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Research Article

Some observations on the Laplacian–energy–like invariant of trees

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Abstract

Let G be a graph of order n. Denote by A the adjacency matrix of G and by $D = diag(d_1, \dots, d_n)$ the diagonal matrix of vertex degrees of G. The Laplacian matrix of G is defined as $L = D - A$. Let $\mu_1, \mu_2, \cdots, \mu_{n-1}, \mu_n$ be eigenvalues of L satisfying $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$. The Laplacian-energy–like invariant is a graph invariant defined as $LEL(G) = \sum_{i=1}^{n-1} \sqrt{\mu_i}$. Improved upper bounds for $LEL(G)$ are obtained and compared when G has a tree structure.

Keywords: Laplacian eigenvalues; energy of graphs; trees.

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1. Introduction

Let $G = (V, E)$ be a graph of order n and size m with the sequence of vertex degrees $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta$, where $V = \{v_1, v_2, \dots, v_n\}$ and $E = \{e_1, e_2, \dots, e_m\}$. In addition, we will use the following notations $\Delta_2 = d_2$, $\delta_2 = d_{n-2}$, and $\delta_p = d_{n-p}$, where p is an integer in the range $1 \leq p \leq n-1$. The diagonal matrix of vertex degrees of G is denoted as $D = diag(d_1, d_2, \dots, d_n)$. Obviously $\det D = \prod_{i=1}^{n} d_i$.

Denote by $A(G)$ the $(0,1)$-adjacency matrix of a graph G. The eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of $A(G)$ represent the eigenvalues of G. The sum of absolute values of these eigenvalues is defined to be the (ordinary) energy of G [4], that is

$$E(G) = \sum_{i=1}^{n} |\lambda_i| .$$

The Laplacian matrix of G is $L = D - A$. Eigenvalues of L, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$, form the Laplacian spectrum of G. The Laplacian-energy–like invariant is a graph invariant defined in terms of Laplacian eigenvalues as [10]

$$LEL(G) = \sum_{i=1}^{n-1} \sqrt{\mu_i} .$$

If vertex v_i is incident with edge e_j in G, it will be denoted as $v_i \sim e_j$. The incidence matrix $B = (b_{ij})$ of order $n \times m$, of graph G is defined as

$$b_{ij} = \begin{cases} 1, & \text{if } v_i \sim e_j \\ 0, & \text{otherwise} . \end{cases}$$

The signless Laplacian matrix of G is defined as $\mathcal{L} = B \cdot B^T = D + A$. The eigenvalues of matrix $\mathcal{L}$, $q_1 \geq q_2 \geq \cdots \geq q_n \geq 0$, are signless Laplacian eigenvalues of G. The corresponding incidence graph energy defined in terms of these eigenvalues is

$$IE(G) = \sum_{i=1}^{n} \sqrt{q_i} .$$

If G is a bipartite graph, then $IE(G)$ coincides with $LEL(G)$, that is $IE(G) = LEL(G)$ [5]. Since any tree is bipartite graph, this means that

$$IE(T) = LEL(T) .$$

More on the mathematical properties of various graph energies can be found in monographs [6, 9], reviews [1, 14] and the references cited therein. In this paper we determine new upper bounds on $LEL(G)$, that is $IE(G)$, when G has a tree structure.

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2. Preliminaries

In this section we recall some results from literature regarding $LEL(G)$ and analytical inequalities that are of interest for the present paper.

Lemma 2.1. [10, 11] *If $\Delta = d_1 \geq \cdots \geq d_n = \delta > 0$ are degrees of vertices of G, then*

$$LEL(G) \leq \sqrt{d_1+1} + \sqrt{d_2} + \cdots + \sqrt{d_{n-1}} + \sqrt{d_n - 1}\,, \tag{1}$$

with equality if and only if $G \cong K_{1,n-1}$.

Lemma 2.2. [2] *If G is a connected, non–complete graph, with at least three vertices, then*

$$LEL(G) \leq \sqrt{1+\Delta} + \sqrt{\delta} + \sqrt{(n-3)(2m-\Delta-\delta-1)}\,, \tag{2}$$

with equality if and only if $G \cong K_{1,n-1}$, $G \cong 2K_1 \bigvee K_{n-2}$ or $G \cong (K_1 \cup K_{n-2}) \bigvee K_1$.

More on the bounds of type (1) and (2) can be found in the review [14] and the papers cited therein.

Remark 2.1. *As can be seen, when $G \cong K_{1,n-1}$ in (1) and (2) equalities occur. Therefore the above inequalities are interesting for the present paper particulary when $G \cong T$. Since any tree T with $n \geq 3$ vertices, has at least two vertices of degree 1, $d_{n-1} = d_n = \delta = 1$, the above inequalities can be considered as*

$$LEL(T) \leq 1 + \sqrt{1+\Delta} + \sum_{i=2}^{n-2} \sqrt{d_i}\,, \tag{3}$$

and

$$LEL(T) \leq 1 + \sqrt{1+\Delta} + \sqrt{(n-3)(2(n-2)-\Delta)}\,, \tag{4}$$

with equalities if and only if $T \cong K_{1,n-1}$.

Lemma 2.3. [3] *Let T be a tree of order n with maximum degree Δ. Then*

$$LEL(T) \leq \sqrt{n} + \sqrt{(n-3)(2n-\Delta-3) + (n-2)\left(\frac{n}{\Delta+1}\right)^{\frac{1}{n-2}}}\,. \tag{5}$$

Equality holds if and only if $T \cong K_{1,n-1}$.

Lemma 2.4. [8] *Let $a = (a_i)$, $i = 1, 2, \ldots, n$, be a sequence of positive real numbers. Then*

$$\left(\sum_{i=1}^{n} \sqrt{a_i}\right)^2 \leq (n-1)\sum_{i=1}^{n} a_i + n\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}. \tag{6}$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Lemma 2.5. [13] *Let $x = (x_i)$ and $a = (a_i)$, $i = 1, 2, \ldots, n$, be two sequences of positive real numbers. Then, for any real r, $r \geq 0$, we have that*

$$\sum_{i=1}^{n} \frac{x_i^{r+1}}{a_i^r} \geq \frac{\left(\sum_{i=1}^{n} x_i\right)^{r+1}}{\left(\sum_{i=1}^{n} a_i\right)^r}\,. \tag{7}$$

Equality holds if and only if $r = 0$, or $\frac{x_1}{a_1} = \cdots = \frac{x_n}{a_n}$.

Remark 2.2. *It is not difficult to observe that inequality (7) is valid if and only if $r \leq -1$ or $r \geq 0$, and equality is attained if and only if either $r = -1$, or $r = 0$, or $\frac{x_1}{a_1} = \cdots = \frac{x_n}{a_n}$. When $-1 \leq r \leq 0$, the opposite inequality is valid in (7).*

3. Main results

Let T be a tree of order $n \geq 3$ with p, $2 \leq p \leq n-1$, pendant vertices. In the next theorem we establish an upper bound on $LEL(T)$ ($IE(T)$) in terms of n, p , maximum degree Δ, second maximum degree Δ_2 and δ_p.

Theorem 3.1. *Let T be a tree of order $n \geq 3$ and p, $2 \leq p \leq n-1$, pendant vertices. Then*

$$LEL(T) \leq p - 1 + \sqrt{1+\Delta} + \sqrt{(n-p-1)\left((2(n-1)-p-\Delta - \frac{1}{2}(\sqrt{\Delta_2} - \sqrt{\delta_p})^2\right)}\,. \tag{8}$$

Equality holds if and only if $T \cong K_{1,n-1}$.

Proof. Let T be a tree of order $n \geq 3$ and p, $2 \leq p \leq n-1$, pendant vertices. Then $d_{n-p+1} = \cdots = d_n = \delta = 1$. If $p = n-1$, then $T \cong K_{1,n-1}$ and equality occurs in (8). Therefore, without affecting the generality, assume that $2 \leq p \leq n-2$. In that case, according to (3) we obtain that

$$LEL(T) \leq p - 1 + \sqrt{1+\Delta} + \sum_{i=2}^{n-p} \sqrt{d_i}\,. \tag{9}$$

On the other hand, based on the Lagrange's identity (see e.g. [12]), we have that

$$\begin{aligned}
(n-p-1)\sum_{i=2}^{n-p} d_i - \left(\sum_{i=2}^{n-p}\sqrt{d_i}\right)^2 &= \sum_{2 \leq i < j \leq n-p}\left(\sqrt{d_i} - \sqrt{d_j}\right)^2 \\
\geq \left(\sqrt{d_2} - \sqrt{d_{n-p}}\right)^2 &+ \sum_{i=3}^{n-p-1}\left(\left(\sqrt{d_2} - \sqrt{d_i}\right)^2 + \left(\sqrt{d_i} - \sqrt{d_{n-p}}\right)^2\right) \\
\geq \left(\sqrt{\Delta_2} - \sqrt{\delta_p}\right)^2 &+ \frac{1}{2}\sum_{i=3}^{n-p-1}\left(\sqrt{\Delta_2} - \sqrt{\delta_p}\right)^2 = \frac{n-p-1}{2}\left(\sqrt{\Delta_2} - \sqrt{\delta_p}\right)^2,
\end{aligned}$$

and therefore

$$\begin{aligned}
\sum_{i=2}^{n-p}\sqrt{d_i} &\leq \sqrt{(n-p-1)(2(n-1)-\Delta-p) - \frac{n-p-1}{2}\left(\sqrt{\Delta_2} - \sqrt{\delta_p}\right)^2} \\
&= \sqrt{(n-p-1)\left(2(n-1)-\Delta-p-\frac{1}{2}\left(\sqrt{\Delta_2} - \sqrt{\delta_p}\right)^2\right)}\,.
\end{aligned} \tag{10}$$

From the above and inequality (9) we arrive at (8).

Equality in (10) holds if and only if $\sqrt{d_3} = \cdots = \sqrt{d_{n-p-1}} = \frac{\sqrt{\Delta_2}+\sqrt{\delta_p}}{2}$, and in (9) if and only if $T \cong K_{1,n-1}$, which implies that equality in (8) holds if and only if $T \cong K_{1,n-1}$. □

Since $(\sqrt{\Delta_2} - \sqrt{\delta_p})^2 \geq 0$, we have the following corollary of Theorem 3.1.

Corollary 3.1. *Let T be a tree of order $n \geq 3$ with p, $2 \leq p \leq n-1$, pendant vertices. Then*

$$LEL(T) \leq p - 1 + \sqrt{1+\Delta} + \sqrt{(n-p-1)(2(n-1)-p-\Delta)}\,. \tag{11}$$

Equality holds if and only if $T \cong K_{1,n-1}$.

The proof of the next theorem is analogous to that of Theorem 3.1, hence omitted.

Theorem 3.2. *Let T be a tree with $n \geq 3$ vertices. Then*

$$LEL(T) \leq 1 + \sqrt{1+\Delta} + \sqrt{(n-3)\left(2(n-2)-\Delta-\frac{1}{2}(\sqrt{\Delta_2} - \sqrt{\delta_2})^2\right)}\,. \tag{12}$$

Equality holds if and only if $T \cong K_{1,n-1}$.

Remark 3.1. *Since $(\sqrt{\Delta_2} - \sqrt{\delta_2})^2 \geq 0$, the following inequalities are valid*

$$\begin{aligned}
LEL(T) &\leq 1 + \sqrt{1+\Delta} + \sqrt{(n-3)\left(2(n-2)-\Delta-\frac{1}{2}(\sqrt{\Delta_2} - \sqrt{\delta_2})^2\right)} \\
&\leq 1 + \sqrt{1+\Delta} + \sqrt{(n-3)(2(n-2)-\Delta)}
\end{aligned}$$

which means that inequality (12) is stronger than (4).

In the next theorem we determine an upper bound on $LEL(T)$ ($IE(T)$) in terms of n, $\det D$, Δ and p.

Theorem 3.3. *Let T be a tree of order $n \geq 3$ with p, $2 \leq p \leq n-1$, pendant vertices. Then*

$$LEL(T) \leq p - 1 + \sqrt{1+\Delta} + + \sqrt{(n-p-2)(2(n-1)-p-\Delta) + (n-p-1)\left(\frac{\det D}{\Delta}\right)^{\frac{1}{n-p-1}}}\,. \tag{13}$$

Equality holds if and only if $T \cong K_{1,n-1}$.

Proof. If $p = n-1$, then $T \cong K_{1,n-1}$ and equality occurs in (13). Therefore, without affecting the generality, assume that $2 \le p \le n-2$. The inequality (6) can be considered in the following form

$$\left(\sum_{i=2}^{n-p} \sqrt{a_i}\right)^2 \le (n-p-2)\sum_{i=2}^{n-p} a_i + (n-p-1)\left(\prod_{i=2}^{n-p} a_i\right)^{\frac{1}{n-p-1}}.$$

For $a_i = d_i$, $i = 2, \ldots, n-p$, the above inequality becomes

$$\left(\sum_{i=2}^{n-p} \sqrt{d_i}\right)^2 \le (n-p-2)\sum_{i=2}^{n-p} d_i + (n-p-1)\left(\prod_{i=2}^{n-p} d_i\right)^{\frac{1}{n-p-1}},$$

that is

$$\left(\sum_{i=2}^{n-p} \sqrt{d_i}\right)^2 \le (n-p-2)(2(n-1)-p-\Delta) + (n-p-1)\left(\frac{\det D}{\Delta}\right)^{\frac{1}{n-p-1}}. \tag{14}$$

From the above and inequality (9) we arrive at (13).

Equality in (14) holds if and only if $\Delta_2 = d_2 = \cdots = d_{n-p} = \delta_p$, whereas in (9) if and only if $T \cong K_{1,n-1}$, which implies that equality in (13) holds if and only if $T \cong K_{1,n-1}$. □

Remark 3.2. *According to arithmetic–geometric mean inequality* (AGM) *(see e.g. [12]) we have that*

$$(n-p-1)\left(\prod_{i=2}^{n-p} d_i\right)^{\frac{1}{n-p-1}} \le \sum_{i=2}^{n-p} d_i = 2(n-1)-p-\Delta\,.$$

Therefore

$$\begin{aligned} LEL(T) &\le p-1+\sqrt{1+\Delta}+\sqrt{(n-p-2)(2(n-1)-p-\Delta)+(n-p-1)\left(\frac{\det D}{\Delta}\right)^{\frac{1}{n-p-1}}} \\ &\le p-1+\sqrt{1+\Delta}+\sqrt{(n-p-2)(2(n-1)-p-\Delta)+2(n-1)-p-\Delta} \\ &= p-1+\sqrt{1+\Delta}+\sqrt{(n-p-1)(2(n-1)-p-\Delta)}\,, \end{aligned}$$

which means that inequality (13) *is stronger than* (11).

The proof of the next Theorem is analogous to that of Theorem 3.3, thus omitted.

Theorem 3.4. *Let T be a tree with $n \ge 4$ vertices. Then*

$$LEL(T) \le 1+\sqrt{1+\Delta}+\sqrt{(n-4)(2(n-2)-\Delta)+(n-3)\left(\frac{\det D}{\Delta}\right)^{\frac{1}{n-3}}}. \tag{15}$$

Equality holds if and only if $T \cong K_{1,n-1}$.

Remark 3.3. *Similarly as in the case of Remark 3.2, the following can be proved*

$$\begin{aligned} LEL(T) &\le 1+\sqrt{1+\Delta}++\sqrt{(n-4)(2(n-2)-\Delta)+(n-3)\left(\frac{\det D}{\Delta}\right)^{\frac{1}{n-3}}} \\ &\le 1+\sqrt{1+\Delta}+\sqrt{(n-3)(2(n-2)-\Delta)}\,. \end{aligned}$$

This means that inequality (15) *is stronger than* (4).

The zeroth order Randić index is a degree based graph invariant introduced in [7] as

$$^0R(G) = \sum_{i=1}^{n} \frac{1}{\sqrt{d_i}}\,.$$

In the next theorem we determine an upper bound on $LEL(T)$ in terms of $^0R(T)$.

Theorem 3.5. *Let T be a tree of order $n \ge 5$ with p, $2 \le p \le n-1$, pendant vertices. Then*

$$LEL(T) \le p-1+\sqrt{1+\Delta}+(n-p-1)(\sqrt{\Delta_2}+\sqrt{\delta_p})-\sqrt{\Delta_2\delta_p}\left({}^0R(T)-\frac{1}{\sqrt{\Delta}}-p\right). \tag{16}$$

Equality holds if and only if $T \cong K_{1,n-1}$.

Proof. For every i, $i = 2, \dots, n-p$, holds

$$(\sqrt{d_i} - \sqrt{\Delta_2})(\sqrt{d_i} - \sqrt{\delta_p}) \le 0\,,$$

$$\sqrt{d_i} + \frac{\sqrt{\Delta_2 \delta_p}}{\sqrt{d_i}} \le \sqrt{\Delta_2} + \sqrt{\delta_p}\,. \tag{17}$$

After summation over i, $i = 2, \dots, n-p$, of the above inequality, we get

$$\sum_{i=2}^{n-p} \sqrt{d_i} + \sqrt{\Delta_2 \delta_p} \sum_{i=2}^{n-p} \frac{1}{\sqrt{d_i}} \le (n-p-1)(\sqrt{\Delta_2} + \sqrt{\delta_p})\,,$$

that is

$$\sum_{i=2}^{n-p} \sqrt{d_i} \le (n-p-1)(\sqrt{\Delta_2} + \sqrt{\delta_p}) - \sqrt{\Delta_2 \delta_p} \left({}^0R(T) - \frac{1}{\sqrt{\Delta}} - p \right)\,.$$

Now, from the above and inequality (9) we obtain (16).

Equality in (17) holds if and only if $d_i \in \{\Delta_2, \delta_p\}$, for $i = 2, \dots, n-p$. On the other hand, equality in (9) holds if and only if $T \cong K_{1,n-1}$, which implies that equality in (16) holds under same condition. □

Corollary 3.2. *Let T be a tree of order $n \ge 5$ with p, $2 \le p \le n-1$, pendant vertices. Then*

$$LEL(T) \le p - 1 + \sqrt{1+\Delta} + \frac{(n-p-1)^2 \left(\sqrt[4]{\frac{\Delta_2}{\delta_p}} + \sqrt[4]{\frac{\delta_p}{\Delta_2}} \right)^2}{4 \left({}^0R(T) - \frac{1}{\sqrt{\Delta}} - p \right)}\,. \tag{18}$$

Equality holds if and only if $T \cong K_{1,n-1}$.

Proof. According to AGM inequality we have that

$$\begin{aligned} 2\sqrt{\sqrt{\Delta_2 \delta_p} \left({}^0R(T) - \frac{1}{\sqrt{\Delta}} - p \right) \sum_{i=2}^{n-p} \sqrt{d_i}} \;&\le\; \sum_{i=2}^{n-p} \sqrt{d_i} + \sqrt{\Delta_2 \delta_p} \left({}^0R(T) - \frac{1}{\sqrt{\Delta}} - p \right) \\ &\le\; (n-p-1)(\sqrt{\Delta_2} + \sqrt{\delta_p})\,, \end{aligned}$$

that is

$$\sum_{i=2}^{n-p} \sqrt{d_i} \le \frac{(n-p-1)^2 \left(\sqrt[4]{\frac{\Delta_2}{\delta_p}} + \sqrt[4]{\frac{\delta_p}{\Delta_2}} \right)^2}{4 \left({}^0R(T) - \frac{1}{\sqrt{\Delta}} - p \right)}\,.$$

From the above and inequality (9) we obtain (18). □

Similarly as in the case of Theorem 3.5 the following theorem can be proved.

Theorem 3.6. *Let T be a tree with $n \ge 5$ vertices. Then*

$$LEL(T) \le 1 + \sqrt{1+\Delta} + (n-3)(\sqrt{\Delta_2} + \sqrt{\delta_2}) - \sqrt{\Delta_2 \delta_2} \left({}^0R(T) - \frac{1}{\sqrt{\Delta}} - 2 \right)\,. \tag{19}$$

Equality holds if and only if $T \cong K_{1,n-1}$.

Corollary 3.3. *Let T be a tree with $n \ge 5$ vertices. Then*

$$LEL(T) \le 1 + \sqrt{1+\Delta} + \frac{(n-3)^2 \left(\sqrt[4]{\frac{\Delta_2}{\delta_2}} + \sqrt[4]{\frac{\delta_2}{\Delta_2}} \right)^2}{4 \left({}^0R(T) - \frac{1}{\sqrt{\Delta}} - 2 \right)}\,.$$

Equality holds if and only if $T \cong K_{1,n-1}$.

Theorem 3.7. *Let T be a tree of order $n \ge 4$ with p pendent vertices. When $p = n-1$, then*

$$LEL(T) = n - 2 + \sqrt{n}\,.$$

When $0 \le p \le n-2$, then

$$LEL(T) \le p - 1 + \sqrt{1+\Delta} + \frac{2(n-1) - \Delta - p}{\sqrt{\Delta_2 \delta_p}} \left(\sqrt{\Delta_2} + \sqrt{\delta_p} - \sqrt{\frac{2(n-1) - \Delta - p}{n-p-1}} \right)\,. \tag{20}$$

Proof. After multiplying (17) by d_i and summation over i, for $i = 2, \dots, n-p$, we obtain

$$\sum_{i=2}^{n-p} d_i^{3/2} + \sqrt{\Delta_2 \delta_p} \sum_{i=2}^{n-p} \sqrt{d_i} \le \left(\sqrt{\Delta_2} + \sqrt{\delta_p} \right) \sum_{i=2}^{n-p} d_i \,,$$

that is

$$\sum_{i=2}^{n-p} d_i^{3/2} + \sqrt{\Delta_2 \delta_p} \sum_{i=2}^{n-p} \sqrt{d_i} \le \left(\sqrt{\Delta_2} + \sqrt{\delta_p} \right) (2(n-1) - \Delta - p) \,. \tag{21}$$

On the other hand, the inequality (7) can be considered in the following form

$$\sum_{i=2}^{n-p} \frac{x_i^{r+1}}{a_i^r} \ge \frac{\left(\sum_{i=2}^{n-p} x_i \right)^{r+1}}{\left(\sum_{i=2}^{n-p} a_i \right)^r} \,, \quad 0 \le p \le n-2 \,.$$

For $r = \frac{1}{2}$, $x_i = d_i$, $a_i = 1$, $i = 2, \dots, n-p$, the above inequality becomes

$$\sum_{i=2}^{n-p} d_i^{3/2} \ge \frac{\left(\sum_{i=2}^{n-p} d_i \right)^{3/2}}{\left(\sum_{i=2}^{n-p} 1 \right)^{1/2}} = \frac{(2(n-1) - \Delta - p)^{3/2}}{(n-p-1)^{1/2}} \,.$$

From the above and inequality (21) we obtain

$$\sum_{i=2}^{n-p} \sqrt{d_i} \le \frac{2(n-1) - \Delta - p}{\sqrt{\Delta_2 \delta_p}} \left(\sqrt{\Delta_2} + \sqrt{\delta_p} - \sqrt{\frac{2(n-1) - \Delta - p}{n-p-1}} \right) \,.$$

Now, from the above and inequality (21) we arrive at (20). □

The proof the next theorem is analogous to that of Theorem 3.7, hence omitted.

Theorem 3.8. *Let T be a tree with $n \ge 4$ vertices. Then*

$$LEL(T) \le 1 + \sqrt{1 + \Delta} + \frac{2(n-2) - \Delta}{\sqrt{\Delta_2 \delta_2}} \left(\sqrt{\Delta_2} + \sqrt{\delta_2} - \sqrt{\frac{2(n-2) - \Delta}{n-3}} \right) \,. \tag{22}$$

Equality holds if and only if $T \cong K_{1,n-1}$.

Corollary 3.4. *Let T be a tree with $n \ge 4$ vertices. Then*

$$LEL(T) \le 1 + \sqrt{1 + \Delta} + \frac{(2(n-2) - \Delta)(\sqrt{\Delta_2} + \sqrt{\delta_2} - 1)}{\sqrt{\Delta_2 \delta_2}} \,. \tag{23}$$

Equality holds if and only if $T \cong K_{1,n-1}$.

Proof. Since $2(n-2) - \Delta \ge n-3$, from the above and (22) we obtain (23). □

4. Comparison and discussion

In this section we compare the upper bounds for $LEL(T)$ obtained by inequalities (12), (15), (19) and (5) and give some numerical results.

Let $T = P_n$, $n \ge 3$. In that case the bounds (15) and (19) coincide, and are equal to

$$LEL(T) \le 1 + \sqrt{3} + \sqrt{2}(n-3) \,.$$

The upper bounds determined by (12) and (19) are incomparable. Namely, when $T = P_n$, $n \ge 3$, the bound (12) is stronger than (19). However, if T is a tree with the vertex degree sequence

$$\left(\frac{n}{2}, \frac{n}{2}, \underbrace{1, \dots, 1}_{n-2} \right),$$

then for $n \ge 6$, the bound (19) is stronger than (12).

Table 1: Numerical values of the bounds (12), (15), (19) and (5) when $T \cong P_n$.

n	Eq. (12)	Eq. (15)	Eq. (19)	Eq. (5)
5	5.5300	5.5605	5.5605	5.9180
10	12.6012	12.6315	12.6315	13.8534
20	26.7433	26.7737	26.7737	29.2713
50	69.1698	69.2001	69.2001	74.2715
100	139.8800	139.9110	139.9110	147.9010

Table 2: Numerical values of the bounds (12), (15), (19) and (5) when T has the degree sequence $(\frac{n}{2}, \frac{n}{2}, \overbrace{1, \ldots, 1}^{n-2})$.

n	Eq. (12)	Eq. (15)	Eq. (19)	Eq. (5)
10	11.9143	12.0987	11.6856	12.7814
20	24.3730	25.1844	23.4789	26.3264
30	36.5527	38.1036	34.8730	39.5747
50	60.5141	63.6866	57.0990	65.660
100	119.3820	126.9560	111.2120	129.8240

Table 1 gives the numerical values for $LEL(T)$ (i.e. $IE(T)$) obtained by inequalities (12), (15), (19) and (5) when tree is a path, $T \cong P_n$, that is for trees defined by the vertex degrees

$$(\underbrace{2, \ldots, 2}_{n-2}, 1, 1)$$

for $n = 5, 10, 20, 50, 100$.

Table 2 gives numerical values for the trees with the vertex degrees $(\frac{n}{2}, \frac{n}{2}, \underbrace{1, \ldots, 1}_{n-2})$ for $n = 10, 20, 30, 50, 100$.

According to Tables 1 and 2 we conclude that the upper bounds on $LEL(T)$ obtained by (12), (15) and (19) are stronger than the one obtained by (5). However, the open question is whether this is true for any tree T.

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