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Research Article

On disjoint cross intersecting families of permutations

Nuttanon Songsuwan, Supida Sengsamak, Nutchapol Jeerawattana, Thiradet Jiarasuksakun, Pawaton Kaemawichanurat*

Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi, Bangkok, Thailand

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Abstract

For the positive integers r and n satisfying $r \leq n$, let $\mathcal{P}_{r,n}$ be the family of partial permutations $\{\{(1,x_1),(2,x_2),\ldots,(r,x_r)\}$: $x_1,x_2,\ldots,x_r$ are different elements of $\{1,2,\ldots,n\}\}$. The subfamilies $\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_k$ of $\mathcal{P}_{r,n}$ are called *cross intersecting* if $A \cap B \neq \emptyset$ for all $A \in \mathcal{A}_i$ and $B \in \mathcal{A}_j$, where $1 \leq i \neq j \leq k$. Also, if $\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_k$ are mutually disjoint, then they are called *disjoint cross intersecting subfamilies* of $\mathcal{P}_{r,n}$. For the disjoint cross intersecting subfamilies $\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_k$ of $\mathcal{P}_{n,n}$, it follows from the AM-GM inequality that $\prod_{i=1}^{k}|\mathcal{A}_i| \leq (n!/k)^k$. In this paper, we present two proofs of the following statement: $\prod_{i=1}^{k}|\mathcal{A}_i| = (n!/k)^k$ if and only if $n = 3$ and $k = 2$.

Keywords: permutations; intersecting families; Erdős-Ko-Rado Theorem.

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1. Introduction and motivation

For the natural numbers n and r such that $r \leq n$, take $[n] = \{1,2,\ldots,n\}$ and let $\mathcal{P}_{r,n}$ be the family of partial permutations $\{\{(1,x_1),(2,x_2),\ldots,(r,x_r)\} : x_1,x_2,\ldots,x_r$ are different elements of $[n]\}$. For convenience, we write $(x_1x_2\ldots x_r)$ to denote $\{(1,x_1),(2,x_2),\ldots,(r,x_r)\}$. Also, we say that x_i is the i^{th} *digit* of $(x_1x_2\ldots x_r)$. For a permutation $\sigma \in \mathcal{P}_{r,n}$, we set $\sigma(i) = x_i$ if $(i,x_i) \in \sigma$. A subfamily $\mathcal{A}$ of $\mathcal{P}_{r,n}$ is *intersecting* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{A}$. For $k \geq 2$, subfamilies $\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_k$ of $\mathcal{P}_{r,n}$ are called *cross intersecting* if $A \cap B \neq \emptyset$ for all $A \in \mathcal{A}_i$ and $B \in \mathcal{A}_j$, where $1 \leq i \neq j \leq k$. Moreover, if $\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_k$ are mutually disjoint, then $\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_k$ are called *disjoint cross intersecting subfamilies* of $\mathcal{P}_{r,n}$. For each $i \in [r]$ and $x_i \in [n]$, the family of all permutations of $\mathcal{P}_{r,n}$ that contains (i,x_i) is called the *star* centered at (i,x_i).

The Erdős-Ko-Rado Theorem [7] is a well-known result in extremal set theory. There have been many generalizations such as cross intersecting families (see [2–4, 10, 11]) and also intersecting families of permutations. Frankl and Deza [8] proved that $|\mathcal{A}| \leq (n-1)!$ for intersecting families $\mathcal{A}$ of $\mathcal{P}_{n,n}$. The extremal case is characterized in [5, 9].

Theorem 1.1 (see [5, 8, 9]). *If $\mathcal{A} \in \mathcal{P}_{n,n}$ is intersecting, then $|\mathcal{A}| \leq (n-1)!$ where the equality holds if and only if $\mathcal{A}$ is the set $\{\sigma \in \mathcal{P}_{n,n} : \sigma(i) = j\}$ for given $i, j \in [n]$.*

In cross intersecting subfamilies $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\mathcal{P}_{n,n}$, when $n = 2$, we see that the maximum of $|\mathcal{A}_1||\mathcal{A}_2|$ is 1 which is when $\mathcal{A}_1 = \mathcal{A}_2 = (12)$ or $\mathcal{A}_1 = \mathcal{A}_2 = (21)$. When $n = 3$, the maximum of $|\mathcal{A}_1||\mathcal{A}_2|$ is 9 which is when $\mathcal{A}_1$ and $\mathcal{A}_2$ are

$$\mathcal{T}_1 = \{(123),(231),(312)\} \quad \text{and} \quad \mathcal{T}_2 = \{(132),(213),(321)\}.$$

When, $n \geq 4$, Leader conjectured in the British Combinatorial Conference 2005 that $|\mathcal{A}_1||\mathcal{A}_2| \leq ((n-1)!)^2$ with equality if and only if $\mathcal{A}_1 = \mathcal{A}_2 = \{\sigma \in \mathcal{P}_{n,n} : \sigma(i) = j\}$ for fixed $i, j \in [n]$. This conjecture was proved by Ellis et. al. [6]. In fact, Ellis et. al. [6] further proved the general result when $|A \cap B| \geq t$ for all $A \in \mathcal{A}_1$ and $B \in \mathcal{A}_2$ when $t \geq 1$. In 2010, Borg [1] established a result concerning the characterization of cross intersecting subfamilies $\mathcal{A}_1,\ldots,\mathcal{A}_k$ of $\mathcal{P}_{r,n}$ when $\sum_{i=1}^{k}|\mathcal{A}_i|$ is maximum as detailed in Theorem 1.2. The following definition was introduced by Borg in his paper [1] too.

Definition 1.1 (see [1]). *For any integer q, $\theta_q : \mathcal{P}_{r,n} \to \mathcal{P}_{r,n}$ is called the translation operator if*

$$\theta_q(A) := \{(a, b + q \bmod n) : (a,b) \in A\},$$

and $\Theta : \mathcal{P}_{r,n} \to \mathcal{P}_{r,n}$ is called the orbit of A if

$$\Theta(A) := \{\theta_q(A) : q \in \mathbb{N}\}.$$

*Corresponding author (pawaton.kae@kmutt.ac.th).

By the property of modulo n, it holds that $|\Theta(A)| = n$.

Lemma 1.1 (see [1]). *If $A \in \mathcal{P}_{r.n}$ then for distinct $x, y \in \Theta(A)$, it holds that $x \cap y = \emptyset$.*

Borg [1] also the next theorem.

Theorem 1.2 (see [1]). *Let $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k$ be cross intersecting subfamilies of $\mathcal{P}_{r,n}$. Then, $\sum_{i=1}^{k} |\mathcal{A}_i|$ is maximum if and only if either of the following five conditions holds.*

(i). *$k < n$ and, for some $i \in [k]$, $\mathcal{A}_i = \mathcal{P}_{r,n}$ and $\mathcal{A}_j = \emptyset$ for all $j \in [k] \setminus \{i\}$.*

(ii). *$k > n$ and $\mathcal{A}_1 = \mathcal{A}_2 = \cdots = \mathcal{A}_k = \{A \in \mathcal{P}_{r,n} : (x, y) \in A\}$ for some $(x, y) \in [r] \times [n]$.*

(iii). *$k = n$ and $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k$ are as in (i) or (ii).*

(iv). *$2 \le k \le 3 = r = n$, $\mathcal{A}_j = \mathcal{T}_1$ and $\mathcal{A}_l = \mathcal{T}_2$ for some $j \in [k]$ and $l \in [k] \setminus \{j\}$, and if $k = 3$, then $\mathcal{A}_p = \emptyset$ for $p \in [k] \setminus \{j, l\}$.*

(v). *$k = r = n = 3$, there exist $j \in [3]$, $i \in [2]$, and $T \in \mathcal{T}_{3-i}$ such that $\mathcal{A}_j = \mathcal{T}_i \cup \{T\}$. Further, $\mathcal{A}_l = \{T\}$ for each $l \in [3] \setminus \{j\}$.*

We remark here that there is no study on disjoint cross intersecting subfamilies $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k$ of $\mathcal{P}_{r,n}$. It is easy to see that $\sum_{i=1}^{k} |\mathcal{A}_i| \le n!$. By the AM-GM inequality, we have that

$$\prod_{i=1}^{k} |\mathcal{A}_i| \le \left(\frac{n!}{k}\right)^k. \tag{1}$$

In this note, we present two proofs of the following theorem

Theorem 1.3. *The equality in (1) holds if and only if $n = 3, k = 2$, for which $\mathcal{A}_1$ and $\mathcal{A}_2$ are $\mathcal{T}_1$ and $\mathcal{T}_2$.*

2. Proof of Theorem 1.3

We observe that when $n = 3$ and $k = 2$, the families $\mathcal{T}_1, \mathcal{T}_2$ give the maximum value of the size product, which is $(\frac{3!}{2})^2$. Thus, in order to prove Theorem 1.3, it suffices to show that if the equality in (1) holds, then $n = 3, k = 2$, for which $\mathcal{A}_1$ and $\mathcal{A}_2$ are $\mathcal{T}_1$ and $\mathcal{T}_2$.

Proof. Assume that the equality in (1) holds. Hence, $\sum_{i=1}^{k} |\mathcal{A}_i| = n!$ which is maximum. Since the subfamilies $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k$ are mutually disjoint, they do not satisfy the conditions (ii) and (v) of Theorem 1.2. Also, if the subfamilies satisfy the condition (i), then $\prod_{i=1}^{k} |\mathcal{A}_i| = 0$ contradicting

$$\prod_{i=1}^{k} |\mathcal{A}_i| = \left(\frac{n!}{k}\right)^k.$$

Hence, the subfamilies do not satisfy the condition (i). This implies that they do not satisfy (iii). Therefore, $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k$ satisfy (iv). Consequently, $n = 3$. Suppose to the contrary that $k \ge 3$. By the condition (iv), we have that $k = 3$ and $\mathcal{A}_3 = \emptyset$. This implies that $\prod_{i=1}^{k} |\mathcal{A}_i| = 0$, which is a contradiction. Hence, $k = 2$ and $\mathcal{A}_1, \mathcal{A}_2$ are $\mathcal{T}_1$ and $\mathcal{T}_2$. □

As we have seen that Theorem 1.3 follows from Theorem 1.2 directly. However, to the best of our knowledge, Theorem 1.3 has not been pointed out in any earlier study.

3. Another proof of Theorem 1.3

Although, the proof of Theorem 1.3 presented in this section is longer than the one given in the previous section, but it does not require the use of Theorem 1.2. All the methods in this proof are just basic tools in Combinatorics. The following proposition is proved simply by induction and hence its proof is omitted.

Proposition 3.1. *For $n \ge 4$, then*

$$\frac{1}{1!} - \frac{1}{2!} + \cdots + \frac{(-1)^n}{(n-1)!} > \frac{1}{2}.$$

Now, we are ready to give another proof of Theorem 1.3.

Proof. Suppose that the equality in (1) holds. Then, $|\mathcal{A}_1| = \cdots = |\mathcal{A}_k| = \frac{n!}{k}$ and $\bigcup_{i=1}^{k} \mathcal{A}_i = \mathcal{P}_{n,n}$. Let $\mathcal{S} = \{S_1, S_2, \ldots, S_{(n-1)!}\}$ be the star centered at $(1,1)$. Note that $\mathcal{P}_{n,n}$ can be partition to $\Theta(S_1), \Theta(S_2), \ldots, \Theta(S_{(n-1)!})$.

Claim 1. If $\mathcal{A}_i \cap \Theta(S_j) \neq \emptyset$, then $\Theta(S_j) \subseteq \mathcal{A}_i$.

Proof of Claim 1. Suppose to the contrary that $\mathcal{A}_i \cap \Theta(S_j) \neq \emptyset$ and there exists $\sigma \in \Theta(S_j)$ but $\sigma \notin \mathcal{A}_i$. Since $\bigcup_{i=1}^{k} \mathcal{A}_i = \mathcal{P}_{n,n}$, $\sigma \in \mathcal{A}_l$ for some l. Thus, $\sigma \cap \gamma = \emptyset$ for all $\gamma \in \mathcal{A}_i \cap \Theta(S_j)$ contradicting $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are cross intersecting families. Hence, $\Theta(S_j) \subseteq \mathcal{A}_i$. This proves Claim 1.

Next, we take $\sigma_i \in \Theta(S_i)$ and $\sigma_j \in \Theta(S_j)$.

Claim 2 If $|\sigma_i \cap \sigma_j| \geq 2$, then there exists $\gamma \in \Theta(S_j)$ such that $\sigma_i \cap \gamma = \emptyset$.

Proof of Claim 2. Suppose that σ_i and σ_j intersects at the i_1^{th} and i_2^{th} digits. Suppose to the contrary that $\gamma \cap \sigma_i \neq \emptyset$ for all $\gamma \in \Theta(S_j) \setminus \{\sigma_j\}$. Since all digits of each permutation in $\Theta(S_j)$ are different, it follows that there are at most $n-2$ remaining digits of σ_i that can intersect these $n-1$ permutations in $\Theta(S_j) \setminus \{\sigma_j\}$. By Pigeonhole Principle, there exists an i_3^{th} digit of σ_i and $\gamma', \gamma'' \in \Theta(S_j) \setminus \{\sigma_j\}$ such that $(i_3, x_{i_3}) \in \sigma_i \cap \gamma' \cap \gamma''$. In particular, $\gamma' \cap \gamma'' \neq \emptyset$ contradicting Lemma 1.1. Therefore, there exists $\gamma \in \Theta(S_j)$ such that $\sigma_i \cap \gamma = \emptyset$. This proves Claim 2.

Now, we suppose to the contrary that $n \geq 4$. Without loss of generality, we assume that $(12\ldots n) \in \mathcal{A}_1$. Let $\mathcal{W}$ be the subfamily of $\mathcal{S}$ such that $S \cap (12\ldots n) = \{(1,1)\}$ for all $S \in \mathcal{W}$. Hence, $(2,2), \ldots, (n,n) \notin S$ for all $S \in \mathcal{W}$. By the formula for the number of derangements, we have

$$|\mathcal{W}| = (n-1)! \sum_{p=2}^{n-1} \frac{(-1)^p}{p!}.$$

Observe that $|S \cap (12\ldots n)| \geq 2$ for each $S \in \mathcal{S} \backslash \mathcal{W}$. By Claim 2, there exists $\gamma \in \Theta(S)$ such that $\gamma \cap (12\ldots n) = \emptyset$. Since $\bigcup_{i=1}^{k} \mathcal{A}_i = \mathcal{P}_{n,n}$, $\gamma \in \mathcal{A}_1$. By Claim 1, $\Theta(S) \subseteq \mathcal{A}_1$. Hence

$$\bigcup_{S \in \mathcal{S} \backslash \mathcal{W}} \Theta(S) \subseteq \mathcal{A}_1$$

and Therefore,

$$\frac{n!}{k} = |\mathcal{A}_1| \geq n\left(|\mathcal{S}| - |\mathcal{W}|\right) = n\left((n-1)! - (n-1)! \sum_{p=2}^{n-1} \frac{(-1)^p}{p!}\right)$$
$$\geq n!\left(1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + \frac{(-1)^n}{(n-1)!}\right).$$

By Proposition 3.1, we have

$$\frac{1}{k} \geq 1 - \frac{1}{2} + \frac{1}{3!} - \cdots + \frac{(-1)^n}{(n-1)!} > \frac{1}{2}$$

which contradicts $k \geq 2$. Thus, $n = 3$. Clearly, $\mathcal{P}_{3,3} = \{(123), (231), (312), (132), (213), (321)\}$. Let

$$\Theta(S_1) = \{(123), (231), (312)\} \quad \text{and} \quad \Theta(S_2) = \{(132), (213), (321)\}.$$

By Claim 1, $\Theta(S_1) \subseteq \mathcal{A}_i$ and $\Theta(S_2) \subseteq \mathcal{A}_j$ where $\{i, j\} = \{1, 2\}$. This implies that $k = 2$ and $\mathcal{A}_1, \mathcal{A}_2$ are $\Theta(S_1)$ and $\Theta(S_2)$. □

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