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Research Article

Computing the sum of k largest Laplacian eigenvalues of tricyclic graphs

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Abstract

Let $G(V,E)$ be a simple graph with $|V(G)|=n$ and $|E(G)|=m$. If $S_k(G)$ is the sum of k largest Laplacian eigenvalues of G, then Brouwer's conjecture states that $S_k(G) \le m + \frac{k(k+1)}{2}$ for $1 \le k \le n$. The girth of a graph G is the length of a smallest cycle in G. If g is the girth of G, then we show that the mentioned conjecture is true for $1 \le k \le \lfloor \frac{g-2}{2} \rfloor$. Wang et al. [*Math. Comput. Model.* **56** (2012) 60–68] proved that Brouwer's conjecture is true for bicyclic and tricyclic graphs whenever $1 \le k \le n$ with $k \ne 3$. We settle the conjecture under discussion also for tricyclic graphs having no pendant vertices when $k = 3$.

Keywords: Laplacian matrix; Laplacian eigenvalues; Brouwer's conjecture; tricyclic graph; degree sequence.

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1. Introduction

Let $G(V,E)$ be a simple graph with order n and size m having vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set $E = \{e_1, e_2, \dots, e_m\}$. We denote a path of length $n-1$ by P_n and a cycle of length n by C_n. The complete graph with n vertices is denoted by K_n. A tricyclic graph is a connected graph with n vertices and $n+2$ edges. In a graph G, the length of a smallest cycle is called the girth of G and is denoted by g. We refer the reader to [14] for other undefined notations and terminology from spectral graph theory. The adjacency matrix of G is defined as $A(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

The *Laplacian matrix* of G is defined as $L(G) = D(G) - A(G)$, where $D = diag(d_1, d_2, \dots, d_n)$ is the diagonal matrix of vertex degrees $d_i = deg(v_i)$. The eigenvalues of $L(G)$ are called the Laplacian eigenvalues (or Laplacian spectrum) of G. Let $\mu_1(G) \ge \mu_2(G) \ge \dots \ge \mu_n(G) = 0$ be the Laplacian eigenvalues of G. For $k = 1,2,3,\dots,n$, let $S_k = \sum_{i=1}^{k} \mu_i(G)$ be the sum of k largest Laplacian eigenvalues of G. Brouwer [1] proposed a conjecture concerning the sum of k largest Laplacian eigenvalues of a graph G, which is stated as

$$S_k(G) \le m + \frac{k(k+1)}{2}, \qquad \text{for} \qquad 1 \le k \le n. \tag{1}$$

Brouwer verified this conjecture for all graphs of order n with $n \le 10$. The conjecture is true for $k = 1$, which follows from the inequality $\mu_1(G) \le n$. In [11], Haemers et al. showed that the conjecture is true for trees and is also true for all graphs when $k = 2$. Du et al. [4] proved that the conjecture is true for unicyclic and bicyclic graphs. Wang et al. [16] showed that the conjecture is also true for tricyclic graphs for $1 \le k \le n-1$ except $k = 3$. Chen [2] showed that if the conjecture is true for a graph G for all k, then it is also true for the complement of G. Ganie et al. [6] improved some previously known upper bounds for $S_k(G)$ in terms of various graph parameters and showed that the conjecture is true for some new families of graphs. Rocha and Trevisan [15] verified that the conjecture is true for all k with $1 \le k \le \lfloor g/5 \rfloor$. This result was improved for $1 \le k \le \lfloor g/4 \rfloor$ by Chen in [3]. For further progress on Brouwer's Conjecture, we refer to [5–10, 13] and the references therein.

In the present work, we show that the conjecture is true for $1 \le k \le \lfloor \frac{g-2}{2} \rfloor$. We also prove that Brouwer's conjecture is true for tricyclic graphs without pendant vertices.

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2. Main results

We will use the following lemmas in the sequel.

Lemma 2.1. *Let A and B be two real $n \times n$ symmetric matrices. Then for any $1 \leq k \leq n$*

$$\sum_{i=1}^{k} \lambda_i(A+B) \leq \sum_{i=1}^{k} \lambda_i(A) + \sum_{i=1}^{k} \lambda_i(B)$$

for $1 \leq k \leq n$

Lemma 2.2. [12] *Let G be an unweighted graph on n vertices with edge set E. Then $\mu_1(G) \leq max\{d_u + d_v : uv \in E\}$.*

Lemma 2.2 implies that every eigenvalue of a graph G', having maximum degree of at most 2, is less than or equal to 4. That is, $\mu_i(G') \leq 4$, for all $1 \leq i \leq n-1$. The following theorem gives all possible degree sequences of a tricyclic graph with no pendant vertices.

Theorem 2.1. *Let $\mathcal{T}$ be a tricyclic graph with no pendent vertices. Then the possible degree sequences of $\mathcal{T}$ are*

$$(2,2,\ldots,2,3,3,3,3),\ (2,2,\ldots,2,2,3,3,4),\ (2,2,\ldots,2,2,4,4),\ (2,2,\ldots,2,2,3,5),\ (2,2,\ldots,2,2,6).$$

Proof. Let $p_2, p_3, \ldots, p_{n-1}$ be the number of vertices with degrees $2, 3, \ldots, n-1$, respectively. So

$$p_2 + p_3 + \ldots + p_{n-1} = n \tag{2}$$

$$2p_2 + 3p_3 + \ldots + (n-1)p_{n-1} = 2|E(G)| = 2(n+2) \tag{3}$$

With the help of equations (2) and (3), we get

$$p_3 + 2p_4 + 3p_5 + 4p_6 + \ldots + (n-3)p_{n-1} = 4 \tag{4}$$

Equation (4) has the following possible solutions:

$$(p_3, p_4, p_5, p_6) = (4,0,0,0), (2,1,0,0), (0,2,0,0), (1,0,1,0), (0,0,0,1)$$

and $p_i = 0$ for all $i \geq 7$. This shows that the maximum degree is at most 6. Thus the possible degree sequences of a tricyclic graph with no pendant vertices are

$$(2,2,\ldots,2,3,3,3,3),\ (2,2,\ldots,2,2,3,3,4),\ (2,2,\ldots,2,2,4,4),\ (2,2,\ldots,2,2,3,5),\ (2,2,\ldots,2,2,6).$$ □

Theorem 2.2. *Let $\mathcal{T}$ be a tricyclic graph of order n with no pendant vertices. Then it has a subgraph with at least $(n-2)$ edges in which all the vertices have degree less than or equal to 2.*

Proof. A tricyclic graph $\mathcal{T}$ with n vertices and no pendant vertices has $n+2$ edges. From Theorem 2.1, it can have one of the following degree sequences:

$$(2,2,\ldots,2,3,3,3,3),\ (2,2,\ldots,2,2,3,3,4),\ (2,2,\ldots,2,2,4,4),\ (2,2,\ldots,2,2,3,5),\ (2,2,\ldots,2,2,6).$$

In each of the above cases, we remove the edges incident to vertices with degree greater than 2 until all the remaining vertices have degree 2 or less. Since in all cases there are at most four vertices having degree greater than 2 and at most four edges are sufficient to be removed so that the remaining graph G' has at least $(n+2)-4 = (n-2)$ edges and each vertex of G' has degree 2 or less. □

Lemma 2.3. *Let G^* be a graph of order n, size $|E|$ and maximum degree at most 2. Then $S_k(G^*) \leq |E|$, where $k \leq \frac{|E|}{4}$.*

Proof. From Lemma 2.2, the Laplacian eigenvalues of G^* are less than or equal to 4 . That is, $\mu_i(G^*) \leq 4$, for all $i \geq 1$. Therefore, $S_k(G^*) \leq 4k$. This implies that $S_k(G^*) \leq |E|$, for $k \leq \frac{|E|}{4}$. □

A graph is said to be planar if it can be embedded on the surface of the plane, that is, it has no crossovers. In a planar graph, a region is characterized by the set of edges forming its boundary.

Lemma 2.4. *Let $\mathcal{T}$ be a tricyclic graph of order n and having no pendant vertex. Then $g \leq \frac{n+2}{2}$, where g is the girth of the graph.*

Proof. Clearly, a tricyclic graph is a planar graph with four regions. Let k_{p_i} be the p_i-sided region for $i = 1, 2, 3, 4$. Since each edge is on the boundary of exactly two regions, therefore $p_1 + p_2 + p_3 + p_4 = 2(n + 2)$. If g is the girth of the graph, then the above equation gives $4g \leq 2(n + 2)$, which implies that $g \leq \frac{n+2}{2}$. □

Theorem 2.3. *Let G be a graph with $c \geq 3$ cycles. Then $S_k(G) \leq |E(G)| + \frac{k(k+1)}{2}$, for $k \leq \lfloor \frac{g-2}{2} \rfloor$, where $g \geq 4$ is the girth of the graph.*

Proof. As G has $c \geq 3$ cycles, so it must have a connected tricyclic subgraph $G_1^* \left(n_1^*, E_1^*\right)$ with no pendant vertex. Then, from Theorem 2.2, $G_1^* \left(n_1^*, E_1^*\right)$ has a subgraph $G'{}_1$ with size $|E_1'|$ and maximum degree at most 2. G_1' is also a subgraph of G. Let G_1 be the graph obtained from G by removing edges of G_1'. If no connected component of G_1 has more than 2 cycles, then stop here, otherwise there exists a connected tricyclic subgraph $G_2^* \left(n_2^*, E_2^*\right)$ with no pendant vertices. Again, by Theorem 2.2, $G_2^* \left(n_2^*, E_2^*\right)$ has a subgraph G_2' with size $|E_2'|$ and maximum degree at most 2. Let G_2 be the graph obtained from G_1 by removing edges of G_2'. Suppose after r steps as above, we get the graph G_r in which no connected component has more than 2 cycles, then

$$S_k(G) \leq S_k\left(G_r\right) + S_k\left(G_1'\right) + S_k\left(G_2'\right) + \ldots + S_k\left(G_r'\right)$$

By Lemma 2.3, we have $S_k\left(G_i'\right) \leq |E_i'|$, for $k \leq \frac{|E_i'|}{4}$ and by Lemma 2.4, we have $\lfloor \frac{g-2}{2} \rfloor \leq \frac{|E_i'|}{4}$. Therefore, $S_k\left(G_i'\right) \leq |E_i'|$, for $k \leq \lfloor \frac{g-2}{2} \rfloor$. Thus, we get

$$\begin{aligned} S_k(G) &\leq \left|E\left(G_r\right)\right| + \frac{k(k+1)}{2} + |E_1'| + |E_2'| + \ldots + |E_r'| \\ S_k(G) &\leq \big(|E(G)| - (|E_1'| + |E_2'| + \ldots + |E_r'|)\big) + \frac{k(k+1)}{2} + |E_1'| + |E_2'| + \ldots + |E_r'| \end{aligned} \tag{5}$$

Hence,

$$S_k(G) \leq |E(G)| + \frac{k(k+1)}{2}$$

for $k \leq \lfloor \frac{g-2}{2} \rfloor$. □

Lemma 2.5. *Let $\mathcal{T}$ be a tricyclic graph with no pendant vertex and $e_1, e_2 \in E(\mathcal{T})$ and $|E(\mathcal{T})| = e(\mathcal{T})$.*

(*a*) *If $\mathcal{T} - \{e_1\}$ consists of two non-trivial components, then $S_3 \leq e(\mathcal{T}) + 6$.*

(*b*) *If $\mathcal{T} - \{e_1, e_2\}$ consists of two non-trivial components with at least three vertices, then $S_3 \leq e(\mathcal{T}) + 6$.*

Proof. **(a)** Let $\mathcal{T} - \{e_1\}$ consist of two non-trivial components G_1 and G_2. Then G_1 and G_2 are either trees or unicyclic or bicyclic graphs. Since the conjecture is true for trees, unicyclic and bicyclic graphs, we have $S_3(G_i) \leq |E(G_i)| + 6$ for $i = 1, 2$. Without loss of generality, if $S_3(G_1 \cup G_2) = S_3(G_1)$, then $S_3(\mathcal{T}) \leq S_3(G_1 \cup G_2) + 2 \Rightarrow S_3(\mathcal{T}) \leq S_3(G_1) + 2 \Rightarrow S_3(\mathcal{T}) \leq (|E(G_1)| + 6) + 2 \Rightarrow S_3(\mathcal{T}) = |E(G_1)| + 8 \Rightarrow S_3(\mathcal{T}) \leq |E(\mathcal{T})| + 6$.

Now, suppose that $S_3(G_1 \cup G_2) \neq S_3(G_i)$, for $i = 1, 2$. Without loss of generality, assume that the first three largest Laplacian eigenvalues of $S_3(G_1 \cup G_2)$ are $\mu_1(G_1)$, $\mu_2(G_1)$and $\mu_3(G_2)$, that is, $S_3(G_1 \cup G_2) \leq S_2(G_1) + S_1(G_2)$. Then $S_3(\mathcal{T}) \leq S_3(G_1 \cup G_2) + 2 \Rightarrow S_3(\mathcal{T}) \leq S_2(G_1) + S_1(G_2) + 2 \Rightarrow S_3(\mathcal{T}) \leq (|E(G_1)| + 3) + (|E(G_2)| + 1) + 2 \Rightarrow S_3(\mathcal{T}) \leq |E(\mathcal{T})| + 5$.

(b) Let $\mathcal{T} - \{e_1, e_2\}$ consist of two non-trivial components G_1 and G_2. Then G_1 and G_2 are either trees or unicyclic or bicyclic graphs. Since the conjecture is true for trees, unicyclic and bicylic graphs, we have $S_3(G_i) \leq |E(G_i)| + 6$, for $i = 1, 2$. Without loss of generality, if $S_3(G_1 \cup G_2) = S_3(G_1)$, then $S_3(\mathcal{T}) \leq S_3(G_1 \cup G_2) + 4 \Rightarrow S_3(\mathcal{T}) \leq S_3(G_1) + 4 \Rightarrow S_3(\mathcal{T}) \leq (|E(G_1)| + 6) + 4 \Rightarrow S_3(\mathcal{T}) = |E(G_1)| + 10 \Rightarrow S_3(\mathcal{T}) \leq |E(\mathcal{T})| + 6$.

Now, assume that $S_3(G_1 \cup G_2) \neq S_3(G_i)$ for $i = 1, 2$. Without loss of generality, assume that the first three largest Laplacian eigenvalues of $S_3(G_1 \cup G_2)$ are $\mu_1(G_1)$, $\mu_2(G_1)$ and $\mu_3(G_2)$, that is, $S_3(G_1 \cup G_2) \leq S_2(G_1) + S_1(G_2)$. Therefore, $S_3(\mathcal{T}) \leq S_3(G_1 \cup G_2) + 4 \Rightarrow S_3(\mathcal{T}) \leq S_2(G_1) + S_1(G_2) + 4 \Rightarrow S_3(\mathcal{T}) \leq (|E(G_1)| + 3) + (|E(G_2)| + 1) + 4 \Rightarrow S_3(\mathcal{T}) \leq |E(\mathcal{T})| + 6$. □

Lemma 2.6. *Let G' be the graph obtained from a tricyclic graph $\mathcal{T}$ by removing any one of the subgraphs $K_2 \cup K_2 \cup K_2 \cup K_2$, or $P_3 \cup K_2 \cup K_2$, or $P_3 \cup P_3$, or $S_4 \cup K_2$ or S_5. If G' has maximum degree at most 2, then Brouwer's conjecture holds for $\mathcal{T}$.*

Proof. Let $\mathcal{T}$ be a tricyclic graph of order n and edge set E. Let G^* be one of the graphs $K_2 \cup K_2 \cup K_2 \cup K_2$, or $P_3 \cup K_2 \cup K_2$, or $P_3 \cup P_3$, or $S_4 \cup K_2$. Let G' be the graph obtained by removing G^* from $\mathcal{T}$. If G' has maximum degree of at most 2, then

$$S_3(\mathcal{T}) \leq S_3(G') + S_3(G^*) \leq 12 + 7 \leq (n + 2) + 6 \tag{6}$$

for $n \geq 11$. The second inequality is due to the Lemma 2.2 and the fact that $S_3(G^*) \leq 7$ for all G^*. For $n \leq 10$, Brouwer has already verified that the conjecture is true. This completes the proof. □

Theorem 2.4. *Let $\mathcal{G}$ be the family of tricyclic graphs with no pendant vertices. Then, for $\mathcal{T} \in \mathcal{G}$, we have*

$$S_k(\mathcal{T}) \leq e(\mathcal{T}) + \frac{k(k+1)}{2}$$

for $1 \leq k \leq n-1$.

Proof. From Theorem 3.4 of [16], the conjecture is true for tricyclic graphs for all k, except for $k = 3$. Therefore it is sufficient to prove the conjecture for $k = 3$. As seen earlier, the possible degree sequences of tricyclic graphs are

$$(2,2,\ldots,2,2,3,3,3,3),\ (2,2,\ldots,2,2,2,3,3,4),(2,2,\ldots,2,2,2,2,4,4),\ (2,2,\ldots,2,2,2,2,3,5),\ (2,2,\ldots,2,2,2,2,2,6).$$

Case I. Assume that any two vertices v_1 and v_2 with $d(v_1), d(v_2) > 2$ are adjacent. Then it is enough to remove at most three edges $\{e_1, e_2, e_3\}$ in order to make the degree of each vertex in the remaining graph $G' = \mathcal{T} - \{e_1, e_2, e_3\}$ less than or equal to 2. Thus, we obtain

$$S_3(\mathcal{T}) \leq S_3(G') + 3S_3(K_2) \leq 12 + 6 = 18$$

and $18 \leq (n+2) + 6$ for $n \geq 10$. Therefore, we get

$$S_3(\mathcal{T}) \leq e(\mathcal{T}) + 6 \quad \text{for} \quad n \geq 10.$$

For $n \leq 10$, Brouwer proved that the conjecture is true.

Case II. Consider the case when no two vertices v_1 and v_2, with $d(v_1), d(v_2) > 2$, are adjacent.
(i) For the degree sequence $(2,2,\ldots,2,2,3,3,3,3)$, let $v_1,\ v_2,\ v_3$ and v_4 be four vertices of degree 3. After removing the path between v_1 and the nearest vertex of degree 3, the remaining graph has either two connected components with at least three vertices or has one connected component. In the first case, from the graph $\mathcal{T}$, just remove one edge from the path between v_1 and the nearest vertex of degree 3. So by Lemma 2.5, $\mathcal{T}$ follows Brouwer's conjecture. In the second case, from the graph $\mathcal{T}$, further remove the path between the remaining two vertices v_3 and v_4 with each of degree. Again there are two possibilities, the remaining graph has either two connected components with at least three vertices or has one connected component. In the first case, just remove one edge from each of the two paths in $\mathcal{T}$. So, by Lemma 2.5, $\mathcal{T}$ follows Brouwer's conjecture. If removing these two paths, the graph still has only one non-empty component, then this component is a cycle C_r with $r \geq 8$ containing vertices $v_1,\ v_2,\ v_3$ and v_4. It means, for this case, in the graph $\mathcal{T}$, there is a cycle C_r, with $r \geq 8$, containing vertices $v_1,\ v_2,\ v_3$. Now we can remove four edges, one edge incident on each v_i, to obtain the graph G' such that degree of all vertices in G' is less than or equal to 2 and no two of the removed edges have a common vertex. So, we have

$$S_3(\mathcal{T}) \leq S_3(G') + S_3(K_2 \cup K_2 \cup K_2 \cup K_2) \leq 12 + 6 \leq (n+2) + 6 \tag{7}$$

for $n \geq 10$. For $n < 10$, the conjecture is true as proved by Brouwer.
(ii) Consider the degree sequence $(2,2,\ldots,2,2,3,3,4)$.
Remove one edge from each of the vertices of degree 3, say e_1 and e_2. Then it is possible to remove two edges incident to vertex of degree 4 such that these two edges do not share a vertex with the edges e_1 or e_2. Therefore the subgraph removed is either $P_3 \cup P_3$, or $P_3 \cup K_2 \cup K_2$ and the remaining graph has vertices with degree less than or equal to 2. From Lemma 2.6, $\mathcal{T}$ follows Brouwer's conjecture.
(iii) Consider the degree sequence $(2,2,\ldots,2,2,4,4)$.
Let v_1 and v_2 be two vertices with degree 4. Now, remove two edges, say $e_1,\ e_2$, incident to vertex v_1. Now, it is possible to remove two edges incident to v_2 such that these two edges have no common vertex with e_1 or e_2. Thus, after removing subgraph $P_3 \cup P_3$, the remaining graph has vertices of degree less than or equal to 2. Hence, using Lemma 2.6, $\mathcal{T}$ follows Brouwer's conjecture.

Similarly, the removal of the subgraph $S_4 \cup K_2$ for the degree sequence $(2,2,\ldots,2,2,2,3,5)$ and the subgraph S_5 for the degree sequence $(2,2,\ldots,2,2,2,2,6)$ leaves the remaining graph with vertices having degree less than or equal to 2. The proof then follows by Lemma 2.6. □

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