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Research Article

Some degree-based topological indices and (normalized Laplacian) energy of graphs

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Abstract

In this paper, by utilizing the concept of the energy of a vertex, connections between some vertex-degree-based topological indices (including the general Randić index, the first Zagreb index, and the forgotten index) and the energy of graphs are established. Several bounds on the energy of the graphs containing no isolated vertices are also given in terms of the first Zagreb index and the forgotten index. Moreover, bounds on the normalized Laplacian energy in terms of two particular cases of the general Randić index are obtained.

Keywords: topological indices; graph energy; normalized Laplacian energy; energy of a vertex.

2020 Mathematics Subject Classification: 05C05, 05C35, 05C69.

1. Introduction

Topological indices are numerical values associated with molecular hydrogen-depleted graphs of chemical compounds. They provide a useful tool for giving a compact and effective description of structural formulas of chemical compounds. Topological indices are utilized for correlating the chemical structures with their physical properties, chemical reactivity, or biological activity [7, 17, 18]. Among such indices, vertex-degree-based topological indices and the graph energy have attracted a lot of attention from researchers.

Let $G = (V(G), E(G))$ be a simple and undirected graph with vertex set the $V(G) = \{v_1, v_2, \cdots, v_n\}$ and the edge set $E(G)$. The degree of a vertex v, denote by $d(v)$, is the number of its neighbors. The maximum and minimum degrees of a graph are denoted by Δ and δ, respectively.

A vertex-degree-based (VDB, for short) topological index is defined for a graph G as

$$TI = TI(G) = \sum_{v_iv_j \in E(G)} F(d(v_i), d(v_j))$$

where $F(x, y)$ is some function with property $F(x, y) = F(y, x)$. Some examples of the function F are listed below:

VDB topological indices	Notation	$F(x, y)$
General Randić index [4]	R_α	$(xy)^\alpha$
First Zagreb index [13]	M_1	$x + y$
Forgotten index [8]	F	$x^2 + y^2$
Sombor index [12]	SO	$\sqrt{x^2 + y^2}$

The ordinary Randić index is equal to $R_{-1/2}$, a famous topological index, which is correlated with a variety of physico-chemical properties of chemical compounds. The adjacency matrix $A(G)$ of the graph G is a matrix whose (i, j)-entry is equal to 1 if $v_iv_j \in E(G)$ and 0 otherwise. For a graph G, the normalized Laplacian matrix $\mathcal{L}(G)$ is a matrix whose (i, j)-entry is

$$\mathcal{L}(G)_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -\frac{1}{\sqrt{d(v_i)d(v_j)}} & \text{if } i \neq j \text{ and } v_iv_j \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

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Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of $A(G)$ and let $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_n = 0$ be the eigenvalues of $\mathcal{L}(G)$. The concept of the (ordinary) energy of a graph G was proposed by Gutman [10] in 1978, which is defined as

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

Cavers [5] considered the energy of a simple and undirected graph with respect to its normalized Laplacian eigenvalues. The normalized Laplacian energy ($\mathcal{L}$-energy) of a graph G is defined as

$$\mathcal{E}_{\mathcal{L}}(G) = \sum_{i=1}^{n} |\lambda_i(\mathcal{L}) - 1| = \sum_{i=1}^{n} |\lambda_i(I - \mathcal{L})|.$$

Finding connections between the graph energy and VDB indices is an interesting issue. For a graph G, Cavers [5] determined some bounds for the $\mathcal{L}$-energy $\mathcal{E}_{\mathcal{L}}(G)$ based on the topological index $R_{-1}(G)$. In the references [6, 11, 14–16], the authors gave relations between VDB topological indices and their corresponding energies; Das et al. [6] refered to these energies as the "degree-based graph energies". In 2018, Arizmendi et al. [2] introduced the concept of the energy of a vertex. By utilizing this new idea, Arizmendi and Arizmendi [1] found a way to relate Randić index to the ordinary energy of a graph G; they showed that $\mathcal{E}(G) \geq 2R_{-1/2}(G)$. Soon after, Yan et al. [21] gave an upper bound on $\mathcal{E}(G)$ in terms of $R_{-1/2}(G)$ and $\Delta(G)$; namely $\mathcal{E}(G) \leq 2\sqrt{\Delta(G)}R_{-1/2}(G)$. Recently, Ülker et al. [19, 20] determined some connections between the topological indices $\mathcal{E}(G)$ and $SO(G)$.

In 2022, Guerrero [9] proposed the idea of the Laplacian energy of a vertex and the normalized Laplacian energy of a vertex. We adopt a new approach (similar to the one used in [1]) to relate $\mathcal{E}_{\mathcal{L}}(G)$ with $R_{-1}(G)$ by utilizing the concept of the normalized Laplacian energy of a vertex.

Motivated by the papers [1, 2, 9, 19–21], in the present study, we establish connections between some vertex-degree-based topological indices (including the general Randić index, the first Zagreb index, and the forgotten index) and the energy of graphs. The structure of this paper is listed below. In Section 2, we give several bounds on the energy of graphs containing no isolated vertices in terms of the first Zagreb and forgotten indices. In Section 3, we relate the normalized Laplacian energy with the topological indices indices $R_{-1/2}$ and R_{-1}.

2. Energy, first Zagreb index, and forgotten index of graphs

In this section, we give some bounds on the energy of graphs in terms of the first Zagreb index (M_1) and the forgotten index (F), by using the concept of the energy of a vertex.

For a matrix M, we denote its trace by $Tr(M)$ and its absolute value $(MM^*)^{\frac{1}{2}}$ by $|M|$. The energy of a graph G can be written as

$$\mathcal{E}(G) = Tr(|A(G)|) = \sum_{i=1}^{n} |A(G)|_{ii}.$$

Definition 2.1 (see [3]). *The energy of the vertex v_i of a graph G is given by*

$$\mathcal{E}(v_i) = |A(G)|_{ii}, \text{ for } i = 1, \cdots, n,$$

where $|A| = (AA^*)^{\frac{1}{2}}$ and A is the adjacency matrix of G.

For an edge $e = (v_i, v_j)$ define $\mathcal{E}(e) = \mathcal{E}(v_i)/d(v_i) + \mathcal{E}(v_j)/d(v_j)$. One can see

$$\begin{aligned}
\sum_{e \in E(G)} \mathcal{E}(e) &= \sum_{v_iv_j \in E(G)} \left(\frac{\mathcal{E}(v_i)}{d(v_i)} + \frac{\mathcal{E}(v_j)}{d(v_j)} \right) \\
&= \sum_{v_iv_j \in E(G)} \frac{\mathcal{E}(v_i)}{d(v_i)} + \sum_{v_iv_j \in E(G)} \frac{\mathcal{E}(v_j)}{d(v_j)} \\
&= \frac{1}{2} \sum_{v_i \in V(G)} \sum_{v_iv_j \in E(G)} \frac{\mathcal{E}(v_i)}{d(v_i)} + \frac{1}{2} \sum_{v_j \in V(G)} \sum_{v_iv_j \in E(G)} \frac{\mathcal{E}(v_j)}{d(v_j)} \\
&= \frac{1}{2} \sum_{v_i \in V(G)} \mathcal{E}(v_i) + \frac{1}{2} \sum_{v_j \in V(G)} \mathcal{E}(v_j) \\
&= \sum_{v \in V(G)} \mathcal{E}(v) = \mathcal{E}(G).
\end{aligned}$$

Before giving the main results, we illustrate some useful lemmas.

Lemma 2.1 (see [2])**.** *Let G be a graph with at least on edge. Then*

$$\mathcal{E}(v) \geq \sqrt{\frac{d(v)}{\Delta(G)}},\ \textit{for all}\ v \in V.$$

Equality holds if and only if $G \cong K_{d,\Delta}$.

Lemma 2.2 (see [2])**.** *For a graph G and a vertex $v \in G$,*

$$\mathcal{E}(v) \leq \sqrt{d(v)},$$

with equality if and only if the connected component containing v_i is isomorphic to S_n and v is its center.

We now prove the following theorems by utilizing Lemmas 2.1 and 2.2.

Theorem 2.1. *Let G be a graph with the energy $\mathcal{E}(G)$ and the first Zagreb index $M_1(G)$, then*

$$\mathcal{E}(G) \geq \sqrt{\frac{\delta(G)}{\Delta^5(G)}M_1(G)},$$

where the equality holds if and only if $G \cong tK_{\frac{n}{2t},\frac{n}{2t}};\ \frac{n}{2t}, t \in \mathbb{N}$. Also, it holds that

$$\mathcal{E}(G) \leq \frac{\sqrt{\Delta(G)}}{\delta^2(G)}M_1(G),$$

with equality if and only if $G \cong mP_2;\ m \in \mathbb{N}$.

Proof. We start from observing that

$$\begin{aligned}
\mathcal{E}(G) &= \sum_{e \in E(G)} \mathcal{E}(e) \\
&= \sum_{v_iv_j \in E(G)} \left(\frac{\mathcal{E}(v_i)}{d(v_i)} + \frac{\mathcal{E}(v_j)}{d(v_j)}\right) \\
&\geq \sum_{v_iv_j \in E(G)} \left(\frac{\sqrt{\frac{d(v_i)}{\Delta(G)}}}{d(v_i)} + \frac{\sqrt{\frac{d(v_j)}{\Delta(G)}}}{d(v_j)}\right) \\
&= \frac{1}{\sqrt{\Delta(G)}} \sum_{v_iv_j \in E(G)} \left(\frac{1}{\sqrt{d(v_i)}} + \frac{1}{\sqrt{d(v_j)}}\right) \\
&\geq \sqrt{\frac{\delta(G)}{\Delta(G)}} \sum_{v_iv_j \in E(G)} \left(\frac{1}{d(v_i)} + \frac{1}{d(v_j)}\right) \\
&\geq \sqrt{\frac{\delta(G)}{\Delta^5(G)}} \sum_{v_iv_j \in E(G)} (d(v_i) + d(v_j)) \\
&= \sqrt{\frac{\delta(G)}{\Delta^5(G)}} M_1(G).
\end{aligned}$$

By Lemma 2.1, the equality holds throughout in the above inequalities if and only if

$$\mathcal{E}(v_i) = \sqrt{\frac{d(v_i)}{\Delta(G)}} = \sqrt{\frac{\delta(G)}{\Delta(G)}}$$

for any vertex $v_i \in V(G)$. Thus, G is the union of complete bipartite graphs with degree $d(v_i) = \frac{n}{2t}$. Analogously, we obtain the upper bound.

Note that

$$\begin{aligned}\mathcal{E}(G) = \sum_{e\in E(G)} \mathcal{E}(e) &= \sum_{v_iv_j\in E(G)} \left(\frac{\mathcal{E}(v_i)}{d(v_i)} + \frac{\mathcal{E}(v_j)}{d(v_j)}\right)\\ &\le \sum_{v_iv_j\in E(G)} \left(\frac{\sqrt{d(v_i)}}{d(v_i)} + \frac{\sqrt{d(v_j)}}{d(v_j)}\right)\\ &\le \sqrt{\Delta(G)} \sum_{v_iv_j\in E(G)} \left(\frac{d(v_i)+d(v_j)}{d(v_i)d(v_j)}\right)\\ &\le \frac{\sqrt{\Delta(G)}}{\delta^2(G)} \sum_{v_iv_j\in E(G)} (d(v_i)+d(v_j))\\ &= \frac{\sqrt{\Delta(G)}}{\delta^2(G)} M_1(G).\end{aligned}$$

By Lemma 2.2, the equality holds throughout in the above inequalities if and only if for any vertex $v_i \in V(G)$,

$$\mathcal{E}(v_i) = \sqrt{d(v_i)} = \sqrt{\delta(G)} = \sqrt{\Delta(G)},$$

which implies that G is the union of the path P_2. □

Theorem 2.2. *Let G be a graph with the energy $\mathcal{E}(G)$ and the forgotten index $F(G)$, then*

$$\mathcal{E}(G) \ge \sqrt{\frac{\delta^3(G)}{\Delta^9(G)}} F(G),$$

with equality if and only if $G \cong tK_{\frac{n}{2t};\frac{n}{2t}},\ \frac{n}{2t}, t\in\mathbb{N}$. Also,

$$\mathcal{E}(G) \le \frac{\sqrt{\Delta^3(G)}}{\delta^4(G)} F(G).$$

with equality if and only if $G \cong mP_2;\ m\in\mathbb{N}$.

Proof. Similar to the proof of Theorem 2.1, we obtain that

$$\begin{aligned}\mathcal{E}(G) &= \sum_{e\in E(G)} \mathcal{E}(e)\\ &= \sum_{v_iv_j\in E(G)} \left(\frac{\mathcal{E}(v_i)}{d(v_i)} + \frac{\mathcal{E}(v_j)}{d(v_j)}\right)\\ &\ge \sum_{v_iv_j\in E(G)} \left(\frac{\sqrt{\frac{d(v_i)}{\Delta(G)}}}{d(v_i)} + \frac{\sqrt{\frac{d(v_j)}{\Delta(G)}}}{d(v_j)}\right)\\ &\ge \sqrt{\frac{\delta^3(G)}{\Delta(G)}} \sum_{v_iv_j\in E(G)} \left(\frac{1}{d^2(v_i)} + \frac{1}{d^2(v_j)}\right)\\ &\ge \sqrt{\frac{\delta^3(G)}{\Delta^9(G)}} \sum_{v_iv_j\in E(G)} (d^2(v_i)+d^2(v_j))\\ &= \sqrt{\frac{\delta^3(G)}{\Delta^9(G)}} F(G).\end{aligned}$$

By Lemma 2.1, the equality holds if and only if

$$\mathcal{E}(v_i) = \sqrt{\frac{d(v_i)}{\Delta(G)}} = \sqrt{\frac{\delta(G)}{\Delta(G)}}$$

for any vertex $v_i \in V(G)$, that is, G is the union of complete bipartite graphs with degree $d(v_i) = \frac{n}{2t}$.

Correspondingly, the upper bound is obtained as follows

$$\begin{aligned}\mathcal{E}(G) = \sum_{e\in E(G)} \mathcal{E}(e) &= \sum_{v_iv_j\in E(G)} \left(\frac{\mathcal{E}(v_i)}{d(v_i)} + \frac{\mathcal{E}(v_j)}{d(v_j)}\right)\\ &\leq \sum_{v_iv_j\in E(G)} \left(\frac{\sqrt{d(v_i)}}{d(v_i)} + \frac{\sqrt{d(v_j)}}{d(v_j)}\right)\\ &\leq \sqrt{\Delta^3(G)} \sum_{v_iv_j\in E(G)} \left(\frac{d^2(v_i)+d^2(v_j)}{d^2(v_i)d^2(v_j)}\right)\\ &\leq \frac{\sqrt{\Delta^3(G)}}{\delta^4(G)} \sum_{v_iv_j\in E(G)} (d^2(v_i)+d^2(v_j))\\ &= \frac{\sqrt{\Delta^3(G)}}{\delta^4(G)} F(G),\end{aligned}$$

where all equalities hold if and only if for any vertex $v_i \in V(G)$,

$$\mathcal{E}(v_i) = \sqrt{d(v_i)} = \sqrt{\delta(G)} = \sqrt{\Delta(G)}.$$

Therefore, G is the union of the path P_2. □

3. Normalized Laplacian energy and general Randić index

In this section, inspired by the papers [1, 21], we relate the Randić index (a classical topological index) with the energy of the normalized Laplacian matrix. We first study the energy of vertices obtained by using the normalized Laplacian matrix $\mathcal{L}$ as defined by Guerrero [9].

Definition 3.1 (see [9]). *For a connected simple graph $G = (V, E)$, the normalized Laplacian energy of a vertex $v_i \in V$ is defined as*

$$\mathcal{E}_{\mathcal{L}}(v_i) := \phi_i(|\mathcal{L} - I|),$$

where the positive linear functional $\phi_i : M_n(\mathbb{R}) \to \mathbb{R}$ is defined by $\phi_i(M_n) = M_{ii}$.

The normalized Laplacian energy can be written as the sum of individual energy of the vertices of G:

$$\mathcal{E}_{\mathcal{L}}(G) = \mathcal{E}_{\mathcal{L}}(v_1) + \mathcal{E}_{\mathcal{L}}(v_2) + \cdots + \mathcal{E}_{\mathcal{L}}(v_n).$$

Along the same line as we obtained bounds on the graph energy in the previous section, we now derive bounds for the normalized Laplacian energy in terms of the topological indices $R_{-1}(G)$ and $R_{-1/2}(G)$.

Lemma 3.1 (see [9]). *If G is a connected graph, then*

$$\mathcal{E}_{\mathcal{L}}(v_i) \geq \frac{1}{d(v_i)} \sum_{v_j\in V(G):v_iv_j\in E(G)} \frac{1}{d(v_j)}.$$

The equality holds if and only if G is a complete bipartite graph.

Lemma 3.2 (see [9]). *If G is a connected graph, then*

$$\mathcal{E}_{\mathcal{L}}(v_i) \leq \sqrt{\frac{1}{d(v_i)} \sum_{v_j\in V(G):v_iv_j\in E(G)} \frac{1}{d(v_j)}}.$$

The equality holds if and only if G is a star and v_i is its center.

Theorem 3.1. *Let G be a graph with Randić index $R_{-\frac{1}{2}}(G)$. Then*

$$\mathcal{E}_{\mathcal{L}}(G) \geq \frac{2}{\Delta^2(G)} R_{-\frac{1}{2}}(G),$$

where the equality holds if and only if $G \cong tK_{\frac{n}{2t},\frac{n}{2t}};\ \frac{n}{2t}, t \in \mathbb{N}$. Also, it holds that

$$\mathcal{E}_{\mathcal{L}}(G) \leq \frac{2}{\delta(G)} R_{-\frac{1}{2}}(G),$$

with equality if and only if $G \cong mP_2;\ m \in \mathbb{N}$.

Proof. For an edge $e=(v_i,v_j)$, define $\mathcal{E}_{\mathcal{L}}(e)=\mathcal{E}_{\mathcal{L}}(v_i)/d(v_i)+\mathcal{E}_{\mathcal{L}}(v_j)/d(v_j)$. Then we have

$$\begin{aligned}\sum_{e\in E(G)}\mathcal{E}_{\mathcal{L}}(e)&=\sum_{v_iv_j\in E(G)}\left(\frac{\mathcal{E}_{\mathcal{L}}(v_i)}{d(v_i)}+\frac{\mathcal{E}_{\mathcal{L}}(v_j)}{d(v_j)}\right)\\&=\sum_{v_iv_j\in E(G)}\frac{\mathcal{E}_{\mathcal{L}}(v_i)}{d(v_i)}+\sum_{v_iv_j\in E(G)}\frac{\mathcal{E}_{\mathcal{L}}(v_j)}{d(v_j)}\\&=\frac{1}{2}\sum_{v_i\in V(G)}\sum_{v_iv_j\in E(G)}\frac{\mathcal{E}_{\mathcal{L}}(v_i)}{d(v_i)}+\frac{1}{2}\sum_{v_j\in V(G)}\sum_{v_jv_i\in E(G)}\frac{\mathcal{E}_{\mathcal{L}}(v_j)}{d(v_j)}\\&=\frac{1}{2}\sum_{v_i\in V(G)}\mathcal{E}_{\mathcal{L}}(v_i)+\frac{1}{2}\sum_{v_j\in V(G)}\mathcal{E}_{\mathcal{L}}(v_j)\\&=\sum_{v_j\in V(G)}\mathcal{E}_{\mathcal{L}}(v_j)=\mathcal{E}_{\mathcal{L}}(G).\end{aligned}$$

By Lemma 3.1, we have

$$\begin{aligned}\mathcal{E}_{\mathcal{L}}(G)&=\sum_{v_iv_j\in E(G)}\mathcal{E}_{\mathcal{L}}(e)=\sum_{v_iv_j\in E(G)}\left(\frac{\mathcal{E}_{\mathcal{L}}(v_i)}{d(v_i)}+\frac{\mathcal{E}_{\mathcal{L}}(v_j)}{d(v_j)}\right)\\&\geq\sum_{v_iv_j\in E(G)}\left(\frac{\frac{1}{d(v_j)}\sum_{v_i\in V(G):v_iv_j\in E(G)}\frac{1}{d(v_i)}}{d(v_j)}+\frac{\frac{1}{d(v_i)}\sum_{v_j\in V(G):v_jv_i\in E(G)}\frac{1}{d(v_j)}}{d(v_i)}\right)\\&\geq\frac{1}{\Delta(G)}\left(\frac{1}{2}\sum_{v_j\in V(G)}\frac{1}{d(v_j)}\sum_{v_iv_j\in E(G)}\frac{1}{d(v_i)}+\frac{1}{2}\sum_{v_i\in V(G)}\frac{1}{d(v_i)}\sum_{v_jv_i\in E(G)}\frac{1}{d(v_j)}\right)\\&\geq\frac{1}{\Delta^2(G)}\sum_{v_i\in V(G)}\frac{1}{\sqrt{d(v_i)}}\sum_{v_jv_i\in E(G)}\frac{1}{\sqrt{d(v_j)}}=\frac{2}{\Delta^2(G)}R_{-\frac{1}{2}}(G),\end{aligned}$$

where all equalities hold if and only if for any vertex,

$$\mathcal{E}_{\mathcal{L}}(v_i)=\frac{1}{d(v_i)}\sum_{v_iv_j\in E(G)}\frac{1}{d(v_j)}=\frac{1}{\Delta(G)}\sum_{v_iv_j\in E(G)}\frac{1}{d(v_j)}.$$

Hence, G is the union of complete bipartite graphs with degree $d(v_i)=\frac{n}{2t}$. Similarly, by Lemma 3.2, we have

$$\begin{aligned}\mathcal{E}_{\mathcal{L}}(G)&=\sum_{v_iv_j\in E(G)}\mathcal{E}_{\mathcal{L}}(e)=\sum_{v_iv_j\in E(G)}\left(\frac{\mathcal{E}_{\mathcal{L}}(v_i)}{d(v_i)}+\frac{\mathcal{E}_{\mathcal{L}}(v_j)}{d(v_j)}\right)\\&\leq\sum_{v_iv_j\in E(G)}\left(\frac{\sqrt{\frac{1}{d(v_j)}\sum_{v_iv_j\in E(G)}\frac{1}{d(v_i)}}}{d(v_i)}+\frac{\sqrt{\frac{1}{d(v_i)}\sum_{v_jv_i\in E(G)}\frac{1}{d(v_j)}}}{d(v_j)}\right)\\&\leq\frac{1}{\delta(G)}\sum_{v_iv_j\in E(G)}\left(\sqrt{\frac{1}{d(v_i)}\sum_{v_iv_j\in E(G)}\frac{1}{d(v_i)}}+\sqrt{\frac{1}{d(v_j)}\sum_{v_jv_i\in E(G)}\frac{1}{d(v_i)}}\right)\\&\leq\frac{1}{\delta(G)}\left(\frac{1}{2}\sum_{v_i\in V(G)}\sqrt{\frac{1}{d(v_i)}}\sum_{v_iv_j\in E(G)}\sqrt{\frac{1}{d(v_i)}}+\frac{1}{2}\sum_{v_j\in V(G)}\sqrt{\frac{1}{d(v_j)}}\sum_{v_jv_i\in E(G)}\sqrt{\frac{1}{d(v_j)}}\right)\\&=\frac{2}{\delta(G)}R_{-\frac{1}{2}}(G),\end{aligned}$$

where all equalities hold if and only if for any vertex,

$$\mathcal{E}_{\mathcal{L}}(v_i)=\sqrt{\frac{1}{d(v_i)}\sum_{v_iv_j\in E(G)}\frac{1}{d(v_j)}}=\sqrt{\frac{1}{\delta(G)}\sum_{v_iv_j\in E(G)}\frac{1}{d(v_j)}}.$$

That is, G is the union of the path P_2. □

In [9], Guerrero determined a lower bound on $\mathcal{E}_{\mathcal{L}}(G)$ by using the normalized Laplacian energy of a vertex.

Theorem 3.2 (see [9]). *If G is a graph then $\mathcal{E}_{\mathcal{L}}(G) \geq 2R_{-1}(G)$.*

Now, we give an upper bound on the normalized Laplacian energy in terms of the minimum degree and topological index R_{-1}.

Theorem 3.3. *If G is a graph then*

$$\mathcal{E}_{\mathcal{L}}(G) \leq \frac{2}{\delta^2(G)} R_{-1}(G),$$

where the equality holds if and only if $G \cong mP_2;\ m \in \mathbb{N}$.

Proof. By Lemma 3.2, we have

$$\begin{aligned}
\mathcal{E}_{\mathcal{L}}(G) &= \sum_{v_iv_j \in E(G)} \mathcal{E}_{\mathcal{L}}(e) = \sum_{v_iv_j \in E(G)} \left(\frac{\mathcal{E}_{\mathcal{L}}(v_i)}{d(v_i)} + \frac{\mathcal{E}_{\mathcal{L}}(v_j)}{d(v_j)} \right) \\
&\leq \sum_{v_iv_j \in E(G)} \left(\frac{\sqrt{\frac{1}{d(v_j)} \sum_{v_iv_j \in E(G)} \frac{1}{d(v_i)}}}{d(v_i)} + \frac{\sqrt{\frac{1}{d(v_i)} \sum_{v_jv_i \in E(G)} \frac{1}{d(v_j)}}}{d(v_j)} \right) \\
&\leq \frac{1}{\delta(G)} \sum_{v_iv_j \in E(G)} \left(\sqrt{\frac{1}{d(v_i)} \sum_{v_iv_j \in E(G)} \frac{1}{d(v_j)}} + \sqrt{\frac{1}{d(v_j)} \sum_{v_jv_i \in E(G)} \frac{1}{d(v_i)}} \right) \\
&\leq \frac{1}{\delta(G)} \left(\frac{1}{2} \sum_{v_i \in V(G)} \sqrt{\frac{1}{d(v_i)}} \sum_{v_iv_j \in E(G)} \sqrt{\frac{1}{d(v_i)}} + \frac{1}{2} \sum_{v_j \in V(G)} \sqrt{\frac{1}{d(v_j)}} \sum_{v_jv_i \in E(G)} \sqrt{\frac{1}{d(v_j)}} \right) \\
&\leq \frac{1}{\delta^2(G)} \left(\frac{1}{2} \sum_{v_i \in V(G)} \frac{1}{d(v_i)} \sum_{v_iv_j \in E(G)} \frac{1}{d(v_j)} + \frac{1}{2} \sum_{v_j \in V(G)} \frac{1}{d(v_j)} \sum_{v_jv_i \in E(G)} \frac{1}{d(v_i)} \right) \\
&= \frac{2}{\delta^2(G)} R_{-1}(G),
\end{aligned}$$

where all equalities hold if and only if for any vertex,

$$\mathcal{E}_{\mathcal{L}}(v_i) = \sqrt{\frac{1}{d(v_i)} \sum_{v_iv_j \in E(G)} \frac{1}{d(v_j)}} = \sqrt{\frac{1}{\delta(G)} \sum_{v_iv_j \in E(G)} \frac{1}{d(v_j)}}.$$

That is, G is the union of the path P_2. □

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