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Research Article

Principal minors of Hermitian (quasi-)Laplacian matrix of second kind for mixed graphs

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Abstract

A mixed graph M_G is obtained from an unoriented simple graph G by giving directions to some edges of G. The Hermitian adjacency matrix of the second kind of M_G is defined as $N(M_G) = (n_{ij})$, in which $n_{ij} = \overline{n_{ji}} = \frac{1+\sqrt{3}\mathbf{i}}{2}$ if $v_i \to v_j$, $n_{ij} = 1$ if $v_i \leftrightarrow v_j$ and 0 otherwise. The Hermitian Laplacian matrix (Hermitian quasi-Laplacian matrix) of the second kind of M_G is defined as $L(M_G) = D(M_G) - N(M_G)$ ($Q(M_G) = D(M_G) + N(M_G)$, respectively), where $D(M_G)$ is the degree diagonal matrix of the underlying graph G of M_G. In this paper, we derive some necessary and sufficient conditions for the singularity of $L(M_G)$ and $Q(M_G)$. We also characterize the principal minor version of Matrix-Tree theorem based on $L(M_G)$ and $Q(M_G)$. As a consequence, we give the explicit expressions for the determinants of two matrices $L(M_G)$ and $Q(M_G)$ for M_G.

Keywords: Hermitian (quasi-)Laplacian matrix; Matrix-Tree theorem; mixed graph.

2020 Mathematics Subject Classification: 05C50, 15A18.

1. Introduction

We only consider simple and finite connected graphs in this paper. Let G be an unoriented graph with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E, in which $|V| = n$ and $|E| = m$. The *degree* d_i of a vertex v_i in G is the number of edges incident with v_i. A *mixed graph* M_G is obtained from an unoriented graph G by giving directions to some edges in G. So G is often referred to the *underlying graph* of M_G. A mixed $M_{G'}$ is called a *mixed subgraph* of M_G if G' is a subgraph of G and the direction of all the edges of $M_{G'}$ is the same as in M_G. For two adjacent vertices v_i and v_j of M_G, we denote an oriented edge from v_i to v_j by $v_i \to v_j$ or $v_j \leftarrow v_i$. Similarly, we denote by $v_i \leftrightarrow v_j$ an unoriented edge between v_i and v_j. Usually, we also use M_C to represent a mixed cycle.

Until now the research on spectral theories of mixed graphs has been paid more and more attention. Recall that, for a mixed graph M_G, its *Hermitian adjacency matrix of the first kind* $H(M_G) = (h_{ij})$ was proposed by Guo and Mohar [5] and independently by Liu and Li [8], in which $h_{ij} = \mathbf{i}$ if $v_i \to v_j$, $h_{ij} = -\mathbf{i}$ if $v_i \leftarrow v_j$, 1 if $v_i \leftrightarrow v_j$, and 0 otherwise. For this matrix associated to the mixed graph, some basic spectral properties were also established in [5, 8]. In 2015, Yu and Qu [15] introduced the incident matrix and Hermitian Laplacian matrix of the first kind for a mixed graph M_G and determined the positive of M_G. Subsequently, Yu et al. [14] characterized the singularity of the Hermitian (quasi-)Laplacian matrix of the first kind for M_G and gave the concise expressions of the determinants of these two matrices. In 2017, an analytical expression for the principal minors of the Hermitian (quasi-)Laplacian matrix of the first kind was derived in [16].

Recently, a *Hermitian adjacency matrix of the second kind*, denoted by $N(M_G)$, for mixed graphs was proposed by Mohar [9], in which n_{ij} is the sixth root of unity $\omega = \frac{1+\sqrt{3}\mathbf{i}}{2}$ if $v_i \to v_j$, $\overline{\omega} = \frac{1-\sqrt{3}\mathbf{i}}{2}$ if $v_i \leftarrow v_j$, 1 if $v_i \leftrightarrow v_j$, and 0 otherwise. It is clear that $N(M_G)$ is Hermitian, so its eigenvalues are all real. At the same time, Mohar [9] pointed out the necessity of studying this novel matrix and gave some basic spectral results in spectral graph theory. The main reason is that the sixth root of unity satisfies $\omega \cdot \overline{\omega} = 1$ and $\omega + \overline{\omega} = 1$, which makes it more natural to study the relationship between eigenvalues and combinatorial properties. Moreover, the sixth root of unity also appears across applications, such as in the definition Eisenstein integers [10, 13], Quantum Field Theory [6] and so on. In 2022, Li and Yu [7] studied the characteristic polynomial of this matrix and obtained an upper bound on its spectral radius. Using switching equivalence, they also studied properties of mixed graphs that are cospectral based on this matrix. For more details about this matrix, we refer readers to [7, 9].

In this paper we give two incidence matrices of the second kind for a mixed graph M_G, denoted by $S(M_G)$ and $T(M_G)$ hereinafter. Using these two matrices, we introduce the concept of *Hermitian Laplacian matrix of the second kind* (resp.

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Hermitian quasi-Laplacian matrix of the second kind) for a mixed graph M_G, which is represented by $L(M_G) = D(M_G) - N(M_G)$ (resp. $Q(M_G) = D(M_G) + N(M_G)$), where $D(M_G)$ is the degree diagonal matrix of M_G. Bapat et al. [1] proposed a real Laplacian matrix for mixed graphs and obtained matrix tree theorem of mixed graphs in 1999. Inspired by this research and [16], we characterize the principal minor version of the matrix tree theorem based on the (quasi-)Hermitian Laplacian matrix of the second kind for M_G. In addition, we also give the explicit expressions about the determinants of these two matrices $L(M_G)$ and $Q(M_G)$ for the mixed graph M_G and present some necessary and sufficient conditions for the Hermitian Laplacian matrix of the second kind $L(M_G)$ to be singular.

It is worth noting that the spectral theories of mixed graphs arising from the Hermitian matrices of first and second kind can be both embodied in the theory of gain graphs. The Hermitian matrix of the first kind for a mixed graph M_G is the (Hermitian) adjacency matrix associated to the gain graph Γ' obtained from G by choosing the weight $e^{\frac{\pi}{2}\mathbf{i}}$ for each oriented edge v_iv_j, and the weight 1 for all the unoriented edges. Similarly, the Hermitian matrix of the second kind for a mixed graph M_G is the (Hermitian) adjacency matrix associated to the gain graph Γ'' obtained from G by choosing the weight $e^{\frac{\pi}{6}\mathbf{i}}$ for each oriented edge v_iv_j, and the weight 1 for all the unoriented edges. It turns out that many results in this paper could be alternatively deduced from the correspondent results involving complex unit gain graphs and quaternion unit gain graphs (see [2–4, 11, 12]). Our proofs ignore the overlapping of the two contexts; thus, no knowledge on gain graphs is required to understand them.

2. Hermitian Laplacian matrix of the second kind for mixed graphs

Suppose that M_G is a mixed graph of order n and size m. An *incidence matrix of the second kind* of M_G is an $n \times m$ matrix $S(M_G) = (s_{ie})$ with entries

$$s_{ie} = \begin{cases} -s_{je}, & \text{if } v_i \leftrightarrow v_j; \\ -\frac{1+\sqrt{3}\mathbf{i}}{2}s_{je}, & \text{if } v_i \rightarrow v_j; \\ -\frac{1-\sqrt{3}\mathbf{i}}{2}s_{je}, & \text{if } v_i \leftarrow v_j; \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

where s_{je} is a complex number such that $|s_{je}| = 1$. Notice that we say "an" incidence matrix of the second kind of M_G because $S(M_G)$ is not unique.

The following lemma is a particular case of Lemma 3.1 in [11]. Considering that the details of the proof are different, we give a complete proof below.

Lemma 2.1. *Let M_G be a mixed graph of order n with edge set E. Then $L(M_G) = S(M_G)S(M_G)^*$, where $S(M_G)$ is as described in (1) and $S(M_G)^*$ denotes the conjugate transpose of $S(M_G)$.*

Proof. First we let $S(M_G)S(M_G)^* = (\sigma_{ij})_{n\times n}$. It is easy to see that $\sigma_{ij} = \sum_{e\in E} s_{ie} \cdot \overline{s_{je}}$. And then we just need to compare the entries between two matrices $L(M_G)$ and $S(M_G)S(M_G)^*$. We divide it into the following two cases.

Case 1: $i \neq j$. For two vertices v_i and v_j, $\sigma_{ij} = 0$ if v_i is not adjacent to v_j. If v_i is adjacent to v_j, then $\sigma_{ij} = s_{ie}\overline{s_{je}}$. Now we consider this value according to the direction of the edge between v_i and v_j. If $v_i \leftrightarrow v_j$, then $s_{ie} = -s_{je}$ and $\sigma_{ij} = s_{ie}\overline{s_{je}} = -1 = -n_{ij}$. If $v_i \rightarrow v_j$, then $s_{ie} = -\frac{1+\sqrt{3}\mathbf{i}}{2}s_{je}$ and $\sigma_{ij} = -\frac{1+\sqrt{3}\mathbf{i}}{2} \cdot s_{je}\overline{s_{je}} = -\frac{1+\sqrt{3}\mathbf{i}}{2} = -n_{ij}$. Finally, if $v_i \leftarrow v_j$, then $s_{ie} = -\frac{1-\sqrt{3}\mathbf{i}}{2}s_{je}$ and $\sigma_{ij} = -\frac{1-\sqrt{3}\mathbf{i}}{2} \cdot s_{je}\overline{s_{je}} = -\frac{1-\sqrt{3}\mathbf{i}}{2} = -n_{ij}$.

Case 2: $i = j$. In this case, it is clear that $\sigma_{ij} = \sum_{e\in E} s_{ie}\overline{s_{je}} = \sum_{e\in E} |s_{ie}|^2 = d_i = n_{ii}$.

Based on the discussion above, we obtain $L(M_G) = S(M_G)S(M_G)^*$. □

The above lemma further implies that $L(M_G)$ is a positive semidefinite matrix.

Let $W = v_{i_1}v_{i_2}\cdots v_{i_k}$ be a mixed walk of length k, and its *weight* is denoted by $n(W) = n_{i_1i_2}n_{i_2i_3}\cdots n_{i_{k-1}i_k}$, in which n_{ij} is the element in the i-th row and j-th column of the Hermitian adjacency matrix of the second kind. It is easy to verify that, if the weight of W is λ for one given direction, then the weight is $\overline{\lambda}$ after the directions of all edges are reversed. Furthermore, let $n(M_C) = n_{12}n_{23}\cdots n_{n-1,n}n_{n1}$ be the weight of a mixed cycle M_C with n vertices. Next we consider four different types of mixed cycles depending on their weights. If $n(M_C) = 1$, then M_C is called a *positive cycle*. If $n(M_C) = -1$, then M_C is called a *negative cycle*. Similarly, if $n(M_C) = \frac{1\pm\sqrt{3}\mathbf{i}}{2}$, then M_C is called a *semi-positive cycle*. If $n(M_C) = \frac{-1\pm\sqrt{3}\mathbf{i}}{2}$, then M_C is called a *semi-negative cycle*. Moreover, we call a mixed graph M_G to be *positive* whenever each mixed cycle of M_G is positive.

Lemma 2.2. *Let M_G be a mixed graph with n vertices and m edges. Then the matrix $L(M_G)$ is singular if and only if any walk $W = v_{i_1}v_{i_2}\cdots v_{i_k}$ has the same weight for $1 \le k \le n$. And at this time, there exists an eigenvector $\eta = (1, \overline{n(W_{12})}, \overline{n(W_{13})}, \ldots, \overline{n(W_{1n})})^\top$ such that $L(M_G)\eta = 0$, where W_{1k} is a walk from v_1 to v_k in M_G.*

Proof. First, assume that the matrix $L(M_G)$ is singular. Then there exists a non-zero vector $\xi^\top = (\xi_1, \xi_2, \ldots, \xi_n)$ such that $L(M_G)\xi = 0$, which implies that $S(M_G)^*\xi = 0$. Now, by some calculations, one has

$$(S(M_G)^*\xi)_e = \overline{s_{ie}}\xi_i + \overline{s_{je}}\xi_j = \begin{cases} (\xi_i - \xi_j)\overline{s_{ie}}, & \text{if } v_i \leftrightarrow v_j; \\ (\xi_i - \frac{1+\sqrt{3}\mathbf{i}}{2}\xi_j)\overline{s_{ie}}, & \text{if } v_i \to v_j; \\ (\xi_i - \frac{1-\sqrt{3}\mathbf{i}}{2}\xi_j)\overline{s_{ie}}, & \text{if } v_i \leftarrow v_j. \end{cases} \tag{2}$$

Since $S(M_G)^*\xi = 0$, then $\xi_i = n_{ij}\xi_j$ for each edge $e = v_iv_j$ of M_G. It follows that

$$\xi_{i_1} = n_{i_1i_2}\xi_{i_2} = n_{i_1i_2}n_{i_2i_3}\xi_{i_3} = \cdots = n_{i_1i_2}n_{i_2i_3}\cdots n_{i_{k-1}i_k}\xi_{i_k} = n(W_{i_1i_k})\xi_{i_k}. \tag{3}$$

Hence, any walk $W = v_{i_1}v_{i_2}\cdots v_{i_k}$ has the same weight for $1 \le k \le n$. Furthermore,

$$\xi^\top = (\xi_1, \xi_2, \ldots, \xi_n) = (\xi_1, \overline{n(W_{12})}\xi_1, \overline{n(W_{13})}\xi_1, \ldots, \overline{n(W_{1n})}\xi_1) = \xi_1\eta^\top. \tag{4}$$

This also implies that $\xi_1 \neq 0$ and $L(M_G)\eta = \frac{1}{\xi_1}S(M_G)S(M_G)^*\xi = 0$.

Conversely, since any walk $W = v_{i_1}v_{i_2}\cdots v_{i_k}$ has the same weight for $1 \le k \le n$, then we may let $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n)^\top$ be a vector such that $\zeta_i = n_{ij}\zeta_j$ for each edge $e = v_iv_j$. Thus, it is easy to see that

$$\zeta^* L(M_G)\zeta = \sum_{e \in E(M_G)} |\zeta_i - n_{ij}\zeta_j|^2 = \sum_{e \in E(M_G)} |n_{ij}\zeta_j - n_{ij}\zeta_j|^2 = 0. \tag{5}$$

Therefore, $L(M_G)$ is a singular matrix. □

Observe that, by some careful checking, we find that Theorem 4 in [15] still holds in our discussion. Thus we have the following theorem from Theorem 4 in [15] and Lemma 2.2. Remark that we can also draw this theorem from Proposition 2.1 in [4].

Theorem 2.1. *A mixed graph M_G is positive if and only if $L(M_G)$ is singular.*

The following lemma can be easily deduced from Lemma 6.7 in [3], but here we give a different proof according to the method of determinant expansion.

Lemma 2.3. *Let M_C be a mixed cycle with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_n\}$. Then* $\det L(M_C) = 2 - (n(M_C) + \overline{n(M_C)})$.

Proof. Remark that the proof is similar to that of Theorem 6 in [14]. Without loss of generality, assume that the edge $e_i = v_iv_{i+1}$ for $1 \le i \le n-1$ and $e_n = v_nv_1$ in the underlying graph of M_C. Clearly, $S(M_C)$ is a square matrix. Then expanding along the first row, one has

$$\det S(M_C) = \prod_{i=1}^{n} s_{ie_i} + (-1)^{n+1}s_{1e_n}\prod_{i=2}^{n} s_{ie_{i-1}}, \tag{6}$$

and

$$\det S(M_C)^* = \prod_{i=1}^{n} \overline{s_{ie_i}} + (-1)^{n+1}\overline{s_{1e_n}}\prod_{i=2}^{n} \overline{s_{ie_{i-1}}}. \tag{7}$$

From the definition of $S(M_C)$, one has $s_{ie}\overline{s_{je}} = -n_{ij}$ for any edge $e = v_iv_j$. Thus it follows from (6) and (7) that

$$\begin{aligned} \det L(M_C) &= \det S(M_C) \cdot \det S(M_C)^* \\ &= 2 + (-1)^{n+1}\overline{s_{1e_n}}\prod_{i=1}^{n} s_{ie_i}\prod_{i=2}^{n}\overline{s_{ie_{i-1}}} + (-1)^{n+1}s_{1e_n}\prod_{i=2}^{n} s_{ie_{i-1}}\prod_{i=1}^{n}\overline{s_{ie_i}} \\ &= 2 + (-1)^{n+1}s_{1e_1}\overline{s_{2e_1}}\cdots s_{ne_n}\overline{s_{1e_n}} + (-1)^{n+1}s_{2e_1}\overline{s_{1e_1}}\cdots s_{1e_n}\overline{s_{ne_n}} \\ &= 2 + (-1)^{n+1}(-1)^n n_{12}n_{23}\cdots n_{n1} + (-1)^{n+1}(-1)^n n_{21}\cdots n_{1n} \\ &= 2 - [n(M_C) + \overline{n(M_C)}]. \end{aligned}$$

This completes the proof. □

The following statement is analogous to an existing result related to the Hermitian Laplacian matrix of the first kind of mixed graphs (see Theorem 5 in [14]), and it can also be obtained from Proposition 2.1 in [4].

Theorem 2.2. *Let M_G be a connected mixed graph. Then $L(M_G)$ is singular if and only if $L(M_C)$ is singular for every mixed cycle M_C in M_G. In particular, the Hermitian Laplacian matrix of the second kind of a mixed tree is always singular.*

Proof. Lemma 2.3 implies that, for a mixed cycle M_C, $L(M_C)$ is singular if and only if M_C is positive. Then, from Theorem 2.1, we get the required result. □

Combining the proof of Theorem 3.1 in [2] with Lemma 2.2 in [12], we can easily get the following conclusion, so we ignore this proof here.

Theorem 2.3. *Suppose that M_G is a mixed unicyclic graph with n vertices and M_C is the unique cycle in M_G. Then* $\det L(M_G) = \det L(M_C) = 2 - [n(M_C) + \overline{n(M_C)}]$.

Next we turn our attention to the principal minor version of matrix tree theorem in Hermitian Laplacian matrix of the second kind for mixed graphs. Before that, we introduce some definitions, which are similar to the ones in [1]. A subgraph Γ of a connected mixed graph M_G is called *essential Laplacian spanning subgraph* of M_G if M_G is positive and Γ is a spanning tree of its underlying graph G; or M_G is not positive, Γ has the same vertices as M_G, its components are unicyclic mixed graphs and the Hermitian Laplacian matrix of the second kind of each cycle is non-singular. An *r-reduced Laplacian spanning substructure* R of M_G is defined as follows: R is a subgraph of M_G with $n-r$ vertices and no Laplacian singular cycles, and the number of vertices and edges in each component of R is the same. It is clear that each component of R is either a rootless tree or a Laplacian non-singular unicyclic graph. Furthermore, we can establish a one-to-one correspondence between R and Γ, in which one vertex is deleted in each positive component of Γ.

The following two lemmas and their proofs are similar to Lemma 2 and Theorem 1 in [16], we ignore some details here.

Lemma 2.4 (see [16]). *For a mixed graph M_G, if a substructure R of M_G is a rootless tree, then* $|\det S(R)| = 1$.

Lemma 2.5 (see [16]). *Let R be a substructure of a connected mixed graph with equal number of vertices and edges. Then the following statements hold:*

1. *If there exists a component of R with distinct numbers of vertices and edges, then* $\det S(R) = 0$.
2. *If every component R has an equal number of vertices and edges, then every component of R is an unicyclic graph or a rootless tree.*
3. *If some component of R is a Laplacian singular unicyclic graph, then* $\det S(R) = 0$*; Otherwise,* $\det S(R) \neq 0$.

Given a mixed graph M_G with vertex set V, let $L[V_{n-r}, V_{n-r}]$ denote principal submatrix of $L(M_C)$ relative to vertex subset V_{n-r}, where V_{n-r} is obtained by deleting r vertices in V.

Theorem 2.4. *Let M_G be a mixed graph with vertex set V and edge set E. Then*

$$\det L[V_{n-r}, V_{n-r}] = \sum_R 3^{\epsilon_1(R)} \cdot 4^{\epsilon_2(R)},$$

where the summation runs over all r-reduced Laplacian spanning substructures R of M_G, and $\epsilon_1(R), \epsilon_2(R)$ are the numbers of semi-negative cycles, negative cycles of R, respectively.

Proof. According to the Cauchy-Binet Theorem and $L(M_G) = S(M_G)S(M_G)^*$, we have

$$\det L[V_{n-r}, V_{n-r}] = \sum_{E_{n-r}} \det S[V_{n-r}, E_{n-r}] \cdot \det S[V_{n-r}, E_{n-r}]^* = \sum_R |\det S(R)|^2,$$

where E_{n-r} is a subset of E with $n-r$ edges and R is the substructure with the pair (V_{n-r}, E_{n-r}).

In the following, we consider the contribution of a substructure R to $\sum_R |\det S(R)|^2$. Since each component of R is either a rootless tree, or a Laplacian non-singular unicyclic graph. Thus we will discuss the following two cases.

Case 1: A rootless tree, denoted by R', is a component of R. In this case, we have $|\det S(R')| = 1$ by Lemma 2.4.

Case 2: A non-singular unicyclic graph, denoted by R'', is a component of R. Now let M_C be the cycle in R''. Theorem 2.3 implies that $\det L(R'') = \det L(M_C) = 2 - [n(M_C) + \overline{n(M_C)}]$. If M_C is a positive cycle, $n(M_C) = 1$, then $\det L(R'') = 0$. If M_C is a negative cycle, $n(M_C) = -1$, then $\det L(R'') = 4$. If M_C is a semi-positive cycle, $n(M_C) = \frac{1 \pm \sqrt{3}\mathbf{i}}{2}$, then $\det L(R'') = 1$. Finally, if M_C is a semi-negative cycle, $n(M_C) = \frac{-1 \pm \sqrt{3}\mathbf{i}}{2}$, then $\det L(R'') = 3$.

Based on the discussion above, we get that a rootless tree contributes 1, and each non-singular unicyclic graph contributes 1, 3 or 4 to the value of determinant. Hence, the required result follows. □

Theorem 2.5. *Let M_G be a connected mixed graph with vertex set V and edge set E, then*

$$\det L(M_G) = \sum_{\tau_1,\tau_2} 3^{\tau_1} 4^{\tau_2} q_{\tau_1,\tau_2},$$

where q_{τ_1,τ_2} is the number of essential Laplacian spanning subgraphs which contain τ_1 semi-negative cycles, τ_2 negative cycles and and we stipulate that $q_{0,0} = 0$.

Proof. If M_G is a mixed tree, then the result holds by Theorem 2.2. Next assume that M_G contains mixed cycles. Then the Cauchy-Binet Theorem implies that $L(M_G) = S(M_G)S(M_G)^* = \sum_{E'} \det S[V,E'] \cdot \det S[V,E']^*$, in which E' is a subset of E with $|E'| = |V|$. It can be verified that all the subgraphs with vertex set V and edge set E' belong to elementary Laplacian spanning subgraph Γ. Hence the required result follows by Theorem 2.4. □

3. Hermitian quasi-Laplacian matrix of the second kind for mixed graphs

Suppose that M_G is a mixed graph of order n and size m. A *quasi-incidence matrix of the second kind* of M_G is an $n \times m$ matrix $T(M_G) = (t_{ie})$ with entries

$$t_{ie} = \begin{cases} t_{je}, & \text{if } v_i \leftrightarrow v_j;\\ \frac{1+\sqrt{3}\mathbf{i}}{2} t_{je}, & \text{if } v_i \to v_j;\\ \frac{1-\sqrt{3}\mathbf{i}}{2} t_{je}, & \text{if } v_i \leftarrow v_j;\\ 0, & \text{otherwise,} \end{cases} \tag{8}$$

where t_{je} is a complex number such that $|t_{je}| = 1$. Notice that we say "an" quasi-incidence matrix of the second kind of M_G as $T(M_G)$ is not unique.

Similar to the proof of Lemma 2.1, we obtain the following lemma.

Lemma 3.1. *Let M_G be a mixed graph of order n with edge set E. Then $Q(M_G) = T(M_G)T(M_G)^*$ is a positive semidefinite matrix, where $Q(M_G)$ is as described in (8).*

Theorem 3.1. *Let M_G be a mixed graph with n vertices and m edges. Then the matrix $Q(M_G)$ is singular if and only if all walks $W = v_{i_1}v_{i_2}\cdots v_{i_k}$ with the same parity have the same weights for $1 \le k \le n$, otherwise they have the opposite weights.*

Proof. First assume that the matrix $Q(M_G)$ is singular. Then there exists a non-zero vector $\xi^\top = (\xi_1, \xi_2, \ldots, \xi_n)$ such that $Q(M_G)\xi = 0$, which implies that $T(M_G)^*\xi = 0$. By some calculations, one obtains

$$(T((M_G)^*\xi)_e = \overline{t_{ie}}\xi_i + \overline{t_{je}}\xi_j = \begin{cases} (\xi_i + \xi_j)\overline{t_{ie}}, & \text{if } v_i \leftrightarrow v_j;\\ (\xi_i + \frac{1+\sqrt{3}\mathbf{i}}{2}\xi_j)\overline{t_{ie}}, & \text{if } v_i \to v_j;\\ (\xi_i + \frac{1-\sqrt{3}\mathbf{i}}{2}\xi_j)\overline{t_{ie}}, & \text{if } v_i \leftarrow v_j. \end{cases} \tag{9}$$

Since $T(M_G)^*\xi = 0$, then $\xi_i = -n_{ij}\xi_j$ for each edge $e = v_iv_j$ of M_G. It follows that

$$\xi_{i_1} = -n_{i_1i_2}\xi_{i_2} = (-1)^2 n_{i_1i_2}n_{i_2i_3}\xi_{i_3} = \cdots = (-1)^k n_{i_1i_2}n_{i_2i_3}\cdots n_{i_{k-1}i_k}\xi_{i_k} = (-1)^k n(W_{i_1i_k})\xi_{i_k},$$

implying that the result holds.

Conversely, assume that any two walks $W = v_{i_1}v_{i_2}\cdots v_{i_k}$ with the same parity have the same weight for $1 \le k \le n$. Now let $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n)^\top$ be a vector such that $\zeta_i = -n_{ij}\zeta_j$ for each edge $e = v_iv_j$. Thus, it is easy to verify that

$$\zeta^* Q(M_G)\zeta = \sum_{e\in E(M_G)} |\zeta_i + n_{ij}\zeta_j|^2 = \sum_{e\in E(M_G)} |-n_{ij}\zeta_j + n_{ij}\zeta_j|^2 = 0.$$

Therefore, $Q(M_G)$ is a singular matrix. □

Lemma 3.2. *Let M_C be a mixed cycle with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_n\}$. Then* $\det Q(M_C) = 2 + (-1)^{n+1}[n(M_C) + \overline{n(M_C)}]$.

Proof. Similar to the proof of Lemma 2.3, we can get

$$\det T(M_C) = \prod_{i=1}^n t_{ie_i} + (-1)^{n+1} t_{1e_n} \prod_{i=2}^n t_{ie_{i-1}}$$

and

$$\det T(M_C)^* = \prod_{i=1}^{n} \overline{t_{ie_i}} + (-1)^{n+1}\overline{t_{1e_n}} \prod_{i=2}^{n} \overline{t_{ie_{i-1}}}.$$

The definition of $T(M_C)$ implies that $t_{ie}\overline{t_{je}} = n_{ij}$ for any edge $e = v_iv_j$. So we have

$$\begin{aligned}\det Q(M_C) &= \det T(M_C) \cdot \det T(M_C)^* \\ &= 2 + (-1)^{n+1}\overline{t_{1e_n}} \prod_{i=1}^{n} t_{ie_i} \prod_{i=2}^{n} \overline{t_{ie_{i-1}}} + (-1)^{n+1} t_{1e_n} \prod_{i=2}^{n} t_{ie_{i-1}} \prod_{i=1}^{n} \overline{t_{ie_i}} \\ &= 2 + (-1)^{n+1} t_{1e_1}\overline{t_{2e_1}} \cdots t_{ne_n}\overline{t_{1e_n}} + (-1)^{n+1} t_{2e_1}\overline{t_{1e_1}} \cdots t_{1e_n}\overline{t_{ne_n}} \\ &= 2 + (-1)^{n+1} n_{12}n_{23} \cdots n_{n1} + (-1)^{n+1} n_{21} \cdots n_{1n} \\ &= 2 + (-1)^{n+1}[n(M_C) + \overline{n(M_C)}].\end{aligned} \tag{10}$$

This completes the proof. □

Theorem 3.2. *Suppose that M_G is a mixed unicyclic graph with n vertices and M_C is the unique cycle in M_G. Then* $\det Q(M_G) = \det Q(M_C) = 2 + (-1)^{n+1}[n(M_C) + \overline{n(M_C)}]$.

Proof. By the same argument as the proof of Theorem 2.3, we get the required result. □

Next we consider the principal minor of Hermitian quasi-Laplacian matrix of the second kind for mixed graphs. First we introduce some definitions. A subgraph Γ of a connected mixed graph M_G is called *essential quasi-Laplacian spanning subgraph* of M_G if M_G is positive and Γ is a spanning tree of its underlying graph G; or M_G is not positive, Γ has the same vertices as M_G, its components are unicyclic mixed graphs and the Hermitian quasi-Laplacian matrix of the second kind of each cycle is non-singular. An *r-reduced quasi-Laplacian spanning substructure* R of M_G is defined as follows: R is a subgraph of M_G with $n - r$ vertices and no quasi-Laplacian singular cycles, and the number of vertices and edges in each component of R is the same. It is clear that each component of R is either a rootless tree or a quasi-Laplacian non-singular unicyclic graph. Furthermore, we can establish a one-to-one correspondence between R and Γ, in which one vertex is deleted in each positive component of Γ.

Theorem 3.3. *Let M_G be a mixed graph with vertex set V and edge set E. Also let V_{n-r} be a subset obtained by deleting r vertices from V. Then*

$$\det Q[V_{n-r}, V_{n-r}] = \sum_{R} 3^{\epsilon_1(R)+\epsilon_2(R)} \cdot 4^{\epsilon_3(R)+\epsilon_4(R)},$$

where the summation runs over all r-reduced quasi-Laplacian spanning substructures R of M_G, and $\epsilon_1(R), \epsilon_2(R), \epsilon_3(R), \epsilon_4(R)$ are the numbers of semi-positive odd cycles, semi-negative even cycles , positive odd cycles, negative even cycles of R, respectively.

Proof. According to the Cauchy-Binet Theorem and $Q(M_G) = T(M_G)T(M_G)^*$, we have

$$\det Q[V_{n-r}, V_{n-r}] = \sum_{E_{n-r}} \det T[V_{n-r}, E_{n-r}] \cdot \det T[V_{n-r}, E_{n-r}]^* = \sum_{R} |\det T(R)|^2, \tag{11}$$

where E_{n-r} is a subset of E with $n - r$ edges and R is the substructure with the pair (V_{n-r}, E_{n-r}).

In the following, we consider the contribution of a substructure R to $\sum_R |\det T(R)|^2$. Since each component of R is either a rootless tree, or a quasi-Laplacian non-singular unicyclic graph, so we will discuss the following two cases.

Case 1: A rootless tree R' is a component of R. In this case, we have $|\det T(R')| = 1$.

Case 2: A non-singular unicyclic graph R'' is a component of R. Now let M_C be the cycle in R''. Theorem 3.2 implies that $\det Q(R'') = \det Q(M_C) = 2 + (-1)^{n+1}[n(M_C) + \overline{n(M_C)}]$. If M_C is a positive odd cycle, $n(M_C) = 1$, then $\det Q(R'') = 4$. If M_C is a negative even cycle, $n(M_C) = -1$, then $\det Q(R'') = 4$. If M_C is a semi-positive cycle, $n(M_C) = \frac{1\pm\sqrt{3}\mathbf{i}}{2}$, then

$$\det Q(R'') = 2 + (-1)^{n+1} = \begin{cases} 1, & \text{if } n \text{ is even;} \\ 3, & \text{if } n \text{ is odd.} \end{cases}$$

Finally, if M_C is a semi-negative cycle, $n(M_C) = \frac{-1\pm\sqrt{3}\mathbf{i}}{2}$, then

$$\det Q(R'') = 2 + (-1)^{n} = \begin{cases} 3, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

According to the discussion above, we get that a rootless tree contributes 1, and each non-singular unicyclic graph contributes 1, 3 or 4 to the value of determinant. It follows from (11) that the result holds. □

The following statement is a direct result of Theorem 3.3, and we ignore its proof.

Theorem 3.4. *Let M_G be a connected mixed graph with vertex set V and edge set E, then*

$$\det Q(M_G) = \sum_{\tau_1,\tau_2,\tau_3,\tau_4} 3^{\tau_1+\tau_2} 4^{\tau_3+\tau_4} q_{\tau_1,\tau_2,\tau_3,\tau_4},$$

where $q_{\tau_1,\tau_2,\tau_3,\tau_4}$ is the number of essential quasi-Laplacian spanning subgraphs which contain τ_1 semi-positive odd cycles, τ_2 semi-negative even cycles, τ_3 positive odd cycles, τ_4 negative even cycles and we stipulate that $q_{0,0,0,0} = 0$.

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