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Research Article

A note on the locally irregular edge colorings of cacti

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Abstract

A graph is locally irregular if the degrees of the end-vertices of every edge are distinct. An edge coloring of a graph G is locally irregular if every color induces a locally irregular subgraph of G. A colorable graph G is any graph which admits a locally irregular edge coloring. The locally irregular chromatic index $\chi'_{\mathrm{irr}}(G)$ of a colorable graph G is the smallest number of colors required by a locally irregular edge coloring of G. The Local Irregularity Conjecture claims that all colorable graphs require at most 3 colors for a locally irregular edge coloring. Recently, it has been observed that the conjecture does not hold for the bow-tie graph B, since B is colorable and requires at least 4 colors for a locally irregular edge coloring. Since B is a cactus graph and all non-colorable graphs are also cacti, this seems to be a relevant class of graphs for the Local Irregularity Conjecture. In this paper, it is proved that $\chi'_{\mathrm{irr}}(G) \leq 4$ for all colorable cactus graphs.

Keywords: locally irregular edge coloring; Local Irregularity Conjecture; cactus graph.

2020 Mathematics Subject Classification: 05C15.

1. Introduction

All graphs mentioned in this paper are considered to be simple, finite and connected, unless explicitly stated otherwise. A *cactus graph* is any graph with edge disjoint cycles. A graph is said to be *locally irregular* if the degrees of the two end-vertices of every edge are distinct. A *locally irregular k-edge coloring*, or *k-liec* for short, is any k-edge coloring of G every color of which induces a locally irregular subgraph of G. Since in this paper we deal only with the locally irregular edge colorings, a graph which admits such a coloring will be called *colorable*. The *locally irregular chromatic index* $\chi'_{\mathrm{irr}}(G)$ of a colorable graph G is defined as the smallest k such that G admits a k-liec. The first question is which graphs are colorable? To answer this question, we first need to introduce a special class $\mathfrak{T}$ of cactus graphs. The class $\mathfrak{T}$ is defined as follows:

- R1. The triangle K_3 is contained in $\mathfrak{T}$;
- R2. For every graph $G \in \mathfrak{T}$, a graph H which also belongs to $\mathfrak{T}$ can be constructed in the following way: a vertex $u \in V(G)$ of degree 2, which belongs to a triangle of G, is identified with an end-vertex of an even length path or with the only vertex of degree one in a graph consisting of a triangle and an odd length path hanging at one of the vertices of the triangle.

Obviously, graphs from $\mathfrak{T}$ are subcubic cacti in which cycles are vertex disjoint triangles, which are connected by an odd length paths, and besides triangles and odd length paths connecting them a cactus graph G from $\mathfrak{T}$ may have only even length paths hanging at any vertex u such that u belongs to a triangle of G and $d_G(u) = 3$. It was established that the only non-colorable graphs are odd length paths, odd length cycles and cacti from $\mathfrak{T}$ [2]. Before we proceed, let us make the following straightforward observation which will be used later.

Observation 1.1. *Let G be a non-colorable graph and let $e \in E(G)$. If e is an edge incident to a leaf or e belongs to a cycle of G, then $G - e$ is colorable.*

For colorable graphs an interesting question is: what is the smallest number of colors required by a locally irregular edge coloring of any graph? Regarding this question, the following conjecture was proposed in [2].

Conjecture 1.1 (Local Irregularity Conjecture)**.** *For every colorable connected graph G, it holds that $\chi'_{\mathrm{irr}}(G) \leq 3$.*

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It was recently shown [8] that the Local Irregularity Conjecture does not hold in general, since there exists a colorable cactus graph, the so called bow-tie graph B shown in Figure 1, for which $\chi'_{\text{irr}}(G) = 4$. There, the following weaker version of Conjecture 1.1 was proposed.

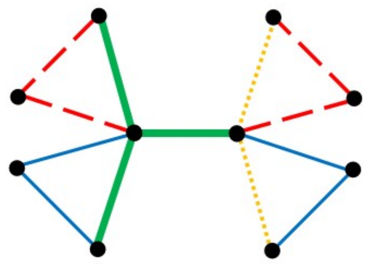

Figure 1: The bow-tie graph B and a 4-liec of it.

Conjecture 1.2. *Every colorable connected graph G satisfies $\chi'_{\text{irr}}(G) \leq 4$.*

Even though the graph B contradicts Local Irregularity Conjecture, many partial results support this conjecture. For example, the conjecture holds for trees [3], unicyclic graphs and cacti with vertex disjoint cycles [8], graphs with minimum degree at least 10^{10} [7], k-regular graphs where $k \geq 10^7$ [2]. The conjecture was also considered for general graphs, but there the upper bound on the number of colors required by a liec of a colorable graph G was first established to be 328 [4], and then it was lowered to 220 [6]. Some other interesting results on special classes of graphs can be found in [1, 5].

The results so far indicate that cacti form a relevant class of graphs for the locally irregular edge colorings, since the non-colorable graphs are cacti and the only known counterexample, graph B, for the Local Irregularity Conjecture is also a cactus graph. Motivated by this, in this paper we further consider this class of graphs. We show that all colorable cacti satisfy Conjecture 1.2.

2. Preliminaries

Let us introduce some results already known in literature, mainly regarding trees and unicyclic graphs, which will be of use in the rest of the paper.

A tree rooted at its leaf will be called a *shrub*. The *root edge* of a shrub T is the edge incident to the root vertex of T. An edge coloring of a shrub T is said to be an *almost locally irregular k-edge coloring*, or *k-aliec* for short, if either it is a k-liec of T or it is a coloring such that only the root edge is locally regular. Let us state few results regarding trees from [3].

Theorem 2.1. *Every shrub admits a 2-aliec.*

Theorem 2.2. *For every colorable tree T, it holds that $\chi'_{\text{irr}}(T) \leq 3$. Moreover, $\chi'_{\text{irr}}(T) \leq 2$ if $\Delta(T) \geq 5$.*

Beside Theorems 2.1 and 2.2, we will also need some specific claims used in [3] to prove our results. In order to state them, let us introduce some more notation. The colors of an edge coloring will be denoted by letters a, b, $c, \ldots$ By $\phi_a(G)$ (resp. $\phi_{a,b}(G)$, $\phi_{a,b,c}(G)$) we will usually denote an edge coloring of G using one (resp. two, three) colors. For an edge coloring $\phi_{a,b,c}$ of G and a vertex $u \in V(G)$, by $\phi_{a,b,c}(u)$ we will denote the set of colors incident to u. If $\phi_{a,b,c}(u)$ contains k colors, then we say u is *k-chromatic*, specifically in the cases of $k = 1$ and $k = 2$ we say u is *monochromatic* and *bichromatic*, respectively. If $\phi_{a,b}$ is an edge coloring of a graph G which uses colors a and b, then $\phi_{c,d}$ will denote the edge coloring of G obtained from $\phi_{a,b}$ by replacing color a with c and color b with d. In particular, the edge coloring $\phi_{b,a}$ obtained from $\phi_{a,b}$ by swapping colors a and b will be called the *inversion* of $\phi_{a,b}$.

We also need to introduce a *sum of colorings*, useful when combining two different colorings. Let G_i, for $i = 1, \ldots, k$, be graphs with pairwise disjoint edge sets, and let G be a graph such that $E(G) = \cup_{i=1}^{k} E(G_i)$. If $\phi^i_{a,b,c}$ is an edge coloring of the graph G_i for $i = 1, \ldots, k$, then $\phi_{a,b,c} = \sum_{i=1}^{k} \phi^i_{a,b,c}$ will denote the edge coloring of G such that $e \in E(G_i)$ implies $\phi_{a,b,c}(e) = \phi^i_{a,b,c}(e)$.

The *a-degree* of a vertex $u \in V(G)$ is defined as the number of edges incident to u which are colored by a, and it is denoted by $d^a_G(u)$. The same name and notation goes for any other color besides a. Now, for a vertex $u \in V(G)$ with k incident edges colored by a, say $uv_1, \ldots, uv_k$, the *a-sequence* of u is defined as $d^a_G(v_1), \ldots, d^a_G(v_k)$. It is usually assumed that vertices v_i are denoted so that the a-sequence is non-increasing.

Regarding shrubs, they admit an 2-aliec according to Theorem 2.1, which will be denoted by $\phi_{a,b}$ where we assume the root edge of the shrub is colored by a. Now, let T be a tree, $u \in V(T)$ a vertex of maximum degree in T, and v_i all the neighbors of u for $i = 1, \dots, k$. Denote by T_i a shrub of T rooted at u with the root edge uv_i. A *shrub based* coloring of T is defined by $\phi_{a,b} = \sum_{i=1}^{k} \phi^i_{a,b}$, where $\phi^i_{a,b}$ is an 2-aliec of T_i. Since we assume that the root edge of T_i is colored by a in $\phi^i_{a,b}$, this implies u is monochromatic in color a by a shrub based coloring $\phi_{a,b}$ of T. Obviously, if a shrub based coloring $\phi_{a,b}$ is not a liec of T, only the edges incident to u may be locally regular by $\phi_{a,b}$. Notice that $\binom{k}{2}$ different 2-edge colorings of T can be obtained from a shrub based coloring $\phi_{a,b}$ by swapping colors a and b in some of the shrubs T_i. If none of those colorings is a liec of T, we say $\phi_{a,b}$ is *inversion resistant*. The following observation was established in [3].

Observation 2.1. *Let T be a tree, $u \in V(T)$ a vertex of maximum degree in T and $\phi_{a,b}$ a shrub based coloring of T rooted at u. A shrub based coloring $\phi_{a,b}$ will be inversion resistant in two cases only:*

- $d_T(u) = 3$ *and the a-sequence of u by $\phi_{a,b}$ is* $3, 2, 2$*;*
- $d_T(u) = 4$ *and the a-sequence of u by $\phi_{a,b}$ is* $4, 3, 3, 2$.

The obvious consequence of the above observation is that in a tree T with $\chi'_{\mathrm{irr}}(T) = 3$ and $\Delta(T) = 3$ (resp. $\Delta(T) = 4$) the vertices of degree 3 (resp. 4) must come in neighboring pairs. Also, if $\chi'_{\mathrm{irr}}(T) = 3$, then a shrub based coloring of the tree T rooted at a vertex u of maximum degree is inversion resistant. A 3-liec of such a tree T is obtained from aliecs of shrubs in a following way: if $d_T(u) = 3$ then $\phi^T_{a,b,c} = \phi^1_{a,b} + \phi^2_{b,a} + \phi^3_{c,b}$ is a 3-liec of T, if $d_T(u) = 4$ then $\phi^T_{a,b,c} = \phi^1_{a,b} + \phi^2_{a,b} + \phi^3_{b,a} + \phi^4_{c,b}$ is a 3-liec of T. Notice that in this 3-liec of T, only the root vertex u is 3-chromatic, all other vertices in T are 1- or 2-chromatic. Hence, we will call u the *rainbow root* of such a liec. Obviously, every vertex u of maximum degree of a tree T with $\chi'_{\mathrm{irr}}(T) = 3$ can be the rainbow root of a 3-liec of T, because either the degree sequence of u by a shrub based coloring is $3, 2, 2$ (resp. $4, 3, 3, 2$) which means the above 3-liecs can be constructed with u being the rainbow root, or the shrub based coloring would not be inversion resistant, which implies $\chi'_{\mathrm{irr}}(G) \leq 2$, a contradiction.

Further, notice that the color c is used by $\phi^T_{a,b,c}$ in precisely one shrub of T. Simply, one can choose that one shrub to be any of the shrubs of T rooted at u. Let us now collect all this in the following formal observation for further referencing.

Observation 2.2. *Let T be a colorable tree with $\chi'_{\mathrm{irr}}(T) = 3$. Then all vertices of maximum degree in T come in neighboring pairs. Also, every vertex of maximum degree in T can be the rainbow root of a 3-liec of T. Finally, for any vertex u of maximum degree in T and for any shrub T_i of T rooted at u, there exists a 3-liec of T such that the color c is used only in T_i.*

Finally, we will also need the following result from [8] on unicyclic graphs.

Theorem 2.3. *Let G be a colorable unicyclic graph. Then $\chi'_{\mathrm{irr}}(G) \leq 3$.*

3. Coloring cacti by four colors

A *grape* G is any cactus graph with at least one cycle in which all cycles share a vertex u, and the vertex u is called the *root* of G. A *berry* G_i of a grape G is any subgraph of G induced by $V(G'_i) \cup \{u\}$, where u is the root of G and G'_i a connected component of $G - u$. Notice that a berry G_i can be either a unicyclic graph in which u is of degree 2 or a tree in which u is a leaf, so such berries will be called *unicyclic berries* and *acyclic berries*, respectively. A unicyclic berry G_i is said to be *triangular* if its cycle is the triangle.

An *end-grape* G_u of a cactus graph G is any subgraph of G such that:

- G_u is a grape rooted at u where u is the only vertex of G_u incident to edges from $G - E(G_u)$, and
- u is incident to either one edge from $G - E(G_u)$ or two such edges which then must belong to a same cycle of $G - E(G_u)$, and such edges are called the *exit edges* of G_u.

This notion is illustrated by Figure 2. Also, for an end-grape G_u rooted at u, the graph $G_0 = G - E(G_u)$ will be called the *root component* of G_u. Notice that $d_{G_0}(u) \leq 2$ and also that $E(G) = E(G_0) \cup E(G_u)$ where $E(G_0) \cap E(G_u) = \emptyset$. Let us first prove the following auxiliary result.

Lemma 3.1. *Let G be a colorable cactus graph with $c \geq 2$ cycles and G_u an end-grape of G. If G_u is comprised of just one berry and that berry is a non-colorable triangular berry which has only one exit edge, then there exists a colorable cactus graph G'_0 with fewer cycles than G, such that $\chi'_{\mathrm{irr}}(G) \leq \max\{\chi'_{\mathrm{irr}}(G'_0), 3\}$.*

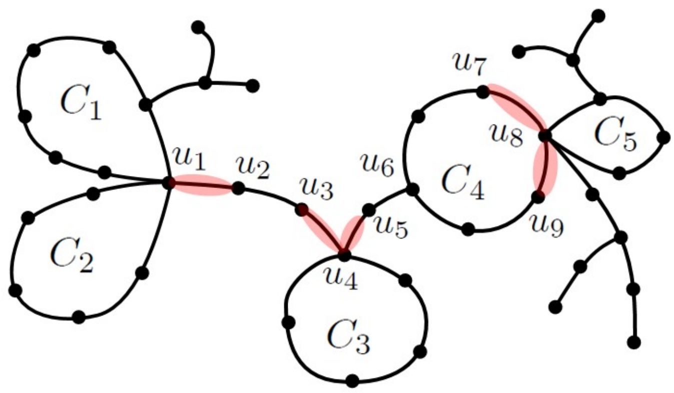

Figure 2: A cactus graph G with five cycles which contains two end-grapes, G_{u_1} and G_{u_8}. The end-grape G_{u_1} has two unicyclic berries and one exit edge u_1u_2. The end-grape G_{u_8} consists of one unicyclic and one acyclic berry and it has two exit edges u_8u_7 and u_8u_9. Notice that the cycle C_3 is not an end-grape of G since it has two exit edges u_4u_3 and u_4u_5 and they do not belong to a same cycle.

Proof. Denote by v and w be the neighbors of u in G_u. Since G_u is a non-colorable triangular berry, it consists of the triangle uvw and possibly an even length path hanging at v and/or w. Let G_0' be the connected component of $G-v$ which contains u. Notice that G_0' consists of a root component G_0 of G_u and a pendant odd length path hanging at u which contains w. If G_0' is non-colorable, since it contains a pendant path we conclude $G_0' \in \mathfrak{T}$, but then G is also non-colorable, a contradiction. So, we may assume G_0' is colorable and admits a liec which uses say colors a and b, so we will denote it by $\phi^0_{a,b}$. Since u and w belong to a pendant path P of G_0', given the parity of their distance to the leaf of the path P, we may assume $\phi^0_{a,b}(u) = \{a\}$ and $\phi^0_{a,b}(w) \subseteq \{a,b\}$. Let $T = G - E(G_0')$ and notice that T is a tree which consists of the path uvw with possibly one even length path hanging at v. Obviously, T admits a 2-liec $\phi^T_{b,c}$ such that $\phi^T_{b,c}(u) = \phi^T_{b,c}(w) = \{c\}$. Since G consists of G_0' and T which meet at vertices u and w, we conclude that $\phi^0_{a,b} + \phi^T_{b,c}$ is a k-liec of G, where $k = \max\{\chi'_{\mathrm{irr}}(G_0'), 3\}$. □

Before we proceed with the main result, let us introduce the notation we will use in a cactus graph with an end-grape G_u and its root component G_0. Let p (resp. q) denote the number of cyclic (resp. acyclic) berries in G_u. We may assume berries of G_u are denoted by G_i, for $i = 1, \ldots, p+q$, so that G_i is a cyclic berry if and only if $i \leq p$. Let v_i be a neighbor of u in a berry G_i for $i = 1, \ldots, p+q$. Let w_i be the other neighbor of u in a cyclic berry G_i for $i = 1, \ldots, p$. The neighbors of u in the root component G_0 will be denoted by u_1 and u_2, where u_2 does not exist if $d_{G_0}(u) = 1$.

Theorem 3.1. *Let G be a colorable cactus graph. Then $\chi'_{\mathrm{irr}}(G) \leq 4$.*

Proof. The proof is by the induction on the number of cycles in G. If G contains less than 2 cycles, then G is a tree or a unicyclic graph, so the claim follows from Theorems 2.2 and 2.3, respectively. Let us assume that the claim holds for all cacti with less than $c \geq 2$ cycles, and let G be a cactus graph with c cycles.

If G is a grape with the root vertex u, let E_u be the set consisting of an edge incident to u from every cycle in G. Since G contains at least two cycles, E_u contains at least two edges, hence E_u induces a locally irregular subgraph of G. Notice that $T = G - E_u$ is a tree. If T is colorable, then it can be colored by 3 colors, so coloring E_u by the fourth color yields a 4-liec of G. If T is non-colorable, then T is a path, and there exists an edge uz in T such that $uz \notin E_u$ and $T' = T - uz$ is a collection of one or two even length paths. Thus T' is colorable and can be colored by 2 colors, then coloring $E_u' = E_u \cup \{uz\}$ by the third color yields a 3-liec of G.

Assume now that G is not a grape, so let G_u be an end-grape of G and let G_0 be the root component of G_u. If G_0 is non-colorable, since G is not a grape, it follows that $G_0 \in \mathfrak{T}$. So, within G_0 there must exist an end-grape of G consisting of a single non-colorable triangular berry with only one exit edge. Then the claim follows from Lemma 3.1 and the induction hypothesis. So, let us assume that G_0 is colorable. Let $E_u = \{uv_1, \ldots, uv_p\}$ and let $T = G_u - E_u$. Notice that T is a tree.

Case 1: *T is not colorable*. Since T is a non-colorable tree, it must be an odd length path. Notice that there exists an edge uz in T, such that $T' = T - uz$ is a collection of one or two even length paths. Hence, T' admits a 2-liec $\phi^{T'}_{a,b}$. Since the degree of u in T' is at most one we may assume $\phi^{T'}_{a,b}(u) \subseteq \{a\}$. Let $E_u' = E_u \cup \{uz\}$ and notice that E_u' contains at least two edges, so it induces a locally irregular subgraph of G which admits a 1-liec $\phi^{E'}_c$. Hence, $\phi^u_{a,b,c} = \phi^{T'}_{a,b} + \phi^{E'}_c$ is a 3-liec of G_u such that $\phi^u_{a,b,c}(u) \subseteq \{a,c\}$.

It remains to consider the root component G_0 of G_u. Since G_0 is colorable, the induction hypothesis implies G_0 admits a 4-liec $\phi^0_{a,b,c,d}$. Also, $d_{G_0}(u) \leq 2$ implies that we may assume $\phi^0_{a,b,c,d}(u) \subseteq \{b,d\}$. Since G is comprised of G_0 and G_u which meet at u, from $\phi^0_{a,b,c,d}(u) \subseteq \{b,d\}$ and $\phi^u_{a,b,c}(u) \subseteq \{a,c\}$ we conclude that $\phi^0_{a,b,c,d} + \phi^u_{a,b,c}$ is the desired 4-liec of G.

Case 2: T *is colorable*. Denote by $\phi^T_{a,b,c}$ a liec of T, and if T is colorable by fewer than 3 colors, we assume the color c (resp. b and c) is not used. We wish to establish that there exists a liec $\phi^T_{a,b,c}$ of T such that

$$\phi^T_{a,b,c}(x) \subseteq \{a,b\} \tag{1}$$

for each $x \in \{u, v_1, \dots, v_p\}$. Now, if $\chi'_{\text{irr}}(T) \leq 2$, then (1) obviously holds. So, let us assume $\chi'_{\text{irr}}(T) = 3$. Notice that vertices $u, v_1, \dots, v_p$ form an independent set in T. Thus, Observation 2.2 implies we can choose the rainbow root z in T distinct from $u, v_1, \dots, v_p$. Notice that u and all v_i for $i \leq p$ belong to at most two distinct shrubs of T rooted at z. Since T has at least three shrubs rooted at z, Observation 2.2 implies the color c can be used only in the shrub not containing vertices u and v_i for $i \leq p$. Thus, we obtain a 3-liec of T for which (1) holds, and the claim is established.

Subcase 2a: $p \geq 2$. Since $|E_u| = p \geq 2$, the set E_u induces a locally irregular subgraph of G which admits a 1-liec ϕ^E_c. Since (1) holds, we conclude that $\phi^u_{a,b,c} = \phi^T_{a,b,c} + \phi^E_c$ is a 3-liec of G_u. It remains to glue $\phi^u_{a,b,c}$ with a coloring of G_0.

Since G_0 is colorable, the induction hypothesis implies there exists a 4-liec $\phi^0_{a,b,c,d}$ of G_0. If u is 1-chromatic by $\phi^0_{a,b,c,d}$, we may assume $\phi^0_{a,b,c,d}(u) = \{d\}$, so $\phi^0_{a,b,c,d} + \phi^u_{a,b,c}$ is the desired 4-liec of G. If u is 2-chromatic by $\phi^0_{a,b,c,d}$, we may assume that u_1u is c-colored, and u_2u is d-colored. When we glue together G_u and G_0 with their respective colorings, the only color incident to u in both graphs is the color c. So, it is important to consider c-degree of u and its neighbors in G_u and in G_0, which will be denoted by $d^c_u(u)$ and $d^c_0(u)$ respectively.

Notice that $d^c_u(u) = p$, and the c-degree of the neighbors of u in G_u is at most one. When G_u and G_0 are glued together, the c-degree of u becomes $p+1$. This will not make c-colored edges of G_u incident to u to become locally regular, the only problem is the edge u_1u in G_0. If $d^c_0(u_1) \neq p+1$, then $\phi^0_{a,b,c,d} + \phi^u_{a,b,c}$ is the desired 4-liec of G. If $d^d_0(u_2) \neq p+1$, we can swap colors c and d in G_0, so the case reduces to the previous case $d^c_0(u_1) \neq p+1$. Finally, if $d^c_0(u_1) = d^d_0(u_2) = p+1$, we have to make a small modification of $\phi^u_{a,b,c}$. Let $\phi'^u_{a,b,c,d}$ be the coloring of G_u obtained from $\phi^u_{a,b,c}$ by changing only the color of uv_1 from c to d. Notice that $\phi'^u_{a,b,c,d}$ is not a liec of G_u, since uv_1 is an isolated edge in color d, but $\phi^0_{a,b,c,d} + \phi'^u_{a,b,c,d}$ is a 4-liec of G, which proves the claim.

Subcase 2b: $p = 1$. Let $G''_0 = G_0 + uv_1$. If G''_0 is colorable, by induction hypothesis it admits a 4-liec $\phi''^0_{a,b,c,d}$. Notice that $d_{G''_0}(u) = d_{G_0}(u) + 1 \leq 3$. Now we argue that u is 1- or 2-chromatic by $\phi''^0_{a,b,c,d}$, and to see this we distinguish the case $d_{G_0}(u) = 1$ and $d_{G_0}(u) = 2$. If $d_{G_0}(u) = 1$, then $d_{G''_0}(u) = 2$, so the claim obviously holds. If $d_{G_0}(u) = 2$, then $d_{G''_0}(u) = 3$, but in this case u is a vertex of a cycle in G''_0 with a pendant edge uv_1. In order for an edge coloring of G''_0 to be locally irregular, the color of the pendant edge uv_1 must be the same as the color of at least one more edge incident to u. From this we deduce that u is 1- or 2-chromatic by $\phi''^0_{a,b,c,d}$, as claimed. Hence, we may assume $\phi''^0_{a,b,c,d}(u) \cup \phi''^0_{a,b,c,d}(v_1) \subseteq \{c,d\}$. Notice that G consists of G''_0 and T which meet at u and v_1. From (1) and $\phi''^0_{a,b,c,d}(u) \cup \phi''^0_{a,b,c,d}(v_1) \subseteq \{c,d\}$ we conclude $\phi''^0_{a,b,c,d} + \phi^T_{a,b,c}$ is the desired 4-liec of G.

If G''_0 is not colorable, then the presence of a leaf in G''_0 implies $G''_0 \in \mathfrak{T}$. Then again the claim follows from Lemma 3.1 and the induction hypothesis. □

4. Concluding remarks

In this paper we established that all colorable cacti require at most 4 colors for a locally irregular edge coloring. This is the best possible upper bound, since there exists the so called bow-tie graph B, which is a colorable cactus graph with $\chi'_{\text{irr}}(B) = 4$. This result can be further extended to a claim that every colorable cactus graph distinct from B requires at most three colors for the locally irregular edge coloring and a paper with this result is in preparation.

Our argument of this claim is lengthy but uses the same approach as Theorem 3.1. The main difference is that in Case 2.a of Theorem 3.1 we do not have to take much care about a- and b-degrees of the neighbors of u in T since we have the fourth color d to use it for at least one of the two edges incident to u in G_0. When the fourth color must not be used, then a great care has to be taken of these a- and b-degrees in T because the same colors must be used for both edges incident to u in G_0. So, one has to avoid colors a and b in G_u to spare them for G_0, i.e. color all edges of G_u incident to u by c and not just E_u. That is not always possible, so special berries and alternative colorings for them need to be introduced. In light of all this, it might be helpful for a reader interested into this to consider Theorem 3.1 as a first step.

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One of the referees indicated that the main result (Theorem 3.1) of this paper had already been independently proved by another group of researchers and that the manuscript containing this result was submitted to a journal in January 2022. The authors remark here that the detail (title, abstract, authors, etc.) of the mentioned manuscript is not yet publicly available (by September 6, 2022).

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