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*Research Article*

# On the comaximal (ideal) graph associated with amalgamated algebra

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**Abstract**

Let $f : A \to B$ be a ring homomorphism of the commutative rings $A$ and $B$ with identities. Let $J$ be an ideal of $B$. The amalgamation of $A$ with $B$ along $J$ with respect to $f$ is a subring of $A \times B$ given by $A \bowtie^f J := \{(a, f(a)+j) \mid a \in A\ ,\ j \in J\}$. In this paper, we investigate the comaximal ideal graph and the comaximal graph of the amalgamated algebra $A \bowtie^f J$. In particular, we determine the Jacobson radical of $A \bowtie^f J$, characterize the diameter of the comaximal ideal graph of $A \bowtie^f J$, and investigate the clique number as well as the chromatic number of this graph.

**Keywords:** comaximal graph; comaximal ideal graph; amalgamated algebra.

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## 1. Introduction

Throughout this paper, $A$ and $B$ denote commutative rings with non-zero identities and $f : A \to B$ is a ring homomorphism with the condition $f(1_A) = 1_B$. Also, assume that $J$ is an ideal of $B$.

The amalgamation of $A$ with $B$ along $J$ with respect to $f$ is a subring of $A \times B$, which is defined as follows:

$$A \bowtie^f J := \{(a, f(a)+j) \mid a \in A\ ,\ j \in J\}.$$

In the case when $f$ is the identity homomorphism and $I$ is an ideal $A$, the subring

$$A \bowtie I := \{(a, a+i) \mid a \in A\ ,\ i \in I\}$$

of $A \times A$ is called the amalgamated duplication of $A$ along $I$.

The concept of amalgamated algebra was introduced and studied by D'Anna, Finocchiaro, and Fontana in [4,5]. Clearly if $J = 0$, then $A \bowtie^f J \cong A$. Also, $A \bowtie^f J \cong A \times B$ whenever $J = B$. For detail about amalgamated algebra, the reader is referred to [4–6].

Let $R$ be a commutative ring with nonzero identity and $U(R)$ be the set of unit elements of $R$. Also, let $\mathrm{Jac}(R)$ denote the Jacobson radical of $R$ and $Z(R)$ be the set of all zero divisors of $R$. Moreover, $V(J)$ stands for the set of all prime ideals of $R$ containing the ideal $J$. In [12], Sharma and Bhatwadekar defined the comaximal graph of $R$, denoted by $\Gamma(R)$, whose vertices are all the elements of $R$ and two distinct vertices $a$ and $b$ are adjacent if and only if $Ra+Rb = R$. In 2008, Maimani et al. in [9], studied and investigated the induced subgraph of $\Gamma(R)$ on the set $R \setminus (U(R) \cup \mathrm{Jac}(R))$, which is denoted by $\Gamma_2'(R)$. In [14], Ye and Wu defined the comaximal ideal graph of the commutative ring $R$, denoted by $\mathcal{C}(R)$. The vertex set of $\mathcal{C}(R)$ consists of all proper ideals which are not contained in the Jacobson radical of $R$, and two distinct vertices $I_1$ and $I_2$ are adjacent if and only if $I_1 + I_2 = R$. Recently, in [13], some properties of the comaximal graph of an amalgamated algebra are investigated. The reader is referred to [8,9,14,15] for further detail on comaximal graphs.

The concept of the zero-divisor graph of a ring $R$ was introduced by Beak in [2] and further studied by Anderson and Naseer in [1]. The definition of the zero-divisor graph of a ring $R$ given by Anderson and Naseer is as follows. The zero-divisor graph of $R$ is the undirected graph with vertices $Z(R) \setminus \{0\}$, and two distinct vertices $x$ and $y$ are adjacent if and only if $xy = 0$. The zero-divisor graph of an amalgamated algebra is studied in several papers; for example, see [7,10,11].

In this paper, we study the comaximal ideal graph and the comaximal graph of the amalgamated algebra. In Section 2, we study some algebraic properties of $A \bowtie^f J$. Especially, we determine the Jacobson radical of $A \bowtie^f J$. In Section 3, we characterize the diameter of $\mathcal{C}(A \bowtie^f J)$ and investigate the clique number and the chromatic number of this graph. Finally, in Section 4, we study some properties of $\Gamma_2'(A \bowtie J)$.

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Now, we recall some definitions and notations on graphs by following the book [3]. The distance between two distinct vertices $a$ and $b$, denoted by $d(a,b)$, is the length of the shortest path connecting $a$ and $b$, if such a path exists; otherwise, we use $d(a,b) := \infty$. The diameter of a graph $G$ is $\mathrm{diam}(G) = \sup\{d(a,b) : a$ and $b$ are distinct vertices of $G\}$. The girth of $G$, denoted by $\mathrm{gr}(G)$, is the length of the shortest cycle in $G$, if such a cycle exists; otherwise, we use $\mathrm{gr}(G) := \infty$. We use the notations $K_n$ and $K_{m,n}$ to denote the complete graph with $n$ vertices and the complete bipartite graph with part sizes $m$ and $n$, respectively. The graph $K_{1,n}$ is known as the star graph. The vertex $x$ of degree $n$ is called the center of $K_{1,n}$. A subset $S \subseteq V(G)$ is called a clique if the subgraph induced by $S$ is complete. The number of vertices in the largest clique of the graph $G$ is called the clique number of $G$ and is denoted by $\omega(G)$. A coloring of a graph $G$ is an assignment of colors (elements of some set) to the vertices of $G$, one color to each vertex, so that adjacent vertices are assigned distinct colors. If $n$ colors are used, the coloring is referred as an $n$-coloring. If there is an $n$-coloring of a graph $G$, then $G$ is called $n$-colorable. The minimum $n$ for which a graph $G$ is $n$-colorable is called the chromatic number of $G$, which is denoted by $\chi(G)$.

## 2. Jacobson radical of $A \bowtie^f J$

In this section, we study the Jacobson radical of $A \bowtie^f J$ and specify the conditions under which $\mathrm{Jac}(A \bowtie^f J)$ is a prime ideal. The results of this section are used in determining the diameter of $\mathcal{C}(A \bowtie^f J)$. In the following proposition, which is taken from [5], the structures of the maximal and prime ideals of $A \bowtie^f J$ are specified.

**Proposition 2.1** (see [5], Proposition 2.6)**.** *Set* $X := \mathrm{Space}(A), Y := \mathrm{Space}(B)$*,* $W := \mathrm{Space}(A \bowtie^f J)$*, and*

$$J_0 := \{0\} \times J(\subseteq A \bowtie^f J).$$

*For all* $P \in X$ *and* $Q \in Y$*, set*

$$P'^f := P \bowtie^f J := \{(p, f(p)+j) \mid p \in P, j \in J\},$$
$$\bar{Q}^f := \{(a, f(a)+j) \mid a \in A, j \in J, f(a)+j \in Q\}.$$

*The following statements hold.*

1. *The map* $P \mapsto P'^f$ *establishes a closed embedding of* $X$ *into* $W$*, so its image, which coincides with* $V(J_0)$*, is homeomorphic to* $X$*.*
2. *The map* $Q \mapsto \bar{Q}^f$ *is a homeomorphism of* $Y \setminus V(J)$ *onto* $W \setminus J_0$*.*
3. *The prime ideals of* $A \bowtie^f J$ *are the type* $P'^f$ *or* $\bar{Q}^f$*, for* $P$ *varying in* $X$ *and* $Q$ *in* $Y \setminus V(J)$*.*
4. *Let* $P \in \mathrm{Space}(A)$*. Then,*$P'^f$ *is a maximal ideal of* $A \bowtie^f J$ *if and only if* $P$ *is a maximal ideal of* $A$*.*
5. *Let* $Q$ *be a prime ideal of* $B$ *not containing* $J$*. Then,* $\bar{Q}^f$ *is a maximal ideal of* $A \bowtie^f J$ *if and only if* $Q$ *is a maximal ideal of* $B$*.*

*In particular,* $\mathrm{Max}(A \bowtie^f J) = \{P'^f \mid P \in \mathrm{Max}(A)\} \cup \{\bar{Q}^f \mid Q \in \mathrm{Max}(B) \setminus V(J)\}$*.*

In [13, Proposition 3.2], the Jacobson radical of the ring $A \bowtie^f J$ is determined as follows.

**Theorem 2.1.** *Let* $f : A \to B$ *be a ring homomorphism and let* $J$ *be an ideal of* $B$*. Then*

$$\mathrm{Jac}(A \bowtie^f J) = \left\{ \big(a, f(a)+j\big) \;\mid\; a \in \mathrm{Jac}(A),\; j \in J,\; f(a)+j \in \bigcap_{Q \in \mathrm{Max}(B) \setminus V(J)} Q \right\}.$$

The following lemma has an important role in Theorem 2.2. Note that when $f : A \to B$ is an isomorphism, then $A \bowtie^f J$ is isomorphic to the amalgamated duplication $B \bowtie J$. So, by assuming this hypothesis, the statements concern essentialy the more special case of the amalgamated duplication (and not the general case of amalgamated algebra).

**Lemma 2.1.** *Suppose that* $f$ *is an isomorphism from* $A$ *to* $B$*. If* $\mathrm{Jac}(A \bowtie^f J) = P \bowtie^f J$*, for some prime ideal* $P$*, then* $P = \mathrm{Jac}(A)$*.*

*Proof.* Assume that $P$ is a prime ideal such that $\mathrm{Jac}(A \bowtie^f J) = P \bowtie^f J$. By Theorem 2.1, we have

$$\mathrm{Jac}(A \bowtie^f J) = \left\{ (a, f(a) + j) \quad | \quad a \in \mathrm{Jac}(A), j \in J, f(a) + j \in \bigcap_{Q \in \mathrm{Max}(B) \setminus V(J)} Q \right\}.$$

Since $f$ is an isomorphism, it's clear that $f(\mathrm{Jac}(A)) \cong \mathrm{Jac}(f(A))$ and $f(A) \cong B$. Now, if $a \in \mathrm{Jac}(A)$, then $f(a) \in \mathrm{Jac}(B)$ and clearly $\mathrm{Jac}(B) \subseteq \cap Q$, where $Q \in \mathrm{Max}(B) \setminus V(J)$. For each $a \in \mathrm{Jac}(A)$ with $f(a) + j \in \cap Q$, where $Q \in \mathrm{Max}(b) \setminus V(j)$, we have

$$j \in J \cap \left( \bigcap_{Q \in \mathrm{Max}(B) \setminus V(J)} Q \right).$$

Suppose that $a$ an arbitrary element of $P$. Since $(a, f(a) + j) \in P \bowtie^f J$, we have $(a, f(a) + j) \in \mathrm{Jac}(A \bowtie^f J)$ for every $j \in J$. So $a \in \mathrm{Jac}(A)$. Hence $P \subseteq \mathrm{Jac}(A)$. Now, assume that $r \in \mathrm{Jac}(A)$ be an arbitrary element. Hence $(r, f(r) + 0) \in \mathrm{Jac}(A \bowtie^f J)$. By our assumption, $\mathrm{Jac}(A \bowtie^f J) = P \bowtie^f J$. Hence $r \in P$ and so we are done. □

In the next two theorems, we describe situations under which $\mathrm{Jac}(A \bowtie^f J)$ is a prime ideal.

**Theorem 2.2.** *Suppose that $f$ is an isomorphism from $A$ to $B$. Then $\mathrm{Jac}(A \bowtie^f J) = P \bowtie^f J$, where $P$ is a prime ideal if and only if $P = \mathrm{Jac}(A)$ is a prime ideal and $J \subseteq \mathrm{Jac}(B)$.*

*Proof.* First, assume that $\mathrm{Jac}(A \bowtie^f J) = P \bowtie^f J$, where $P$ is a prime ideal. By Lemma 2.1, we have $\mathrm{Jac}(A) = P$. Now, we show that $J \subseteq \mathrm{Jac}(B)$. Since $(0, j) \in P \bowtie^f J = \mathrm{Jac}(A \bowtie^f J)$ for every $j \in J$, we have $J \subseteq \cap Q$, where $Q \in \mathrm{Max}(B) \setminus V(J)$. Thus $\mathrm{Max}(B) \setminus V(J) = \emptyset$. Hence $J \subseteq Q$, for every $Q \in \mathrm{Max}(B)$. Therefore $J \subseteq \mathrm{Jac}(B)$.

Conversely, suppose that $\mathrm{Jac}(A) = P$ is a prime ideal and $J \subseteq \mathrm{Jac}(B)$. Then $\mathrm{Max}(B) \setminus V(J) = \emptyset$. Hence $\cap Q = B$, where $Q \in \mathrm{Max}(B) \setminus V(J)$. Since $\mathrm{Jac}(A)$ is a prime ideal, one can easily see that

$$\mathrm{Jac}(A \bowtie^f J) = \{(a.f(a) + j) \mid a \in \mathrm{Jac}(A), \quad j \in J, \quad f(a) + j \in B\}$$

is a prime ideal which is equal to $P \bowtie^f J$. □

**Example 2.1.** *Let $A = B = \mathbb{Z}_8$, $P = J =< 2 >$. In this case, we can describe explicitly and easily the prime spectrum of Spec$(A)$ (or Spec$(B)$) and the prime spectrum of Spec$(A \bowtie^f J)$. Clearly $P = \mathrm{Jac}(A)$ is a prime ideal and $J \subseteq \mathrm{Jac}(B)$. Hence, by Theorem 2.2, $\mathrm{Jac}(A \bowtie^f J) = P \bowtie^f J$ which is a prime ideal.*

**Lemma 2.2.** *Assume that $f$ is an isomorphism from $A$ to $B$. If $J \cap (\cap Q) \notin \{\{0\}, J\}$, where $Q \in \mathrm{Max}(B) \setminus V(J)$, then $\mathrm{Jac}(A \bowtie^f J)$ is not a prime ideal of $A \bowtie^f J$.*

*Proof.* Assume that $J \cap (\cap Q) \notin \{\{0\}, J\}$, where $Q \in \mathrm{Max}(B) \setminus V(J)$. So there exist non-zero element $j_1$ in $J \cap (\cap Q)$ and $j_2 \in J \setminus \cap Q$, where $Q \in \mathrm{Max}(B) \setminus V(J)$. Let $a \in \mathrm{Jac}(A)$. We consider the following two cases.

*Case 1.* There exists $b \in \cap Q \setminus \mathrm{Jac}(B)$, where $Q \in \mathrm{Max}(B) \setminus V(J)$. Then $(a, f(a) + j_2)(f^{-1}(b), b + j_1) \in \mathrm{Jac}(A \bowtie^f J)$. Since $b \notin \mathrm{Jac}(B)$ and $\mathrm{Jac}(B) \cong \mathrm{Jac}(A)$, we have $(f^{-1}(b), b + j_1) \notin \mathrm{Jac}(A \bowtie^f J)$. Since $a \in \mathrm{Jac}(A)$ and $f$ is an isomorphism, $f(a) \in \mathrm{Jac}(B) \subseteq \cap Q$, where $Q \in \mathrm{Max}(B) \setminus V(J)$. Thus $f(a) + j_2 \notin \cap Q$, where $Q \in \mathrm{Max}(B) \setminus V(J)$. Otherwise $j_2 \in \cap Q$ which is impossible. Hence $(a, f(a) + j_2) \notin \mathrm{Jac}(A \bowtie^f J)$. Therefore in this case $\mathrm{Jac}(A \bowtie^f J)$ is not a prime ideal.

*Case 2.* $\cap Q \subseteq \mathrm{Jac}(B)$, where $Q \in \mathrm{Max}(B) \setminus V(J)$. Since $\mathrm{Jac}(B) \subseteq \cap Q$, we have $\cap Q = \mathrm{Jac}(B)$, where $Q \in \mathrm{Max}(B) \setminus V(J)$. Hence $V(J) = \emptyset$ which implies that $J = B$. Thus $A \bowtie^f J \cong A \times B$. Since $f$ is an isomorphism, $\mathrm{Jac}(A \bowtie^f J) \cong \mathrm{Jac}(A \times A) \cong \mathrm{Jac}(A) \times \mathrm{Jac}(A)$. It is straight forward to cheek that $\mathrm{Jac}(A \bowtie^f J)$ is not a prime ideal. □

**Theorem 2.3.** *Assume that $f$ is an isomorphism from $A$ to $B$. Then $\mathrm{Jac}(A \bowtie^f J) = \bar{P^f}$, where $P$ is a prime ideal of $A$ not containing $J$, if and only if $P = \mathrm{Jac}(B)$ is a prime ideal and $J \cap (\cap Q) \in \{\{0\}, J\}$, where $Q \in \mathrm{Max}(B) \setminus V(J)$.*

*Proof.* First, suppose that $\mathrm{Jac}(A \bowtie^f J) = \bar{P^f}$ where $P$ is a prime ideal of $A$ not containing $J$ . By Lemma 2.2,

$$J \cap (\cap Q) \in \{\{0\}, J\},$$

where $Q \in \mathrm{Max}(B) \setminus V(J)$. Now, Assume that $p$ is an arbitrary element in $P$. Since $(f^{-1}(p), p + 0) \in \bar{P^f}$ and $\mathrm{Jac}(A \bowtie^f J) = \bar{P^f}$, we have $f^{-1}(p) \in \mathrm{Jac}(A)$. Thus $p \in f(\mathrm{Jac}(A))$. Since $f$ is an isomorphism, $p \in \mathrm{Jac}(B)$. Let $x \in \mathrm{Jac}(B)$ be an arbitrary element. Clearly $(f^{-1}(x), x + 0) \in \mathrm{Jac}(A \bowtie^f J) = \bar{P^f}$. Then $x \in P$. Hence $P = \mathrm{Jac}(B)$.

Conversely, suppose that $P = \mathrm{Jac}(B)$ is a prime ideal and $J \cap (\cap Q) \in \{\{0\}, J\}$, where $Q \in \mathrm{Max}(B) \setminus V(J)$. We show that $\mathrm{Jac}(A \bowtie^f J) = \bar{P^f}$. We have two cases.

*Case 1.* $J \cap (\cap Q) = \{0\}$, where $Q \in \mathrm{Max}(B) \setminus V(J)$. If there exist $a \in \mathrm{Jac}(A)$ and $j \in J$ such that $f(a) + j \in \cap Q$, then $j \in \cap Q$, where $Q \in \mathrm{Max}(B) \setminus V(J)$. Thus $j = 0$ which implies that $\mathrm{Jac}(A \bowtie^f J) = \{(a, f(a)) \mid a \in \mathrm{Jac}(A)\}$. Clearly, in this situation $\mathrm{Jac}(A \bowtie^f J)$ is a prime ideal. If $J = 0$, then it is easy to see that $\mathrm{Jac}(A \bowtie^f J) = \bar{P^f}$. Now, suppose that $J \neq 0$. By Theorem 2.2, $\mathrm{Jac}(A \bowtie^f J) \neq K \bowtie^f J$, where $K$ is a prime ideal of $A$. Assume that $\mathrm{Jac}(A \bowtie^f J) = \bar{Q^f}$, for some prime ideal $Q$. We prove that $Q = P$. We have $(f^{-1}(p), p + 0) \in \bar{Q^f}$ for every $p \in P$. So $p \in Q$. Hence $P \subseteq Q$. Also, $(f^{-1}(q), q + 0) \in \bar{Q^f}$ for every $q \in Q$. Thus $f^{-1}(q) \in \mathrm{Jac}(A)$. Since $f$ is an isomorphism, $q \in \mathrm{Jac}(B)$. Therefore $Q = P$.

*Case 2.* $J \subseteq \cap Q$, where $Q \in \mathrm{Max}(B) \setminus V(J)$. Then $J \subseteq \mathrm{Jac}(B)$. In this situation, by Theorem 2.2, $\mathrm{Jac}(A \bowtie^f J) = P \bowtie^f J$. One can easily prove that $\bar{P^f} = P \bowtie^f J$. □

**Example 2.2.** *Let $A = B = \mathbb{Z}_6$, $p = J =< 2 >$, and $q =< 3 >$. In this case, we can describe explicitly and easily the prime spectrum of Spec$(A)$ (or Spec$(B)$) and the prime spectrum of Spec$(A \bowtie^f J)$. Clearly $\mathrm{Jac}(A \bowtie^f J) = 0$ which is not a prime ideal. One can easily see that none of the conditions of Theorems 2.2 and 2.3 hold for $\mathrm{Jac}(A \bowtie^f J)$.*

We end this section with the following lemma which explains a situation when $A \bowtie^f J$ has no proper ideal.

**Lemma 2.3.** *Assume that $I \trianglelefteq A \bowtie^f J$, $J \subseteq \mathrm{Jac}(B)$ and $(1, 1 + j) \in I$. Then $I = A \bowtie^f J$, where $f(1_A) = 1_B$.*

*Proof.* Assume that $j \in J$. Since $J \subseteq \mathrm{Jac}(B)$, $1 + j$ is a unit element in $B$. So there exists $u \in B$ such that $(1 + j)u = 1$. Clearly $(1, 1 + (-ju)) \in A \bowtie^f J$. We have

$$(1, 1 + j)(1, 1 + (-ju)) = (1, 1 - ju + j - j^2u) = (1, 1 + 0) = 1_{A \bowtie^f J},$$

because

$$-ju + j - j^2u = -ju(1 + j) + j = 0.$$

Hence $(1, 1) \in I$ and therefore $I = A \bowtie^f J$. □

## 3. Diameter, clique number, and chromatic number

In this section, we study some basic properties of the comaximal ideal graph of $A \bowtie^f J$.

**Remark 3.1.** *Suppose that $I \trianglelefteq A \bowtie^f J$. Then one can easily cheek that*

$$I_1 = \{\alpha \in A \mid (\alpha, f(\alpha) + \beta) \in I, \text{for some } \beta \in J\}$$

*is an ideal of $A$. Also, if $I$ is an ideal of $A$, then it is easy to see that $I \bowtie^f J$ is an ideal of $A \bowtie^f J$.*

**Lemma 3.1.** *The graph $\mathcal{C}(A)$ is isomorphic to an induced subgraph of $\mathcal{C}(A \bowtie^f J)$.*

*Proof.* Assume that $\{J_i\}_{i \in I}$ is a family of distinct vertices of $\mathcal{C}(A)$. Then, by Remark 3.1, $\{J_i \bowtie^f J\}_{i \in I}$ are distinct vertices in $\mathcal{C}(A \bowtie^f J)$. Now, let $G$ be the induce subgraph of $\mathcal{C}(A \bowtie^f J)$ with vertex set $\{J_i \bowtie^f J\}_{i \in I}$. Clearly $|V(\mathcal{C}(A))| = |V(G)|$. Also, one can easily cheek that $J_i$ is adjacent to $J_j$ in $\mathcal{C}(A)$ if and only if $J_i \bowtie^f J$ is adjacent to $J_j \bowtie^f J$ in $\mathcal{C}(A \bowtie^f J)$, for each distinct $i$ and $j$ in $I$. Hence the result holds. □

In the following, we investigate the completeness of $\mathcal{C}(A \bowtie^f J)$.

**Proposition 3.1.** *If $\mathcal{C}(A \bowtie^f J)$ is a complete graph, then $\mathcal{C}(A)$ is a complete graph.*

*Proof.* If $A$ is a local ring, then clearly $\mathcal{C}(A)$ is a null graph, and so there is nothing to prove. Now, assume that $A$ is not local and that $I$ and $I'$ are two distinct vertices in $\mathcal{C}(A)$. Then $I \bowtie^f J$ and $I' \bowtie^f J$ are two distinct vertices in $\mathcal{C}(A \bowtie^f J)$. Since $\mathcal{C}(A \bowtie^f J)$ is a complete graph, we have $I \bowtie^f J + I' \bowtie^f J = A \bowtie^f J$. So there are elements $(a, f(a) + j) \in I' \bowtie^f J$ and $(b, f(b) + \gamma) \in I \bowtie^f J$ such that

$$(a, f(a) + j) + (b, f(b) + \gamma) = 1_{A \bowtie^f J} = (1, 1).$$

Hence $a + b = 1_A$, which implies that $I + I' = A$. Therefore $\mathcal{C}(A)$ is a complete graph. □

In [14], the completeness of the comaximal ideal graph is investigated.

**Theorem 3.1** (see [14], Theorem 4.6)**.** *Let $R$ be a commutative ring. Then the following statements are equivallent:*

*(i)* $\mathcal{C}(R)$ *is a complete graph.*

*(ii)* $\mathrm{diam}(\mathcal{C}(R))=1$*.*

*(iii)* $\mathcal{C}(R) \cong K_2$*.*

*(iv)* $R \cong F_1 \times F_2$*, where $F_1$ and $F_2$ are fields.*

It is easy to deduce the following corollary from Proposition 3.1 and Theorem 3.1.

**Corollary 3.1.** *Let $A$ be a non-local ring such that $\mathcal{C}(A \bowtie^f J)$ is a complete graph. Then $A \cong F_1 \times F_2$, where $F_1$ and $F_2$ are fields.*

**Lemma 3.2.** *Suppose that $A$ is a non-local ring such that $\mathcal{C}(A \bowtie^f J)$ is a complete graph. Then $J \subseteq \mathrm{Jac}(B)$.*

*Proof.* Assume that $\mathcal{C}(A \bowtie^f J)$ is a complete graph. Then, by [14, Theorem 4.6], $\mathcal{C}(A \bowtie^f J) \cong E_1 \times E_2$, where $E_1$ and $E_2$ are two fields. Therefore $A \bowtie^f J$ has exactly two maximal ideals. Also, by Corollary 3.1, $A \cong F_1 \times F_2$, where $F_1$ and $F_2$ are fields. So, by Proposition 2.1, $F_1 \times \{0\} \bowtie^f J$ and $\{0\} \times F_2 \bowtie^f J$ are two maximal ideals of $A \bowtie^f J$. Thus again, by Proposition 2.1, there exists no maximal ideal of type $\bar{Q}^f$, or $\bar{Q}^f$ is one of the maximal ideals $F_1 \times \{0\} \bowtie^f J$ or $\{0\} \times F_2 \bowtie^f J$, where $Q \in \mathrm{Max}(B) \setminus V(J)$.

Now, if there exists $Q \in \mathrm{Max}(B) \setminus V(J)$, then $J \nsubseteq Q$ and also $\bar{Q}^f$ is a maximal ideal of $A \bowtie^f J$. So there is $\beta \in J \setminus Q$, which implies that $(0, \beta) \notin \bar{Q}^f$. Since $(0, \beta) \in F_1 \times \{0\} \bowtie^f J$ and $(0, \beta) \in \{0\} \times F_2 \bowtie^f J$, we have $\bar{Q}^f \neq F_1 \times \{0\} \bowtie^f J$ and $\overline{Q}^f \neq \{0\} \times F_2 \bowtie^f J$, which is a contradiction. Thus $Q \notin \mathrm{Max}(B) \setminus V(J)$. Therefore $\mathrm{Max}(B) \subseteq V(J)$. Hence, for each ideal $Q$ of $\mathrm{Max}(B)$, we have $J \subseteq Q$, which implies that $J \subseteq \mathrm{Jac}(B)$. □

**Proposition 3.2.** *Assume that $\mathcal{C}(A)$ is a complete graph, and that $J \subseteq \mathrm{Jac}(B)$. Then $\mathcal{C}(A \bowtie^f J)$ is a complete bipartite graph, where $f(1_A) = 1_B$.*

*Proof.* Let $\mathcal{C}(A)$ be a complete graph, and suppose that $J \subseteq \mathrm{Jac}(B)$. Then $A \cong F_1 \times F_2$, where $F_1$ and $F_2$ are fields. Hence $F_1 \times \{0\} \bowtie^f J$ and $\{0\} \times F_2 \bowtie^f J$ are two maximal ideals of $A \bowtie^f J$. We claim that $A \bowtie^f J$ has exactly two maximal ideals. Assume on the contrary that there is a maximal ideal $M$ such that $M \nsubseteq F_1 \times \{0\} \bowtie^f J$ and $M \nsubseteq \{0\} \times F_2 \bowtie^f J$. Thus there exists an element $((e_1, e_1), f(e_1, e_2) + j)$ in $M$, where $e_1 \neq 0$ and $e_2 \neq 0$. Since

$$((e_1, e_1), f(e_1, e_2) + j).((e_1^{-1}, e_1^{-1})f(e_1^{-1}, e_2^{-1}) + 0) \in M,$$

we have $((1, 1), f(1, 1) + jf(e_1^{-1}, e_2^{-1})) \in M$. So, by Lemma 2.3, we have $M = A \bowtie^f J$, which is a contradiction. Therefore $|\mathrm{Max}(A \bowtie^f J)| = 2$, and so $\mathcal{C}(A \bowtie^f J)$ is a complete bipartite graph. □

In the following theorem, we completely determine the case under which the graph $\mathcal{C}(A \bowtie^f J)$ is complete.

**Theorem 3.2.** *Suppose that $A$ is a local ring and $f$ is an isomorphism. Then $\mathcal{C}(A \bowtie^f J)$ is a complete graph if and only if $A$ is a field and $J = B$.*

*Proof.* Let $(A, M)$ be a local ring and let $\mathcal{C}(A \bowtie^f J)$ be a complete graph. So, by [14, Theorem 4.6], we have $|\mathrm{Max}(A \bowtie^f J)| = 2$. It is clear that $J \neq \{0\}$. Otherwise $A \bowtie^f J \cong A$ which is a contradiction with the fact that $|\mathrm{Max}(A \bowtie^f J)| = 2$. If $M \neq \{0\}$, then $0 \bowtie^f J$ is an ideal of $A \bowtie^f J$ which is contained in $M \bowtie^f J$. We know that $J \nsubseteq \mathrm{Jac}(B)$. Otherwise, $|\mathrm{Max}(A \bowtie^f J)| = 1$. So $0 \bowtie^f J \nsubseteq \mathrm{Jac}(A \bowtie^f J)$. Thus $0 \bowtie^f J$ and $M \bowtie^f J$ are two vertices which are not adjacent and this is a contradiction. So $M = \{0\}$. Hence $A$ is a field. Since $J \neq 0$ and $f$ is an isomorphism, we have $J = B$.

Conversely, assume that $A$ is a field and $J = B$. Since $f$ is an isomorphism, $B$ is a field. Hence it's clear that $\mathcal{C}(A \bowtie^f J)$ is a complete graph. □

**Theorem 3.3.** *Assume that $A$ is a non-local ring and $f(1_A) = 1_B$. Then $\mathcal{C}(A \bowtie^f J)$ is a complete graph if and only if $\mathcal{C}(A)$ is a complete graph and $J = 0$.*

*Proof.* First suppose that $\mathcal{C}(A \bowtie^f J)$ is a complete graph. Then, by Proposition 3.1, $\mathcal{C}(A)$ is a complete graph. So $A \cong F_1 \times F_2$, where $F_1$ and $F_2$ are fields. Also $A \bowtie^f J \cong E_1 \times E_2$, where $E_1$ and $E_2$ are fields. Hence $(F_1 \times F_2) \bowtie^f J \cong E_1 \times E_2$, which implies that the maximal ideals $(F_1 \times \{0\}) \bowtie^f J$ and $(\{0\} \times F_2) \bowtie^f J$ of the ring $(F_1 \times F_2) \bowtie^f J$ are in the correspondence with the maximal ideals $E_1 \times \{0\}$ and $\{0\} \times E_2$ of $E_1 \times E_2$. Since $(0, 1) \in \{0\} \times E_2$ and $(1, 0) \in E_1 \times \{0\}$, we have that the elements $((0,1), f(0,1)+j)$ and $((1,0), f(1,0)+t)$, belonging to $(\{0\} \times F_2) \bowtie^f J$ and $(F_1 \times \{0\}) \bowtie^f J$, respectively, for some $j, t \in J$. Hence, for every $a \in F_1$, $b \in F_2$ and $l \in J$, we must have the following equations:

$$((0,1), f(0,1)+j)((0,b), f(0,b)+l) = ((0,b), f(0,b)+l) \tag{1}$$

$$((1,0), f(1,0)+t)((a,0), f(a,0)+l) = ((a,0), f(a,0)+l). \tag{2}$$

From Equation (1), we have

$$f(0,b) + lf(0,1) + jf(0,b) + jl = f(0,b) + l \quad \forall \; b \in F_2, \; \forall \; l \in J. \tag{3}$$

In the Equation (3), by putting $l = 0$, we get the relation $jf(0,b) = 0$, for all $b \in F_2$. Now, since $f(1,0) + f(0,1) = (1,1)$, by Equation (3), we have $jl = lf(1,0)$, for all $l \in J$. Thus $l(j - f(1,0)) = 0$ for all $l \in J$. So $J(j - f(1,0)) = 0$.

Now, from the Equation (2), we have the relation

$$f(a,0) + lf(1,0) + tf(a,0) + lt = f(a,0) + l \quad \forall \; a \in F_1, \; \forall \; l \in J.$$

By applying a similar method which was used in the previous paragraph, we have $tf(a,0) = 0$, for all $a \in F_1$, and $J(t - f(0,1)) = 0$. Since $jt - jf(0,1) = 0$ and $jf(0,1) = 0$, we have $jt = 0$. Also, we must have

$$((0,1), f(0,1)+j)((1,0), f(1,0)+jt) = (0,0).$$

Thus, $jf(1,0) + tf(0,1) = 0$, and so $j(1 - f(0,1)) + t(1 - f(1,0)) = 0$. Hence, $j = -t$. Now, we have

$$\begin{aligned} 0 &= J(t - f(0,1)) \\ &= J(-j - 1 + f(1,0)) \\ &= J(-1 - (j - f(1,0))) = -J \end{aligned}$$

Therefore $J = 0$, and the result holds.

For the converse implication, assume that $J = 0$ and that $\mathcal{C}(A)$ is a complete graph. Then $A \bowtie^f J \cong A$, which implies that $\mathcal{C}(A \bowtie^f J) \cong \mathcal{C}(A)$. Therefore $\mathcal{C}(A \bowtie^f J)$ is also a complete graph. □

By [14, Theorem 4.5], $\mathcal{C}(A \bowtie^f J)$ is a complete bipartite graph if and only if $|\mathrm{Max}(A \bowtie J)| = 2$. In the following proposition, we determine the conditions under which $|\mathrm{Max}(A \bowtie J)| = 2$.

**Proposition 3.3.** *The graph $\mathcal{C}(A \bowtie J)$ is complete bipartite if and only if one of the following conditions holds.*

1. *$A$ is a local ring with $|\mathrm{Max}(B)| - |\mathrm{Max}(B/J)| = 1$.*
2. *$\mathcal{C}(A)$ is a complete bipartite graph and $J \subseteq \mathrm{Jac}(B)$.*

*Proof.* Suppose that $\mathcal{C}(A \bowtie J)$ is a complete bipartite graph. Then $|\mathrm{Max}(A \bowtie J)| = 2$. If $A$ is local, then we can obtain the maximal ideal $\bar{Q}^f$, where $Q \in \mathrm{Max}(B) \setminus V(J)$. So $|\mathrm{Max}(B)| - |\mathrm{Max}(B/J)| = 1$. Otherwise, if the statement (1) is not hold, then we have $|\mathrm{Max}(A)| = 2$. Thus $|\mathrm{Max}(B) \setminus V(J)| = 0$, and so $J \subseteq \mathrm{Jac}(B)$. Conversely, if one of the conditions (1) or (2) holds, then $|\mathrm{Max}(A \bowtie J)| = 2$. Therefore, $\mathcal{C}(A \bowtie J)$ is a complete bipartite graph. □

Assume that $A \bowtie^f J$ is a ring with $|\mathrm{Max}(A \bowtie^f J)| = n$, where $2 \le n < \infty$, and $f$ is an isomorphism. If $|\mathrm{Max}(A \bowtie^f J)| = 2$, then $\mathcal{C}(A \bowtie^f J)$ is a complete bipartite graph. Now, if $A \cong F_1 \times F_2$ and $J = 0$, where $F_1$ and $F_2$ are fields, then, by Theorem 3.3, $\mathrm{diam}(\mathcal{C}(A \bowtie^f J)) = 1$. Also if $A$ is a field and $J \neq 0$, then by Theorem 3.2, we have $\mathrm{diam}(\mathcal{C}(A \bowtie^f J)) = 1$. Otherwise, $\mathrm{diam}(\mathcal{C}(A \bowtie^f J)) = 2$. Now, assume that $|\mathrm{Max}(A \bowtie^f J)| \ge 3$. By [15, Proposition 2.6] we have $\mathrm{diam}(\mathcal{C}(A \bowtie^f J)) = 3$.

Now, assume that the ring $A \bowtie^f J$ has infinity many maximal ideals. It is easy to see that in this situation, $\mathrm{diam}(\mathcal{C}(A \bowtie^f J)) \ge 1$. In the following theorem, we characterize the situations under which the diameter of $\mathcal{C}(A \bowtie^f J)$ is two or three.

**Theorem 3.4.** *Let $A \bowtie^f J$ be a ring with infinity many maximal ideals and $f$ be an isomorphism. Then $\mathrm{diam}(\mathcal{C}(A \bowtie^f J)) = 2$ if and only if one of the following conditions holds.*

1. *$\mathrm{Jac}(A)$ is a prime ideal and $J \subseteq \mathrm{Jac}(A)$.*
2. *$\mathrm{Jac}(B)$ is a prime ideal and $J \cap (\cap Q) \in \{\{0\}, J\}$, where $Q \in \mathrm{Max}(B) \setminus V(J)$.*

*Also, $\mathrm{diam}(\mathcal{C}(A \bowtie^f J)) = 3$ if and only if none of the conditions (1) and (2) hold.*

*Proof.* By [15, Corollary 3.6], we have $\mathrm{diam}(\mathcal{C}(A \bowtie^f J)) = 2$ if and only if $\mathrm{Jac}(A \bowtie^f J)$ is a prime ideal and, by [15, Corollary 3.7], we have $\mathrm{diam}(\mathcal{C}(A \bowtie^f J)) = 3$ if and only if $\mathrm{Jac}(A \bowtie^f J)$ is not a prime ideal. Now, by Theorems 2.2 and 2.3, we have $\mathrm{Jac}(A \bowtie^f J)$ is a prime ideal if and only if one of the conditions (1) or (2) holds, and so we are done. □

In the next theorem, we determine the chromatic and clique numbers of $\mathcal{C}(A \bowtie^f J)$, denoted by the symbols $\chi$ and $\omega$.

**Theorem 3.5.** *In the comaximal ideal graph of $A \bowtie^f J$, the following statements hold.*

1. $\chi(\mathcal{C}(A \bowtie^f J)) = \chi(\mathcal{C}(A)) + \chi(\mathcal{C}(B)) - \chi(\mathcal{C}(B/J))$.
2. $\omega(\mathcal{C}(A \bowtie^f J)) = \omega(\mathcal{C}(A)) + \omega(\mathcal{C}(B)) - \omega(\mathcal{C}(B/J))$.

*Proof.* By [15, Proposition 2.6], we have $\chi(\mathcal{C}(A \bowtie^f J)) = \omega(\mathcal{C}(A \bowtie^f J)) = |\mathrm{Max}(A \bowtie^f J)|$. So it is enough to compute the number of maximal ideals of $A \bowtie^f J$. We know that

$$\mathrm{Max}(A \bowtie^f J) = \{P \bowtie^f J \mid P \in \mathrm{Max}(A)\} \ \cup \ \{\bar{Q}^f \mid Q \in \mathrm{Max}(B) \setminus V(J)\},$$

where $\bar{Q}^f = \{(a, f(a) + j) \mid a \in A,\ j \in J,\ f(a) + j \in Q\}$. Since $|\{P \bowtie^f J \mid P \in \mathrm{Max}(A)\}| = |\mathrm{Max}(A)|$ and

$$|\{\bar{Q}^f \mid Q \in \mathrm{Max}(B) \setminus V(J)\}| = |\mathrm{Max}(B) \setminus V(J)|,$$

we have $|\ \mathrm{Max}(A \bowtie^f J)| = |\mathrm{Max}(A)| + |\mathrm{Max}(B) \setminus V(J)|$. Now, consider an arbitrary maximal ideal $M \in \mathrm{Max}(B)$. Since $J \trianglelefteq B$, we have either $J \subseteq M$ or $J \nsubseteq M$. If $J \subseteq M$, then $M/J$ is a maximal ideal of $B/J$. Hence

$$|\mathrm{Max}(B) \setminus V(J)| = |\mathrm{Max}(B)| - |\mathrm{Max}(B/J)|.$$

Thus, $|\mathrm{Max}(A \bowtie^f J)| = |\mathrm{Max}(A)| + |\mathrm{Max}(B)| - |\mathrm{Max}(B/J)|$. We claim that do not exist maximal ideal $P \in \mathrm{Max}(A)$ and $Q \in \mathrm{Max}(B) \setminus V(J)$ such that $\bar{Q}^f = P \bowtie^f J$. By contrary, assume that there exist maximal ideals $P \in \mathrm{Max}(A)$ and $Q \in \mathrm{Max}(B) \setminus V(J)$ such that $\bar{Q}^f = P \bowtie^f J$. Since $(0, j) = (0, f(0) + j) \in P \bowtie^f J$, for every $j \in J$, we have $(0, j) \in \bar{Q}^f$, for every $j \in J$. So $f(0) + j \in Q$, for every $j \in J$. Hence $J \subseteq Q$, which is a contradiction. Therefore

$$\chi(\mathcal{C}(A \bowtie^f J)) = |\mathrm{Max}(A \bowtie^f J)| = \chi(\mathcal{C}(A)) + \chi(\mathcal{C}(B)) - \chi(\mathcal{C}(B/J))$$

and

$$\omega(\mathcal{C}(A \bowtie^f J)) = |\mathrm{Max}(A \bowtie^f J)| = \omega(\mathcal{C}(A)) + \omega(\mathcal{C}(B)) - \omega(\mathcal{C}(B/J)).$$ □

## 4. Some properties of $\Gamma_2'(A \bowtie J)$

In this section, we study some properties of the comaximal graph of amalgamated duplication $\Gamma_2'(A \bowtie J)$. In the next two propositions, we provide the condition under which $\Gamma_2'(A \bowtie J)$ is isomorphic to $K_{1,n}$ or $K_{2,n}$.

**Proposition 4.1.** *If $J = 0$, then $\Gamma_2'(A \bowtie J)$ is a star graph if and only if $\Gamma_2'(A)$ is a star graph. Otherwise, $\Gamma_2'(A \bowtie J)$ is a star graph if and only if $A \cong \mathbb{Z}_2$.*

*Proof.* If $J = 0$, then $A \bowtie J \cong A$. Hence $\Gamma_2'(A \bowtie J) \cong \Gamma_2'(A)$. Therefore $\Gamma_2'(A \bowtie J)$ is a star graph if and only if $\Gamma_2'(A)$ is a star graph. Next, assume that $J \neq 0$ and that $A \cong \mathbb{Z}_2$. Hence $J \cong \mathbb{Z}_2$. Since $A \bowtie A \cong A \times A$, we have $A \bowtie J \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus $\Gamma_2'(A \bowtie J) \cong \Gamma_2'(\mathbb{Z}_2 \times \mathbb{Z}_2)$, which is isomorphic to the star graph $K_2$. Conversely, suppose that $\Gamma_2'(A \bowtie J)$ is a star graph. So $\Gamma_2'(A \bowtie J)$ has a center vertex, say $(a, a + j)$, for some $a \in A$ and $j \in J$. Since

$$J \nsubseteq \bigcap_{Q \in \mathrm{Max}(A) \setminus V(J)} Q,$$

there is $(0, j_1) \notin \mathrm{Jac}(A \bowtie J)$. Since $(0, j_1)$ is not unit, $(0, j_1) \in V(\Gamma_2'(A \bowtie J))$. Thus $(a, a + j)$ is adjacent to $(0, j_1)$,which implies that $a \in U(A)$. Clearly $a + j$ is not unit, because $(a, a + j)$ is a vertex. If $a + j \neq 0$, then since $(a, 0) \in V(\Gamma_2'(A \bowtie J))$, we have $(a, a + j)$ is adjacent to $(a, 0)$. Hence $a + j$ is unit, which is impossible. Thus $a + j = 0$ and the center vertex of $\Gamma_2'(A \bowtie J)$ is of the form $(a, 0)$, where $a \in U(A)$. Now, since $\Gamma_2'(A \bowtie J)$ is a star graph, $|U(A)| = 1$. If $A \setminus U(A) \neq \{0\}$, then there exists a non-zero $b \in A \setminus U(A)$ such that $b \notin \mathrm{Jac}(A)$. So we find a vertex $(b, b)$ in $V(\Gamma_2'(A \bowtie J))$ which is not adjacent to $(a, 0)$, and this is impossible. Thus $A \setminus U(A) = \{0\}$. Therefore $A \cong \mathbb{Z}_2$ and the result holds. □

**Proposition 4.2.** *Assume that $n$ is a positive integer with $n > 1$. If $J = 0$, then $\Gamma_2'(A \bowtie J)$ is $K_{2,n}$ if and only if $\Gamma_2'(A)$ is $K_{2,n}$. Otherwise, $\Gamma_2'(A \bowtie J)$ is $K_{2,n}$ if and only if $A \cong \mathbb{Z}_3$.*

*Proof.* If $J = 0$, then $A \bowtie J \cong A$. Hence $\Gamma'_2(A \bowtie J) \cong \Gamma'_2(A)$. Therefore $\Gamma'_2(A \bowtie J)$ is $K_{2,n}$ if and only if $\Gamma'_2(A)$ is $K_{2,n}$. Now, assume that $J \neq 0$ and that $\Gamma'_2(A \bowtie J)$ is a complete bipartite graph $K_{2,n}$. Suppose that $X$ is the part of size two in $K_{2,n}$. Let $(a, a+j_1), (a, a+j_2) \in X$, for some $a \in A$ and $j_1, j_2 \in J$. Then there exists $j \in J$ such that $(0, j) \notin \mathrm{Jac}(A \bowtie J)$. Since $(0, j)$ is not unit, $(0, j) \in V(\Gamma'_2(A \bowtie J))$. Thus $(a, a+j_1)$ is adjacent to $(0, j)$. Hence $a \in U(A)$ and $a + j_1$ is not unit, because $(a, a+j_1)$ is a vertex. If $a + j \neq 0$, then since $(a, 0) \in V(\Gamma'_2(A \bowtie J))$, we have $(a, a+j_1)$ is adjacent to $(a, 0)$. This implies that $a + j_1$ is unit and it is impossible. Thus $a + j_1 = 0$. Similarly $a + j_2 = 0$. Since $\mid X \mid = 2$, one can see that $\mid U(A) \mid = 2$. Let $X = \{(a, 0), (b, 0)\}$, where $a, b \in U(A)$. If there exists a non-zero element $x$ in $A \setminus U(A)$, then $(a, x)$ is a vertex in $\Gamma'_2(A \bowtie J)$ which is adjacent to neither $(a, 0)$ nor $(b, 0)$, which is impossible. So $|A \setminus U(A)| = 1$, which implies that $A \cong \mathbb{Z}_3$. Clearly, if $A \cong \mathbb{Z}_3$, then $\Gamma'_2(A \bowtie J)$ is isomorphic to $K_{2,2}$. □

In the following, we study the girth of $\Gamma'_2(A \bowtie J)$. By [15, Proposition 2.6], we have $\mathrm{gr}(\Gamma'_2(A \bowtie J)) = 3$ if and only if $\mid \mathrm{Max}(A \bowtie J) \mid \geq 3$. Also, if $|\mathrm{Max}(A \bowtie J)| = 1$, then the vertex set $\Gamma'_2(A \bowtie J)$ is empty.

**Proposition 4.3.** *Suppose that* $|\mathrm{Max}(A \bowtie J)| = 2$. *Then* $\mathrm{gr}(\Gamma'_2(A \bowtie J)) = \infty$ *if and only if one of the following conditions holds.*

1. $J = 0$ *and* $A \cong \mathbb{Z}_2 \times F$, *where* $F$ *is a field.*
2. $A \cong \mathbb{Z}_2$ *and* $J \cong \mathbb{Z}_2$.

*Also,* $\mathrm{gr}(\Gamma'_2(A \bowtie J)) = 4$ *if and only if none of the above conditions hold.*

*Proof.* Assume that $|\mathrm{Max}(A \bowtie J)| = 2$. Then $\Gamma'_2(A \bowtie J)$ is a complete bipartite graph. Hence, by Proposition 4.1 and [9, Proposition 2.4], $\Gamma'_2(A \bowtie J)$ is isomorphic to $K_{1,n}$, where $n \geq 1$ if and only if one of the conditions (1) or (2) holds. So $\mathrm{gr}(\Gamma'_2(A \bowtie J)) = \infty$ if and only if one of the conditions (1) or (2) holds. Otherwise $\mathrm{gr}(\Gamma'_2(A \bowtie J)) = 4$. □

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