Some observations on the Laplacian–energy–like invariant of trees

Marjan Matejić¹, Ş. B. Bozkurt Altındağ², Igor Milovanović¹,*; Emina Milovanović¹

¹Faculty of Electronic Engineering, University of Niš, Niš, Serbia
²Department of Mathematics, Kamil Özdag Science Faculty, Karamanoğlu Mehmetbey University, Karaman, Turkey

(Received: 20 June 2022. Accepted: 11 November 2022. Published online: 23 November 2022.)

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Abstract

Let G be a graph of order n. Denote by A the adjacency matrix of G and by \( D = \text{diag}(d_1, \ldots, d_n) \) the diagonal matrix of vertex degrees of G. The Laplacian matrix of G is defined as \( L = D - A \). Let \( \mu_1, \mu_2, \ldots, \mu_n \) be eigenvalues of L satisfying \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0 \). The Laplacian–energy–like invariant is a graph invariant defined as \( LEL(G) = \sum_{i=1}^{n-1} \sqrt{\mu_i} \). Improved upper bounds for \( LEL(G) \) are obtained and compared when G has a tree structure.

Keywords: Laplacian eigenvalues; energy of graphs; trees.

2020 Mathematics Subject Classification: 15A18, 05C50.

1. Introduction

Let \( G = (V, E) \) be a graph of order n and size m with the sequence of vertex degrees \( \Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta \), where \( V = \{v_1, v_2, \ldots, v_n\} \) and \( E = \{e_1, e_2, \ldots, e_m\} \). In addition, we will use the following notations \( \Delta_2 = d_2, \delta_2 = d_{n-2}, \) and \( \delta_p = d_{n-p}, \) where \( p \) is an integer in the range \( 1 \leq p \leq n-1 \). The diagonal matrix of vertex degrees of G is denoted as \( D = \text{diag}(d_1, d_2, \ldots, d_n) \). Obviously \( D = \prod_{i=1}^{n} d_i \).

Denote by \( A(G) \) the \((0,1)\)-adjacency matrix of a graph G. The eigenvalues \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \) of \( A(G) \) represent the eigenvalues of G. The sum of absolute values of these eigenvalues is defined to be the (ordinary) energy of G [4], that is

\[
E(G) = \sum_{i=1}^{n} |\lambda_i|.
\]

The Laplacian matrix of G is \( L = D - A \). Eigenvalues of \( L, \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq 0 \), form the Laplacian spectrum of G. The Laplacian-energy–like invariant is a graph invariant defined in terms of Laplacian eigenvalues as [10]

\[
LEL(G) = \sum_{i=1}^{n-1} \sqrt{\mu_i}.
\]

If vertex \( v_i \) is incident with edge \( e_j \) in G, it will be denoted as \( v_i \sim e_j \). The incidence matrix \( B = (b_{ij}) \) of order \( n \times m \), of graph G is defined as

\[
b_{ij} = \begin{cases} 1, & \text{if } v_i \sim e_j \\ 0, & \text{otherwise} \end{cases}.
\]

The signless Laplacian matrix of G is defined as \( L = B \cdot B^T = D + A \). The eigenvalues of matrix \( L, q_1 \geq q_2 \geq \cdots \geq q_n \geq 0 \), are signless Laplacian eigenvalues of G. The corresponding incidence graph energy defined in terms of these eigenvalues is

\[
IE(G) = \sum_{i=1}^{n} \sqrt{q_i}.
\]

If G is a bipartite graph, then \( IE(G) \) coincides with \( LEL(G) \), that is \( IE(G) = LEL(G) \) [5]. Since any tree is bipartite graph, this means that

\[
IE(T) = LEL(T).
\]

More on the mathematical properties of various graph energies can be found in monographs [6, 9], reviews [1, 14] and the references cited therein. In this paper we determine new upper bounds on \( LEL(G) \), that is \( IE(G) \), when G has a tree structure.

* Corresponding author (igor.milovanovic@elfak.ni.ac.rs).
2. Preliminaries

In this section we recall some results from literature regarding $LEL(G)$ and analytical inequalities that are of interest for the present paper.

Lemma 2.1. \([10, 11]\) If $\Delta = d_1 \geq \cdots \geq d_n = \delta > 0$ are degrees of vertices of $G$, then

$$LEL(G) \leq \sqrt{d_1 + 1} + \sqrt{d_2 + \cdots + \sqrt{d_{n-1}} + \sqrt{d_n - 1}},$$

with equality if and only if $G \cong K_{1,n-1}$.

Lemma 2.2. \([2]\) If $G$ is a connected, non-complete graph, with at least three vertices, then

$$LEL(G) \leq \sqrt{1 + \Delta + \sqrt{\delta + \sqrt{(n-3)(2m-\Delta - \delta - 1)}}},$$

with equality if and only if $G \cong K_{1,n-1}$, $G \cong 2K_1 \vee K_{n-2}$ or $G \cong (K_1 \cup K_{n-2}) \vee K_1$.

More on the bounds of type (1) and (2) can be found in the review \([14]\) and the papers cited therein.

Remark 2.1. As can be seen, when $G \cong K_{1,n-1}$ in (1) and (2) equalities occur. Therefore the above inequalities are interesting for the present paper particularly when $G \cong T$. Since any tree $T$ with $n \geq 3$ vertices, has at least two vertices of degree 1, $d_{n-1} = d_n = \delta = 1$, the above inequalities can be considered as

$$LEL(T) \leq 1 + \sqrt{1 + \Delta + \sum_{i=2}^{n-2} \sqrt{d_i}},$$

and

$$LEL(T) \leq 1 + \sqrt{1 + \Delta + (n-3)(2(n-2) - \Delta)},$$

with equalities if and only if $T \cong K_{1,n-1}$.

Lemma 2.3. \([3]\) Let $T$ be a tree of order $n$ with maximum degree $\Delta$. Then

$$LEL(T) \leq \sqrt{n} + \sqrt{(n-3)(2n-\Delta - 3) + (n-2) \left(\frac{n}{\Delta + 1}\right)^{\frac{1}{2}}}.$$  \hspace{1cm} (5)

Equality holds if and only if $T \cong K_{1,n-1}$.

Lemma 2.4. \([8]\) Let $a = (a_i), i = 1, 2, \ldots, n$, be a sequence of positive real numbers. Then

$$\left(\sum_{i=1}^{n} \sqrt{a_i}\right)^2 \leq (n-1) \sum_{i=1}^{n} a_i + n \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}.$$  \hspace{1cm} (6)

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Lemma 2.5. \([13]\) Let $x = (x_i)$ and $a = (a_i), i = 1, 2, \ldots, n$, be two sequences of positive real numbers. Then, for any real $r$, $r \geq 0$, we have that

$$\sum_{i=1}^{n} \frac{x_i^{r+1}}{a_i} \geq \frac{\left(\sum_{i=1}^{n} x_i\right)^{r+1}}{\left(\sum_{i=1}^{n} a_i\right)}.$$  \hspace{1cm} (7)

Equality holds if and only if $r = 0$, or $\frac{x_1}{a_1} = \cdots = \frac{x_n}{a_n}$.

Remark 2.2. It is not difficult to observe that inequality (7) is valid if and only if $r \leq -1$ or $r \geq 0$, and equality is attained if and only if either $r = -1$, or $r = 0$, or $\frac{x_1}{a_1} = \cdots = \frac{x_n}{a_n}$, when $-1 \leq r \leq 0$, the opposite inequality is valid in (7).

3. Main results

Let $T$ be a tree of order $n \geq 3$ with $p, 2 \leq p \leq n-1$, pendant vertices. In the next theorem we establish an upper bound on $LEL(T)$ ($IE(T)$) in terms of $n, p$, maximum degree $\Delta$, second maximum degree $\Delta_2$ and $\delta_p$.

Theorem 3.1. Let $T$ be a tree of order $n \geq 3$ and $p, 2 \leq p \leq n-1$, pendant vertices. Then

$$LEL(T) \leq p - 1 + \sqrt{1 + \Delta + \sqrt{(n-p-1) \left(2(n-1) - p - \Delta - \frac{1}{2}(\sqrt{\Delta_2} - \sqrt{\delta_p})^2\right)}}.$$  \hspace{1cm} (8)

Equality holds if and only if $T \cong K_{1,n-1}$.
Proof. Let \( T \) be a tree of order \( n \geq 3 \) and \( p \), \( 2 \leq p \leq n-1 \), pendant vertices. Then \( d_{n-p+1} = \cdots = d_{n} = \delta = 1 \). If \( p = n-1 \), then \( T \cong K_{1,n-1} \) and equality occurs in (8). Therefore, without affecting the generality, assume that \( 2 \leq p \leq n-2 \). In that case, according to (3) we obtain that

\[
LEL(T) \leq p - 1 + \sqrt{1 + \Delta} + \sum_{i=2}^{n-p} \sqrt{d_i}.
\]  

(9)

On the other hand, based on the Lagrange’s identity (see e.g. [12]), we have that

\[
(n-p-1) \sum_{i=2}^{n-p} d_i - \left( \sum_{i=2}^{n-p} \sqrt{d_i} \right)^2 = \sum_{2 \leq i < j \leq n-p} \left( \sqrt{d_i} - \sqrt{d_j} \right)^2
\]

\[
\geq \left( \sqrt{d_2} - \sqrt{d_{n-p}} \right)^2 + \sum_{i=1}^{n-p-1} \left( \left( \sqrt{d_2} - \sqrt{d_i} \right)^2 + \left( \sqrt{d_i} - \sqrt{d_{n-p}} \right)^2 \right)
\]

\[
\geq \left( \sqrt{\Delta_2} - \sqrt{\delta_p} \right)^2 + \frac{1}{2} \sum_{i=3}^{n-p-1} \left( \sqrt{\Delta_2} - \sqrt{\delta_p} \right)^2 = \frac{n-p-1}{2} \left( \sqrt{\Delta_2} - \sqrt{\delta_p} \right)^2,
\]

and therefore

\[
\sum_{i=2}^{n-p} \sqrt{d_i} \leq \sqrt{(n-p-1)(2(n-1) - \Delta - p)} - \frac{n-p-1}{2} \left( \sqrt{\Delta_2} - \sqrt{\delta_p} \right)^2
\]

\[
= \sqrt{(n-p-1) \left( 2(n-1) - \Delta - p - \frac{1}{2} \left( \sqrt{\Delta_2} - \sqrt{\delta_p} \right)^2 \right)}.
\]  

(10)

From the above and inequality (9) we arrive at (8).

Equality in (10) holds if and only if \( \sqrt{\Delta_3} = \cdots = \sqrt{d_{n-p-1}} = \frac{\sqrt{\Delta_2} + \sqrt{\delta_p}}{2} \), and in (9) if and only if \( T \cong K_{1,n-1} \), which implies that equality in (8) holds if and only if \( T \cong K_{1,n-1} \).

Since \( (\sqrt{\Delta_2} - \sqrt{\delta_p})^2 \geq 0 \), we have the following corollary of Theorem 3.1.

Corollary 3.1. Let \( T \) be a tree of order \( n \geq 3 \) with \( p \), \( 2 \leq p \leq n-1 \), pendant vertices. Then

\[
LEL(T) \leq p - 1 + \sqrt{1 + \Delta} + \sqrt{(n-p-1)(2(n-1) - \Delta - p)}.
\]  

(11)

Equality holds if and only if \( T \cong K_{1,n-1} \).

The proof of the next theorem is analogous to that of Theorem 3.1, hence omitted.

Theorem 3.2. Let \( T \) be a tree with \( n \geq 3 \) vertices. Then

\[
LEL(T) \leq 1 + \sqrt{1 + \Delta} + \sqrt{(n-3) \left( 2(n-2) - \Delta - \frac{1}{2} \left( \sqrt{\Delta_2} - \sqrt{\delta_2} \right)^2 \right)}.
\]  

(12)

Equality holds if and only if \( T \cong K_{1,n-1} \).

Remark 3.1. Since \( (\sqrt{\Delta_2} - \sqrt{\delta_2})^2 \geq 0 \), the following inequalities are valid

\[
LEL(T) \leq 1 + \sqrt{1 + \Delta} + \sqrt{(n-3) \left( 2(n-2) - \Delta - \frac{1}{2} \left( \sqrt{\Delta_2} - \sqrt{\delta_2} \right)^2 \right)}
\]

\[
\leq 1 + \sqrt{1 + \Delta} + \sqrt{(n-3)(2(n-2) - \Delta)}
\]

which means that inequality (12) is stronger than (4).

In the next theorem we determine an upper bound on \( LEL(T) (IE(T)) \) in terms of \( n, \det D, \Delta \) and \( p \).

Theorem 3.3. Let \( T \) be a tree of order \( n \geq 3 \) with \( p \), \( 2 \leq p \leq n-1 \), pendant vertices. Then

\[
LEL(T) \leq p - 1 + \sqrt{1 + \Delta} + \sqrt{(n-p-2)(2(n-1) - p - \Delta) + (n-p-1) \left( \frac{\det D}{\Delta} \right)^{\frac{1}{p-1}}}
\]  

(13)

Equality holds if and only if \( T \cong K_{1,n-1} \).
Proof: If \( p = n - 1 \), then \( T \cong K_{1,n-1} \) and equality occurs in (13). Therefore, without affecting the generality, assume that \( 2 \leq p \leq n - 2 \). The inequality (6) can be considered in the following form
\[
\left( \sum_{i=2}^{n-p} a_i \right)^2 \leq (n-p-2) \sum_{i=2}^{n-p} a_i + (n-p-1) \left( \prod_{i=2}^{n-p} a_i \right)^{\frac{1}{n-p-1}}.
\]
For \( a_i = d_i, i = 2, \ldots, n-p \), the above inequality becomes
\[
\left( \sum_{i=2}^{n-p} d_i \right)^2 \leq (n-p-2) \sum_{i=2}^{n-p} d_i + (n-p-1) \left( \prod_{i=2}^{n-p} d_i \right)^{\frac{1}{n-p-1}},
\]
that is
\[
\left( \sum_{i=2}^{n-p} \sqrt{d_i^2} \right)^2 \leq (n-p-2) \sum_{i=2}^{n-p} d_i + (n-p-1) \left( \prod_{i=2}^{n-p} d_i \right)^{\frac{1}{n-p-1}}.
\]
From the above and inequality (9) we arrive at (13).

Equality in (14) holds if and only if \( \Delta = d_2 = \cdots = d_{n-p} = \delta_p \), whereas in (9) if and only if \( T \cong K_{1,n-1} \), which implies that equality in (13) holds if and only if \( T \cong K_{1,n-1} \). \( \square \)

Remark 3.2. According to arithmetic–geometric mean inequality (AGM) (see e.g. [12]) we have that
\[
(n-p-1) \left( \prod_{i=2}^{n-p} d_i \right)^{\frac{1}{n-p-1}} \leq \sum_{i=2}^{n-p} d_i = 2(n-1) - p - \Delta.
\]
Therefore
\[
LEL(T) \leq p - 1 + \sqrt{1 + \Delta + \sqrt{(n-p-2)(2(n-1) - p - \Delta) + (n-p-1) \left( \frac{\det D}{\Delta} \right)^{\frac{1}{n-p-1}}}}.
\]
\[
LEL(T) \leq p - 1 + \sqrt{1 + \Delta + \sqrt{(n-p-2)(2(n-1) - p - \Delta) + 2(n-1) - p - \Delta}} = p - 1 + \sqrt{1 + \Delta + \sqrt{(n-3)(2(n-2) - \Delta)}}.
\]
which means that inequality (13) is stronger than (11).

The proof of the next Theorem is analogous to that of Theorem 3.3, thus omitted.

Theorem 3.4. Let \( T \) be a tree with \( n \geq 4 \) vertices. Then
\[
LEL(T) \leq 1 + \sqrt{1 + \Delta + \sqrt{(n-4)(2(n-2) - \Delta) + (n-3) \left( \frac{\det D}{\Delta} \right)^{\frac{1}{n-3}}}}.
\]
Equality holds if and only if \( T \cong K_{1,n-1} \).

Remark 3.3. Similarly as in the case of Remark 3.2, the following can be proved
\[
LEL(T) \leq 1 + \sqrt{1 + \Delta + \sqrt{(n-4)(2(n-2) - \Delta) + (n-3) \left( \frac{\det D}{\Delta} \right)^{\frac{1}{n-3}}}} \leq 1 + \sqrt{1 + \Delta + \sqrt{(n-3)(2(n-2) - \Delta)}}.
\]
This means that inequality (15) is stronger than (4).

The zeroth order Randić index is a degree based graph invariant introduced in [7] as
\[
0^0R(G) = \sum_{i=1}^{n} \frac{1}{\sqrt{d_i}}.
\]
In the next theorem we determine an upper bound on \( LEL(T) \) in terms of \( 0^0R(T) \).

Theorem 3.5. Let \( T \) be a tree of order \( n \geq 5 \) with \( p, 2 \leq p \leq n - 1 \), pendant vertices. Then
\[
LEL(T) \leq p - 1 + \sqrt{1 + \Delta + (n-p-1)(\sqrt{\Delta_2 + \sqrt{\Delta_3}} - \sqrt{\Delta_2}\Delta_p) \left( 0^0R(T) - \frac{1}{\sqrt{\Delta}} - p \right)}.
\]
Equality holds if and only if \( T \cong K_{1,n-1} \).
Proof. For every \( i, i = 2, \ldots, n-p \), holds
\[
(\sqrt{d_i} - \sqrt{\Delta_2})(\sqrt{d_i} - \sqrt{\delta_p}) \leq 0,
\]
\[
\sqrt{d_i} + \frac{\sqrt{\Delta_2} \delta_p}{\sqrt{d_i}} \leq \sqrt{\Delta_2} + \sqrt{\delta_p}.
\] (17)

After summation over \( i, i = 2, \ldots, n-p \), of the above inequality, we get
\[
\sum_{i=2}^{n-p} \sqrt{d_i} + \frac{\sqrt{\Delta_2} \delta_p}{\sqrt{d_i}} \sum_{i=2}^{n-p} \frac{1}{\sqrt{d_i}} \leq (n-p-1)(\sqrt{\Delta_2} + \sqrt{\delta_p}),
\]
that is
\[
\sum_{i=2}^{n-p} \sqrt{d_i} \leq (n-p-1)(\sqrt{\Delta_2} + \sqrt{\delta_p}) - \sqrt{\Delta_2} \delta_p \left( \frac{\delta_p}{\sqrt{\Delta_2}} - \frac{1}{\sqrt{\Delta_2}} - p \right).
\]

Now, from the above and inequality (9) we obtain (16).

Equality in (17) holds if and only if \( d_i \in \{\Delta_2, \delta_p\}, \) for \( i = 2, \ldots, n-p \). On the other hand, equality in (9) holds if and only if \( T \cong K_{1,n-1} \), which implies that equality in (16) holds under same condition. \( \square \)

Corollary 3.2. Let \( T \) be a tree of order \( n \geq 5 \) with \( 2 \leq p \leq n-1 \), pendant vertices. Then
\[
LEL(T) \leq p-1 + \sqrt{1+\Delta} + \frac{(n-p-1)^2 \left( \frac{\Delta_2}{\delta_p} + \frac{\delta_p}{\Delta_2} \right)^2}{4 \left( \frac{\delta_p}{\sqrt{\Delta_2}} - \frac{1}{\sqrt{\Delta_2}} - p \right)}.
\] (18)

Equality holds if and only if \( T \cong K_{1,n-1} \).

Proof. According to AGM inequality we have that
\[
2 \sqrt{\Delta_2 \delta_p \left( \frac{\delta_p}{\sqrt{\Delta_2}} - \frac{1}{\sqrt{\Delta_2}} - p \right)} \sum_{i=2}^{n-p} \sqrt{d_i} \leq \sum_{i=2}^{n-p} \sqrt{d_i} + \sqrt{\Delta_2 \delta_p} \left( \frac{\delta_p}{\sqrt{\Delta_2}} - \frac{1}{\sqrt{\Delta_2}} - p \right)
\]
\[
\leq (n-p-1)(\sqrt{\Delta_2} + \sqrt{\delta_p}),
\]
that is
\[
\sum_{i=2}^{n-p} \sqrt{d_i} \leq \frac{(n-p-1)^2 \left( \frac{\Delta_2}{\delta_p} + \frac{\delta_p}{\Delta_2} \right)^2}{4 \left( \frac{\delta_p}{\sqrt{\Delta_2}} - \frac{1}{\sqrt{\Delta_2}} - p \right)}.
\]

From the above and inequality (9) we obtain (18). \( \square \)

Similarly as in the case of Theorem 3.5 the following theorem can be proved.

Theorem 3.6. Let \( T \) be a tree with \( n \geq 5 \) vertices. Then
\[
LEL(T) \leq 1 + \sqrt{1+\Delta} + (n-3)(\sqrt{\Delta_2} + \sqrt{\delta_2}) - \sqrt{\Delta_2} \delta_2 \left( \frac{\delta_2}{\sqrt{\Delta_2}} - \frac{1}{\sqrt{\Delta_2}} - 2 \right).
\] (19)

Equality holds if and only if \( T \cong K_{1,n-1} \).

Corollary 3.3. Let \( T \) be a tree with \( n \geq 5 \) vertices. Then
\[
LEL(T) \leq 1 + \sqrt{1+\Delta} + \frac{(n-3)^2 \left( \sqrt{\Delta_2} + \sqrt{\delta_2} \right)^2}{4 \left( \frac{\delta_2}{\sqrt{\Delta_2}} - \frac{1}{\sqrt{\Delta_2}} - 2 \right)}.
\]

Equality holds if and only if \( T \cong K_{1,n-1} \).

Theorem 3.7. Let \( T \) be a tree of order \( n \geq 4 \) with \( p \) pendent vertices. When \( p = n-1 \), then
\[
LEL(T) = n - 2 + \sqrt{n}.
\]

When \( 0 \leq p \leq n-2 \), then
\[
LEL(T) \leq p-1 + \sqrt{1+\Delta} + \frac{2(n-1) - \Delta - p}{\sqrt{\Delta_2 \delta_p}} \left( \sqrt{\Delta_2} + \sqrt{\delta_p} - \sqrt{\frac{2(n-1) - \Delta - p}{n-p-1}} \right).
\] (20)
Proof. After multiplying (17) by \(d_i\) and summation over \(i\), for \(i = 2, \ldots, n - p\), we obtain

\[
\sum_{i=2}^{n-p} d_i^{3/2} + \sqrt{\Delta_2 \delta_p} \sum_{i=2}^{n-p} \sqrt{d_i} \leq \left( \sqrt{\Delta_2} + \sqrt{\delta_p} \right) \sum_{i=2}^{n-p} d_i ,
\]

that is

\[
\sum_{i=2}^{n-p} d_i^{3/2} + \sqrt{\Delta_2 \delta_p} \sum_{i=2}^{n-p} \sqrt{d_i} \leq \left( \sqrt{\Delta_2} + \sqrt{\delta_p} \right) (2(n-1) - \Delta - p) .
\]  \hspace{1cm} (21)

On the other hand, the inequality (7) can be considered in the following form

\[
\sum_{i=2}^{n-p} \frac{d_i^{r+1}}{a_i^r} \geq \left( \frac{\sum_{i=2}^{n-p} d_i}{\sum_{i=2}^{n-p} a_i} \right)^{r+1} , \quad 0 \leq p \leq n - 2 .
\]

For \(r = \frac{1}{2}\), \(x_i = d_i\), \(a_i = 1\), \(i = 2, \ldots, n - p\), the above inequality becomes

\[
\sum_{i=2}^{n-p} d_i^{3/2} \geq \left( \frac{\sum_{i=2}^{n-p} d_i}{\sum_{i=2}^{n-p} 1} \right)^{3/2} = \frac{(2(n-1) - \Delta - p)^{3/2}}{(n-p-1)^{3/2}} .
\]

From the above and inequality (21) we obtain

\[
\sum_{i=2}^{n-p} \sqrt{d_i} \leq \frac{2(n-1) - \Delta - p}{\sqrt{\Delta_2 \delta_p}} \left( \sqrt{\Delta_2} + \sqrt{\delta_p} - \sqrt{\frac{2(n-1) - \Delta - p}{n-p-1}} \right) .
\]

Now, from the above and inequality (21) we arrive at (20).

The proof the next theorem is analogous to that of Theorem 3.7, hence omitted.

**Theorem 3.8.** Let \(T\) be a tree with \(n \geq 4\) vertices. Then

\[
\text{LEL}(T) \leq 1 + \sqrt{1 + \Delta} + \frac{2(n-2) - \Delta}{\sqrt{\Delta_2 \delta_2}} \left( \sqrt{\Delta_2} + \sqrt{\delta_2} - \sqrt{\frac{2(n-2) - \Delta}{n-3}} \right) .
\]  \hspace{1cm} (22)

Equality holds if and only if \(T \cong K_{1,n-1}\).

**Corollary 3.4.** Let \(T\) be a tree with \(n \geq 4\) vertices. Then

\[
\text{LEL}(T) \leq 1 + \sqrt{1 + \Delta} + \frac{(2(n-2) - \Delta)(\sqrt{\Delta_2} + \sqrt{\delta_2} - 1)}{\sqrt{\Delta_2 \delta_2}} .
\]  \hspace{1cm} (23)

Equality holds if and only if \(T \cong K_{1,n-1}\).

**Proof.** Since \(2(n-2) - \Delta \geq n-3\), from the above and (22) we obtain (23).

\(\square\)

### 4. Comparison and discussion

In this section we compare the upper bounds for \(\text{LEL}(T)\) obtained by inequalities (12), (15), (19) and (5) and give some numerical results.

Let \(T = P_n\), \(n \geq 3\). In that case the bounds (15) and (19) coincide, and are equal to

\[
\text{LEL}(T) \leq 1 + \sqrt{3} + \sqrt{2}(n-3) .
\]

The upper bounds determined by (12) and (19) are incomparable. Namely, when \(T = P_n\), \(n \geq 3\), the bound (12) is stronger than (19). However, if \(T\) is a tree with the vertex degree sequence

\[
\left( \frac{n}{2}, \frac{n}{2}, 1, \ldots, 1 \right) ,
\]

then for \(n \geq 6\), the bound (19) is stronger than (12).
Table 1: Numerical values of the bounds (12), (15), (19) and (5) when $T \cong P_n$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Eq. (12)</th>
<th>Eq. (15)</th>
<th>Eq. (19)</th>
<th>Eq. (5)</th>
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<td>50</td>
<td>69.1698</td>
<td>69.2001</td>
<td>69.2001</td>
<td>74.2713</td>
</tr>
<tr>
<td>100</td>
<td>139.8800</td>
<td>139.9110</td>
<td>139.9110</td>
<td>147.9010</td>
</tr>
</tbody>
</table>

Table 2: Numerical values of the bounds (12), (15), (19) and (5) when $T$ has the degree sequence $(\frac{n}{2}, \frac{n}{2}, 1, \ldots, 1)$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Eq. (12)</th>
<th>Eq. (15)</th>
<th>Eq. (19)</th>
<th>Eq. (5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>11.9143</td>
<td>12.0987</td>
<td>11.6856</td>
<td>12.7814</td>
</tr>
<tr>
<td>20</td>
<td>24.3730</td>
<td>25.1844</td>
<td>23.4789</td>
<td>26.3264</td>
</tr>
<tr>
<td>30</td>
<td>36.5527</td>
<td>38.1036</td>
<td>34.8730</td>
<td>39.5747</td>
</tr>
<tr>
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<td>60.5141</td>
<td>63.6866</td>
<td>57.0990</td>
<td>65.660</td>
</tr>
<tr>
<td>100</td>
<td>119.3820</td>
<td>126.9560</td>
<td>111.2120</td>
<td>129.8240</td>
</tr>
</tbody>
</table>

Table 1 gives the numerical values for $LEL(T)$ (i.e. $IE(T)$) obtained by inequalities (12), (15), (19) and (5) when tree is a path, $T \cong P_n$, that is for trees defined by the vertex degrees $(\frac{n}{2}, \frac{n}{2}, n-2, 1, \ldots, 1)$ for $n = 5, 10, 20, 50, 100$.

Table 2 gives numerical values for the trees with the vertex degrees $(\frac{n}{2}, \frac{n}{2}, 1, \ldots, 1)$ for $n = 10, 20, 30, 50, 100$.

According to Tables 1 and 2 we conclude that the upper bounds on $LEL(T)$ obtained by (12), (15) and (19) are stronger than the one obtained by (5). However, the open question is whether this is true for any tree $T$.

References


