{n\geq 0}(M_{k,r})^n x^n\right)f_r=e_r(1-xM_{k,r})^{-1}f_r.$$

Note that it is not hard to see the first row of the matrix $(1-xM_{k,r})^{-1}$ is given by

$$\Bigg(1,\frac{kx}{1-x},\frac{k(k-1)qx^2}{(1-x)(1-(k-1)q(x+\cdots+x^r))},$$
$$\dots,\frac{k(k-1)qx^{r+1}}{(1-x)(1-(k-1)q(x+\cdots+x^r))},\frac{k(k-1)qx^{r+2}}{(1-x)^2(1-(k-1)q(x+\cdots+x^r))}\Bigg).$$

Hence, we can state the following result.

Theorem 2.3. *The generating function $G_{k,r}(x,q)$ is a rational generating function given by*

$$1+\frac{kx}{1-x}+\frac{k(k-1)qx^2}{(1-x)^2(1-(k-1)q(x+\cdots+x^r))}.$$

For example, for $k=3$ and $r=1$ we obtain the generating function

$$\begin{aligned}G_{3,1}(x,q)&=\frac{1+(1-2q)x-(2-4q)x^2+4qx^3}{(1-x)^2(1-2qx)}\\ &=1+3x+(3+6q)x^2+(3+12q+12q^2)x^3+(3+18q+\mathbf{24q^2}+24q^3)x^4+\cdots\end{aligned}$$

The corresponding words to the bold coefficient in the above series are

1121, 1123, 1131, 1132, 1211, 1233, 1311, 1322, 2122, 2133, 2212, 2213,

2231, 2232, 2311, 2322, 3122, 3133, 3211, 3233, 3312, 3313, 3321, 3323.

Corollary 2.1. *The generating function of the number of changes of the r-smooth k-ary words is*

$$\left.\frac{\partial G_{k,r}(x,q)}{\partial q}\right|_{q=1}=\frac{(k-1)kx^2}{(1-kx+(k-1)x^{r+1})^2}.$$

There is an alternative way to obtain the generating function by means of the symbolic method. Remember that $\mathcal{S}_k^{(r)}$ denotes the family (combinatorial class) of all r-smooth k-ary words, then we can write a symbolic equation:

$$\mathcal{S}_k^{(r)}=\epsilon+\bigoplus_{i=1}^{k}\mathrm{SEQ}_{\geq 1}(\mathcal{Z}_i)+\bigoplus_{\substack{\ell\geq 0,\\ i_s\neq i_{s+1},1\leq s\leq\ell}}\mathrm{SEQ}_{\geq 1}(\mathcal{Z}_{i_1})\mathrm{SEQ}_{\leq r}(\mathcal{Z}_{i_2})\cdots\mathrm{SEQ}_{\leq r}(\mathcal{Z}_{i_\ell})\mathrm{SEQ}_{\geq 1}(\mathcal{Z}_{i_{\ell+1}}),$$

where $\mathcal{Z}_i$ denotes an atomic element (a symbol of the alphabet). For a general background about the symbolic method see the book [3]. In terms of generating functions, the last equation translates into

$$\begin{aligned}G_{k,r}(x,q)&=1+k\frac{x}{1-x}+\sum_{\ell\geq 0}\frac{k(k-1)x^2q}{(1-x)^2}((k-1)q)^\ell(x+x^2+\cdots+x^r)^\ell\\ &=1+k\frac{x}{1-x}+\frac{k(k-1)x^2q}{(1-x)^2(1-(k-1)q(x+x^2+\cdots+x^r))}.\end{aligned}$$

Combinatorial expressions

In the following theorems we study the number of r-smooth words of length n. This new counting sequence is denoted by $s_{k,r}(n)$.

Theorem 2.4. *The number of r-smooth k-ary words of length n is given by*

$$s_{k,r}(n) = k + k(k-1)\sum_{i=0}^{n}\sum_{\ell=0}^{n-i-2}(i+1)(k-1)^{n-\ell-i-2}\left(\!\!\binom{n-\ell-i-2}{\ell}\!\!\right)_{r-1},$$

where $\left(\!\binom{u}{v}\!\right)_r$ is the multinomial coefficient, that is

$$\left(\!\!\binom{u}{v}\!\!\right)_r = \sum_{j_1+j_2+\cdots+j_r=v}\binom{u}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{r-1}}{j_r}.$$

Proof. From Theorem 2.3 we have that for all positive integers n

$$\begin{aligned} s_{k,r}(n) &= [x^n]G_{k,r}(x,1) = [x^n]\left(\frac{kx}{1-x} + \frac{k(k-1)x^2}{(1-x)^2(1-(k-1)(x+\cdots+x^r))}\right)\\ &= k + k(k-1)[x^{n-2}]\frac{1}{(1-x)^2(1-(k-1)(x+\cdots+x^r))}\\ &= k + k(k-1)[x^{n-2}]\sum_{i\geq 0}(i+1)x^i\sum_{i\geq 0}(k-1)^i(1+x+\cdots+x^{r-1})x^i\\ &= k + k(k-1)[x^{n-2}]\sum_{i\geq 0}\sum_{j\geq 0}(i+1)(k-1)^j\sum_{\ell=0}^{(r-1)j}\left(\!\!\binom{j}{\ell}\!\!\right)_{r-1}x^{\ell+i+j}. \end{aligned}$$

This with $n-2=\ell+i+j$ implies the desired result. □

Combinatorial proof. A r-smooth k-ary word w can be factored as $w = a_1^{\alpha_1}\cdots a_\ell^{\alpha_\ell}$, where $\ell = c(w)+1$ ($c(w)$ is the number of changes of w), $a_i \in [k]$ such that $a_m \neq a_{m+1}$ for $m \in [\ell-1]$, and $\alpha_1+\cdots+\alpha_\ell = n$ with $\alpha_j \leq r$ for $1 < j < \ell-1$. That is, the r-smooth k-ary words of size n are in bijection with the union of (2ℓ)-tuples $(a_1,\ldots,a_\ell,\alpha_1,\ldots,\alpha_\ell) \in [k]^\ell \times [n]^\ell$ for $\ell \in [n]$ satisfying the properties above described. The first ℓ components of the tuple can be chosen in $k(k-1)^{\ell-1}$ ways by choosing the first letter and then choosing one of the remaining $(k-1)$ letters at each step. The second part of the tuple is a composition of n, where the first and last exponents are not restricted, and the remaining ones have to be in $[r]$. Let $i = \alpha_1+\alpha_2$, so choosing (α_1,α_ℓ) can be done in $(i-1)$ ways. The rest of the tuple is a restricted composition of $n-i$ of size $\ell-2$ with parts in $[r]$, using the principle of inclusion-exclusion choosing indices $\{x_1,\ldots,x_j\} \subseteq [\ell-2]$ such that $\alpha_{x_j} > r$, one can see that there are

$$\sum_{j=0}^{\ell-2}(-1)^j\binom{\ell-2}{j}\binom{(n-i)-j\cdot r-1}{(\ell-2)-1}$$

ways to choose them. Adding over all possible $2 \leq i \leq n$ and $3 \leq \ell \leq n$ yields

$$s_{k,r}(n) = k + k(k-1)(n-1) + \sum_{\substack{2\leq i\leq n\\ 3\leq \ell\leq n-i}} k(k-1)^{\ell-1}(i-1)\sum_{j=0}^{\ell-2}(-1)^j\binom{\ell-2}{j}\binom{(n-i)-j\cdot r-1}{(\ell-2)-1}.$$

By the change of variable $i \mapsto i+2$ and $\ell \mapsto n-i-\ell$ the result follows using that the multinomial coefficient can be evaluated with the combinatorial sum (cf. [1])

$$\left(\!\!\binom{u}{v}\!\!\right)_r = \sum_{j=0}^{\lfloor v/(r+1)\rfloor}(-1)^j\binom{u}{j}\binom{v-j(r+1)+u-1}{u-1}.$$

□

In particular, for $r=1$ and $k=2$, $s_{2,1}(n) = n(n-1)+2$ and for $k>2$

$$s_{k,1}(n) = \frac{k((k-1)^{n+1}-(k-1)(n(k-2)+5)+k^2)}{(k-2)^2}.$$

For $r=2$ and $k=2$, $s_{2,2}(n) = 2+2\sum_{i=0}^{n}(n-i)F_i = 2F_{n+3}-2n-2$, where F_n is the n-th Fibonacci number. For $r=2$ and $k>2$, we have

$$s_{k,2}(n) = k + k(k-1)\sum_{i=0}^{n}(n-i)a_k(i),$$

where $a_k(n) = (k-1)(a_k(n-1) + a_k(n-2))$ for $n \geq 2$ and the initial values $a_k(0) = 0$ and $a_k(1) = 1$.

In the following theorem we generalize the above recurrence relation. Let $a_{k,r}(n)$ be the recurrence relation of order r defined by

$$a_{k,r}(n) = (k-1)(a_{k,r}(n-1) + a_{k,r}(n-2) + \cdots + a_{k,r}(n-r)) \tag{1}$$

for $n \geq r$, and the initial values $a_{k,r}(n) = 0$ for $n \leq r-2$ and $a_{k,r}(r-1) = 1$.

Theorem 2.5. *For all $n \geq 1$ we have*

$$s_{k,r}(n) = k + (k-1)k \sum_{i=0}^{n} (n-i) a_{k,r}(i+r-2).$$

Proof. From Theorem 2.3 we have that for all positive integer n

$$s_{k,r}(n) = [x^n]G_{k,r}(x,1) = k + k(k-1)[x^{n-2}] \frac{1}{(1-x)^2(1-(k-1)(x+\cdots+x^r))}.$$

Notice that the generating function of the sequence $(a_{k,r}(n))_{n\geq 0}$ is the rational function

$$\frac{1}{1-(k-1)(x+\cdots+x^r)} = \frac{1-x}{1-kx+(k-1)x^{r+1}}.$$

Therefore, the Cauchy product implies the desired result. □

Combinatorial proof. Notice that $a_{k,r}(m+r-1)$ is the number of colored compositions of m with colors in $[k-1]$ such that each part is in $[r]$. This can be seen by considering the recursion on those colored partitions when selecting the last part of the component. Adding over all possibilities for the last part, one gets the recursion for $a_{k,r}(n)$. Using this, and the decomposition of a r-smooth k-ary word w given by $w = a_1^{\alpha_1} \ldots a_\ell^{\alpha_\ell}$ with $a_j \neq a_{j+1}$ and $(\alpha_1, \ldots, \alpha_\ell)$ a composition of n with $\alpha_1 + \alpha_\ell = i$, one can add over all possible choices of $i = \alpha_1 + \alpha_\ell$ which gives

$$\begin{aligned} s_{k,r}(n) &= k + k(k-1)\sum_{i=2}^{n}(i-1)a_{k,r}(n-i+r-1) \\ &= k + k(k-1)\sum_{i=0}^{n} i \cdot a_{k,r}(n-i+r-2), \end{aligned}$$

where $k(k-1)$ is multiplied for choosing the first and last letters from $[k]$. The results follow from the change of variable $i \mapsto n-i$. □

Table 1 shows the number of r-smooth k-ary words for small values of r and k.

r	k	n	1	2	3	4	5	6	7	8	9	10
1	2		2	4	8	14	22	32	44	58	74	92
1	3		3	9	27	69	159	345	723	1485	3015	6081
1	4		4	16	64	220	700	2152	6520	19636	58996	177088
2	2		2	4	8	16	30	54	94	160	268	444
2	3		3	9	27	81	231	645	1779	4881	13359	36525
2	4		4	16	64	256	988	3772	14332	54376	206200	781816
3	2		2	4	8	16	32	62	118	222	414	768
3	3		3	9	27	81	243	717	2103	6153	17979	52509
3	4		4	16	64	256	1024	4060	16060	63484	250876	991336

Table 1: Values of $s_{k,r}(n)$ for $1 \leq n \leq 10$, $2 \leq k \leq 4$, and $1 \leq r \leq 3$.

Corollary 2.2. *The number of r-smooth k-ary words of length n with m changes $(m \geq 1)$ is given by*

$$s_{k,r}(n,m) = k(k-1)^m \sum_{i=0}^{n}(i+1)\left(\!\!\binom{m-1}{n-m-i-1}\!\!\right)_{r-1}.$$

Asymptotic analysis

In Theorem 2.6 we give an asymptotic expression for the number r-smooth k-ary words. Before, we need some auxiliary results.

Lemma 2.1. *For integers $k > 1$ and $r \geq 1$, the polynomial*

$$p_{k,r}(x) = 1 - (k-1)(x + x^2 + \cdots + x^r)$$

has an unique simple positive zero of smallest modulus $1/\Phi_{k-1,r}$, where

$$\Phi_{k-1,r} = \lim_{n\to\infty} \frac{a_{k,r}(n+1)}{a_{k,r}(n)}$$

and $a_{k,r}(n)$ is the sequence defined in (1).

Proof. The recurrence relation $(a_{k,r}(n))_{n\geq 0}$ has characteristic polynomial

$$x^r - (k-1)(x^{r-1} + \cdots + 1) = p_{k,r}(x)^R,$$

where $p_{k,r}(x)^R$ denotes the reciprocal polynomial of $p_{k,r}(x)$. From a result of Ostrowski [10], the polynomial $p_{k,r}(x)^R$ is asymptotically simple. Therefore, this polynomial has the unique simple positive dominant root $\Phi_{k-1,r}$. Szczyrba et al. [12] showed that

$$\Phi_{k-1,r} = \lim_{n\to\infty} \frac{a_{k,r}(n+1)}{a_{k,r}(n)}, \quad n > n_0,$$

where n_0 is the largest index for which $a_{k,r}(n) = 0$. Since β is a root of a polynomial $p(x)$ if and only if β^{-1} is a root of $p(x)^R$, we conclude that $1/\Phi_{k-1,r}$ is the unique simple positive root of $p_{k,r}(x)$ of smallest modulus. □

We also need the following for the asymptotics of linear recurrences relations (see [11]). Assume that a rational generating function $f(x)/g(x)$, with $f(x)$ and $g(x)$ relatively prime and $g(0) \neq 0$, has a unique pole $1/\beta$ with the smallest modulus. Then, if the multiplicity of $1/\beta$ is ν, we have

$$[x^n]\frac{f(x)}{g(x)} \sim \nu \frac{(-\beta)^\nu f(1/\beta)}{g^{(\nu)}(1/\beta)} \beta^n n^{\nu-1}. \tag{2}$$

Theorem 2.6. *The number $s_{k,r}(n)$ of r-smooth k-ary words of length n is asymptotically given by*

$$k + k(k-1)\frac{(\Phi_{k-1,r})^{n-1}}{h_{k,r}(1/\Phi_{k-1,r})}, \quad \textit{as} \quad n \to \infty,$$

where

$$h_{k,r}(x) = 1 + k - 2kx - (r+1)(k-1)x^r + (r+2)(k-1)x^{r+1}.$$

Proof. The unique pole $1/\beta$ of the rational generating function

$$\frac{1}{(1-x)^2(1-(k-1)(x+\cdots+x^r))}$$

is the smallest real positive root of $p_{k,r}(x)$. From the Lemma 2.1 we have $1/\beta = 1/\Phi_{k-1,r}$. From the equality

$$s_{k,r}(n) = k + k(k-1)[x^{n-2}]\frac{1}{(1-x)^2(1-(k-1)(x+\cdots+x^r))}$$

and (2) we conclude the desired result. □

For example, for $k = 5$ and $r = 3$ we have

$$s_{5,3}(n) \sim 5 + 20\frac{(\Phi_{4,3})^{n-1}}{6 - 10x - 16x^3 + 20x^4}\Big|_{x=1/\Phi_{4,3}} := b_{5,3}(n)$$

and

$$\Phi_{4,3} = \lim_{n\to\infty} \frac{a_{5,3}(n+1)}{a_{5,3}(n)} \approx 4.9673651413960243510.$$

Table 2 shows some numerical values.

From a similar argument as in Theorem 2.6 we can prove the following result.

Theorem 2.7. *The number of changes of the r-smooth k-ary words of length n is asymptotically given by*

$$k(k-1)n\frac{(\Phi_{k-1,r})^n}{g_{k,r}(1/\Phi_{k-1,r})}, \quad \textit{as} \quad n \to \infty,$$

where

$$g_{k,r}(x) = (k - ((k-1)(r+1))x^r)^2 + (k-1)r(r+1)x^{r-1}(1 - x^{r+1} - kx(1-x^r)).$$

n	$s_{5,3}(n)$	$b_{5,3}(n)$	$s_{5,3}(n)/b_{5,3}(n)$
1	5	10.142	2.028497050
2	25	30.545	1.221784079
3	125	131.89	1.055114923
4	625	635.31	1.016489297
5	3125	3135.96	1.003506916
6	15545	15557.6	1.000811653
7	77245	77260.5	1.000201026
8	383745	383761.4	1.000042784
9	1906245	1906263.2	1.000009576
10	9469065	9469085.8	1.000002197

Table 2: Some numerical values of the sequences $s_{5,3}(n)$ and $b_{5,3}(n)$.

The r-rough words

Changing the inequality that defines r-smooth words, one gets r-*rough words*, i.e., k-ary words of length n such that if $\mathcal{P}(w) = (p_1, \dots, p_{c(w)})$ then either $c(w) \leq 1$, or $p_{i+1} - p_i \geq r$ for $i \in [c(w) - 1]$. The set of r-rough k-ary words of size n is denoted as $\mathcal{T}_{k,n}^{(r)}$, and $\mathcal{T}_k^{(r)} = \bigcup_{n\geq 0} \mathcal{T}_{k,n}^{(r)}$. For example, for $k = 3$ and $r = 2$, the 33 2-rough words of length 4 are

1111, 1112, 1113, 1122, 1133, 1221, 1222, 1223, 1331, 1332, 1333,

2111, 2112, 2113, 2211, 2221, 2222, 2223, 2233, 2331, 2332, 2333,

3111, 3112, 3113, 3221, 3222, 3223, 3311, 3322, 3331, 3332, 3333.

Theorem 2.8. *The number of r-rough k-ary words of length n is given by*

$$t_{k,r}(n) = k \sum_{\ell=0}^{\left\lfloor \frac{n+r-2}{r} \right\rfloor} (k-1)^\ell \binom{n - (\ell-1)(r-1) - 1}{\ell}.$$

Proof. As in the proof of Theorem 2.4, one can understand a r-rough k-ary word $w \in [k]^n$ as two tuples $(a_1, \dots, a_\ell)$ and $(\alpha_1, \dots, \alpha_\ell)$ with $\alpha_1 + \alpha_\ell = i$, $a_j \neq a_{j+1}$ for $j \in [\ell - 1]$, and $\alpha_j \geq r$ for $1 < j < \ell$. Using stars and bars, one can see that if ℓ and i are fixed, the number of exponents are given by $\binom{((n-i)-(\ell-2)(r-1))-1}{(\ell-2)-1}$. Adding over all possibilities of ℓ and i gives

$$t_{k,r}(n) = k + k(k-1)(n-1) + \sum_{\substack{2\leq i\leq n\\ 3\leq \ell\leq n-i}} k(k-1)^{\ell-1}(i-1)\binom{(n-i) - (\ell-2)(r-1) - 1}{\ell - 3}.$$

Using the Chu-Vandermonde identity, one gets that the sum over i gives

$$\sum_{2\leq i\leq n} \binom{i-1}{1}\binom{(n - (\ell-2)(r-1) - 2) - (i-1)}{\ell - 3} = \binom{n - (\ell-2)(r-1) - 2 + 1}{((\ell-3)+1)+1}.$$

The final summation is given when applied the change of variable $\ell \mapsto \ell + 1$. □

Notice that k divides $t_{k,r}(n)$. The sequence $t_{k,r}(n)/k$ has been studied before (cf. [2]).

Finally, we define the bivariate generating function

$$T_{k,r}(x, q) := \sum_{w\in\mathcal{T}_k^{(r)}} x^{|w|} q^{c(w)}.$$

We can write a symbolic equation for the combinatorial class $\mathcal{T}_k^{(r)}$:

$$\mathcal{T}_k^{(r)} = \epsilon + \bigoplus_{i=1}^{k} \mathbf{SEQ}_{\geq 1}(\mathcal{Z}_i) + \bigoplus_{\substack{\ell\geq 0,\\ i_s\neq i_{s+1}, 1\leq s\leq \ell}} \mathbf{SEQ}_{\geq 1}(\mathcal{Z}_{i_1})\mathbf{SEQ}_{\geq r}(\mathcal{Z}_{i_2})\cdots\mathbf{SEQ}_{\geq r}(\mathcal{Z}_{i_\ell})\mathbf{SEQ}_{\geq 1}(\mathcal{Z}_{i_{\ell+1}}),$$

where $\mathcal{Z}_i$ denotes an atomic element (a symbol of the alphabet). In terms of generating functions, the last equation translates into

$$\begin{aligned} T_{k,r}(x, q) &= 1 + k\frac{x}{1-x} + \sum_{\ell\geq 0} \frac{k(k-1)x^2 q}{(1-x)^2} ((k-1)q)^\ell (x^r + x^{r+1} + \cdots)^\ell \\ &= 1 + k\frac{x}{1-x} + \frac{k(k-1)x^2 q}{(1-x)^2(1 - (k-1)q(x^r + x^{r+1} + \cdots))} \\ &= 1 + k\frac{x}{1-x} + \frac{k(k-1)x^2 q}{(1-x)(1 - x - (k-1)qx^r)}. \end{aligned}$$

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References

[1] H. Belbachir, S. Bouroubi, A. Khelladi, Connection between ordinary multinomials, Fibonacci numbers, Bell polynomials and discrete uniform distribution, *Ann. Math. Inform.* **35** (2008) 21–30.

[2] D. Birmajer, J. B. Gil, M. D. Weiner, On the enumeration of restricted words over a finite alphabet, *J. Integer Seq.* **19** (2016) #16.1.3.

[3] P. Flajolet, R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, Cambridge, 2009.

[4] S. Kitaev, *Patterns in Permutations and Words*, Monographs in Theoretical Computer Science, Springer, 2011.

[5] A. Knopfmacher, T. Mansour, A. Munagi, Smooth compositions and smooth words, *Pure Math. Appl. (P.U.M.A)* **11** (2011) 209–226.

[6] A. Knopfmacher, T. Mansour, A. Munagi, H. Prodinger, Staircase words and Chebyshev polynomials, *Appl. Anal. Discrete Math.* **4** (2010) 81–95.

[7] T. Mansour, Smooth partitions and Chebyshev polynomials, *Bull. Lond Math. Soc.* **41** (2010) 961–970.

[8] T. Mansour, A. S. Shabani, Smooth column convex polyominoes, *Discrete Comput. Geom.* **68** (2022) 525–539.

[9] T. Mansour, M. Shattuck, Counting inversion sequences by parity successions and runs, *Discrete Appl. Math.* **330** (2023) 24–39.

[10] A. M. Ostrowski, *Solutions of Equations and Systems of Equations*, Academic Press, 1966.

[11] R. Sedgewick, P. Flajolet, *An Introduction to the Analysis of Algorithms*, 2nd Edition, Addison-Wesley, 2013.

[12] I. Szczyrba, R. Szczyrba, M. Burtscher, Analytic representations of the n-anacci constants and generalizations thereof, *J. Integer Seq.* **18** (2015) #15.4.5.