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Research Article

Bounds on the general eccentric distance sum of graphs

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Abstract

Some sharp bounds on the general eccentric distance sum are presented for (i) graphs with given order and chromatic number, (ii) trees with given bipartition, and (iii) bipartite graphs with given order and matching number. Bounds for bipartite graphs hold also if the matching number is replaced by the independence number, vertex cover number or edge cover number.

Keywords: general eccentric distance sum; distance-based index; Wiener index; bipartite graph.

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1. Introduction

Let $V(G)$ and $E(G)$ be the vertex set and edge set of a graph G. The number of vertices is called the order and the number of edges is the size of G. The number of edges incident with a vertex u is the degree $deg_G(u)$ of u. The number of edges in a shortest path between vertices u and v is the distance $d_G(u,v)$ between u and v. The distance between u and a vertex farthest from u in G is the eccentricity $ecc_G(u)$ of u in G. The diameter of G is the maximum eccentricity among eccentricities of the vertices in G.

A matching is a set of edges of a graph G such that no two edges in that set have a vertex in common. A vertex independent set is a set of vertices of a graph G such that no two vertices in that set are adjacent in G. The cardinality of a maximum matching/independent set is the matching number/independence number of G, respectively. A vertex cover of a graph G is a set of vertices such that each edge of G is incident with at least one vertex from that set. An edge cover of G is a set of edges such that each vertex of G is incident with at least one edge from that set. The vertex/edge cover number is the cardinality of a minimum vertex cover/edge cover, respectively.

For $k \geq 2$, a graph is called k-partite if its vertex set can be partitioned into k sets, where any two vertices from the same set are non-adjacent. A complete k-partite graph $K_{p_1,p_2,\dots,p_k}$ is a k-partite graph with partite sets of orders $p_1, p_2, \dots, p_k$, where any two vertices from different partite sets are adjacent. A 2-partite graph is called a bipartite graph.

A connected graph containing no cycles is a tree. A leaf is a vertex of a tree having degree 1. The double star S_{p_1,p_2} is a tree containing exactly two vertices which are not leaves, and their degrees are p_1 and p_2, respectively. So S_{p_1,p_2} contains $p_1 + p_2 - 2$ leaves. For the complement $\overline{G}$ of G, we have $V(\overline{G}) = V(G)$ and $uv \in E(\overline{G})$ if and only if vertices u and v are not adjacent in G.

For a connected graph G and $a, b \in \mathbb{R}$, the general eccentric distance sum is defined as

$$EDS_{a,b}(G) = \sum_{u \in V(G)} [ecc_G(u)]^a [D_G(u)]^b,$$

where $D_G(u) = \sum_{v \in V(G)} d_G(u,v)$. This index generalizes several distance-based indices. We obtain the classical eccentric distance sum for $a = 1$ and $b = 1$, the total eccentricity index for $a = 1$ and $b = 0$, and the first Zagreb eccentricity index of G for $a = 2$ and $b = 0$. For $a = 0$ and $b = 1$, we get $EDS_{0,1}(G) = 2W(G)$, where $W(G)$ is the Wiener index.

The eccentric distance sum EDS belongs to topological indices which have been investigated extensively. An upper bound on the EDS for graphs of given order and minimum degree was obtained by Mukungunugwa and Mukwembi [16]. A lower bound for trees with prescribed order was given by Yu, Feng, and Ilić [24], and also by Hua, Xu, and Wen [11]. The EDS for trees was studied also in [8, 18]. The EDS was investigated for several basic graphs in [17], graphs related to groups in [1], cubic transitive graphs in [23], graph operations in [2], bipartite graphs in [4, 15], fullerances in [9], and

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Shahin

Sierpiński networks in [3]. Relations between the EDS and a few other indices were studied in [10]. Interesting results on the EDS were presented also in [12–14]. Bounds on $EDS_{a,b}$ for trees, bipartite graphs and general graphs with given order as well as for graphs with given order and number of pendant vertices/vertex connectivity were presented in [20]. Another general index was studied for example in [6] and some distance-based indices were investigated also in [5, 7, 19].

We give bounds on $EDS_{a,b}$ for trees with given bipartition and bipartite graphs with given order and matching number. Lower bounds are obtained for $a \geq 0$ and $0 < b \leq 1$. Upper bounds are obtained for $a \leq 0$ and $b < 0$. Bounds for bipartite graphs hold also if the matching number is replaced by the independence number, vertex cover number or edge cover number. A lower bound on $EDS_{a,b}$ for graphs with given order and chromatic number, where each color is used for at least two vertices, is presented for $a \geq 0$ and $b \geq 1$. Finally, for $a \geq 0$, we present a lower bound on $EDS_{a,1}(G)$ for graphs G of given order and size containing no vertex adjacent to all the other vertices, and a lower bound on $EDS_{a,1}(G) + EDS_{a,1}(\overline{G})$ for graphs G of given order. All the bounds are sharp and extremal graphs are presented.

2. Results

First, we present Lemma 2.1 which was proved in [21]. We use this lemma in the proofs of Theorems 2.1, 2.2, 2.3, 2.4, and 2.5 to compare $EDS_{a,b}$ of some graphs.

Lemma 2.1. *Let $1 \leq x < y$ and $c > 0$. Then for $b > 1$ and $b < 0$,*

$$(x+c)^b - x^b < (y+c)^b - y^b.$$

If $0 < b < 1$, then

$$(x+c)^b - x^b > (y+c)^b - y^b.$$

A bipartite graph with two partite sets U_1 and U_2 has an (s,t)-bipartition if $|U_1| = s$ and $|U_2| = t$. Clearly, for the order n of G, we have $n = s + t$. In Theorems 2.1 and 2.2, we consider trees having an (s,t)-bipartition with $s \geq t \geq 2$, because the unique tree having an $(s,1)$-bipartition is the star with $s+1$ vertices. For $a = b = 1$, Theorem 2.1 was presented in [8]. For $a = 2$ and $b = 0$, Theorem 2.1 was given in [19].

Theorem 2.1. *Let $a \geq 0$, $0 < b \leq 1$ and $s \geq t \geq 2$. For any tree T with an (s,t)-bipartition,*

$$EDS_{a,b}(T) \geq (s-1)3^a(3t+2s-4)^b + (t-1)3^a(3s+2t-4)^b + 2^a(2t+s-2)^b + 2^a(2s+t-2)^b,$$

with equality if and only if $T \cong S_{s,t}$.

Proof. Among trees with an (s,t)-bipartition, we denote a tree with the smallest $EDS_{a,b}$ by T'. Let us prove by contradiction that $T' \cong S_{s,t}$.

Assume that $T' \ncong S_{s,t}$. A tree with diameter $d \leq 2$ does not exist for $s \geq t \geq 2$ and the only tree having diameter 3 is $S_{s,t}$. Thus $d \geq 4$. We denote a diametral path in T' by $u_0u_1 \dots u_d$ (so $d_{T'}(u_0, u_d) = d$) and the leaves adjacent to u_{d-1} by $w_1, w_2, \dots, w_p$, where u_d is one of them and $p \geq 1$. Without loss of generality, we assume that $D_{T'}(u_1) \leq D_{T'}(u_{d-1})$. Let $T'' = T' - \{u_{d-1}w_1, u_{d-1}w_2, \dots, u_{d-1}w_p\} + \{u_{d-3}w_1, u_{d-3}w_2, \dots, u_{d-3}w_p\}$. Clearly, T'' has an (s,t)-bipartition. Let us use u_1 and u_{d-1} to obtain a contradiction. We have $ecc_{T'}(u_1) = ecc_{T'}(u_{d-1}) = ecc_{T''}(u_{d-1}) = d-1$ and $d-2 \leq ecc_{T''}(u_1) \leq d-1$. We obtain

$$D_{T''}(u_1) = D_{T'}(u_1) - 2p$$

and

$$D_{T''}(u_{d-1}) = D_{T'}(u_{d-1}) + 2p.$$

For any vertex $z \in V(T') \setminus \{u_1, u_{d-1}\}$, we have $ecc_{T'}(z) \geq ecc_{T''}(z)$ and $D_{T'}(z) \geq D_{T''}(z)$, therefore

$$[ecc_{T'}(z)]^a[D_{T'}(z)]^b \geq [ecc_{T''}(z)]^a[D_{T''}(z)]^b$$

for $a \geq 0$ and $0 < b < 1$. Moreover, there are some vertices $z \in V(T') \setminus \{u_1, u_{d-1}\}$ (for example u_0), for which $D_{T'}(z) > D_{T''}(z)$, therefore

$$[ecc_{T'}(z)]^a[D_{T'}(z)]^b > [ecc_{T''}(z)]^a[D_{T''}(z)]^b$$

for those vertices.

Thus,

$$\begin{aligned}
&EDS_{a,b}(T') - EDS_{a,b}(T'') \\
&= \sum_{z \in V(T') \setminus \{u_1, u_{d-1}\}} ([ecc_{T'}(z)]^a [D_{T'}(z)]^b - [ecc_{T''}(z)]^a [D_{T''}(z)]^b) \\
&\quad + [ecc_{T'}(u_1)]^a [D_{T'}(u_1)]^b - [ecc_{T''}(u_1)]^a [D_{T''}(u_1)]^b + [ecc_{T'}(u_{d-1})]^a [D_{T'}(u_{d-1})]^b - [ecc_{T''}(v_{u-1})]^a [D_{T''}(u_{d-1})]^b \\
&> [ecc_{T'}(u_1)]^a [D_{T'}(u_1)]^b - [ecc_{T''}(u_1)]^a [D_{T''}(u_1)]^b + [ecc_{T'}(u_{d-1})]^a [D_{T'}(u_{d-1})]^b - [ecc_{T''}(v_{u-1})]^a [D_{T''}(u_{d-1})]^b \\
&= (d-1)^a [D_{T'}(u_1)]^b - [ecc_{T''}(u_1)]^a [D_{T'}(u_1) - 2p]^b + (d-1)^a ([D_{T'}(u_{d-1})]^b - [D_{T'}(u_{d-1}) + 2p]^b) \\
&\geq (d-1)^a ([D_{T'}(u_1)]^b - [D_{T'}(u_1) - 2p]^b) + (d-1)^a ([D_{T'}(u_{d-1})]^b - [D_{T'}(u_{d-1}) + 2p]^b) \\
&\geq 0,
\end{aligned}$$

because for $b = 1$,

$$[D_{T'}(u_1)]^b - [D_{T'}(u_1) - 2p]^b + [D_{T'}(u_{d-1})]^b - [D_{T'}(u_{d-1}) + 2p]^b = 0,$$

and for $0 < b < 1$, from Lemma 2.1, we obtain

$$[D_{T'}(u_1)]^b - [D_{T'}(u_1) - 2p]^b > [D_{T'}(u_{d-1}) + 2p]^b - [D_{T'}(u_{d-1})]^b.$$

Therefore $EDS_{a,b}(T') > EDS_{a,b}(T'')$. Hence T' does not have the minimum $EDS_{a,b}$. We have a contradiction.

So $T' \cong S_{s,t}$ which contains two vertices which are not leaves, say v and v', where v is adjacent to $s-1$ leaves v_i, $i = 1, 2, \ldots, s-1$, and v' is adjacent to $t-1$ leaves v'_j, $j = 1, 2, \ldots, t-1$. We have

$$\begin{aligned}
ecc_{S_{s,t}}(v) &= 2, \quad D_{S_{s,t}}(v) = s + 2(t-1), \\
ecc_{S_{s,t}}(v') &= 2, \quad D_{S_{s,t}}(v') = t + 2(s-1), \\
ecc_{S_{s,t}}(v_i) &= 3, \quad D_{S_{s,t}}(v_i) = 1 + 2(s-1) + 3(t-1), \\
ecc_{S_{s,t}}(v'_j) &= 3, \quad D_{S_{s,t}}(v'_j) = 1 + 2(t-1) + 3(s-1).
\end{aligned}$$

Hence

$$EDS_{a,b}(S_{s,t}) = (s-1)3^a(3t+2s-4)^b + (t-1)3^a(3s+2t-4)^b + 2^a(2t+s-2)^b + 2^a(2s+t-2)^b.$$

□

Theorem 2.2. *Let $a \leq 0$, $b < 0$ and $s \geq t \geq 2$. For any tree T with an (s,t)-bipartition,*

$$EDS_{a,b}(T) \leq (s-1)3^a(3t+2s-4)^b + (t-1)3^a(3s+2t-4)^b + 2^a(2t+s-2)^b + 2^a(2s+t-2)^b,$$

with equality if and only if $T \cong S_{s,t}$.

Proof. Only those parts of the proof are presented which differ from the proof of Theorem 2.1. Among trees with an (s,t)-bipartition, we denote a tree with the largest $EDS_{a,b}$ by T'. Since $ecc_{T''}(u_1) \leq d-1$, we have

$$[ecc_{T''}(u_1)]^a \geq (d-1)^a \quad \text{for } a \leq 0.$$

For any vertex $z \in V(T') \setminus \{u_1, u_{d-1}\}$, we have $ecc_{T'}(z) \geq ecc_{T''}(z)$ and $D_{T'}(z) \geq D_{T''}(z)$, therefore $[ecc_{T'}(z)]^a \leq [ecc_{T''}(z)]^a$ for $a \leq 0$ and $[D_{T'}(z)]^b \leq [D_{T''}(z)]^b$ for $b < 0$, so $[ecc_{T'}(z)]^a[D_{T'}(z)]^b \leq [ecc_{T''}(z)]^a[D_{T''}(z)]^b$. Thus,

$$\begin{aligned}
&EDS_{a,b}(T') - EDS_{a,b}(T'') \\
&\leq [ecc_{T'}(u_1)]^a [D_{T'}(u_1)]^b - [ecc_{T''}(u_1)]^a [D_{T''}(u_1)]^b + [ecc_{T'}(u_{d-1})]^a [D_{T'}(u_{d-1})]^b - [ecc_{T''}(v_{u-1})]^a [D_{T''}(u_{d-1})]^b \\
&= (d-1)^a [D_{T'}(u_1)]^b - [ecc_{T''}(u_1)]^a [D_{T'}(u_1) - 2p]^b + (d-1)^a ([D_{T'}(u_{d-1})]^b - [D_{T'}(u_{d-1}) + 2p]^b) \\
&\leq (d-1)^a ([D_{T'}(u_1)]^b - [D_{T'}(u_1) - 2p]^b) + (d-1)^a ([D_{T'}(u_{d-1})]^b - [D_{T'}(u_{d-1}) + 2p]^b) \\
&< 0,
\end{aligned}$$

because for $b < 0$, from Lemma 2.1, we obtain

$$[D_{T'}(u_1)]^b - [D_{T'}(u_1) - 2p]^b < [D_{T'}(u_{d-1}) + 2p]^b - [D_{T'}(u_{d-1})]^b.$$

Therefore $EDS_{a,b}(T') < EDS_{a,b}(T'')$. Hence T' does not have the maximum $EDS_{a,b}$. We have a contradiction. □

The proofs of Theorems 2.3, 2.4 and 2.5 use the next lemma which was proved in [20].

Lemma 2.2. *Let G be a connected graph with two non-adjacent vertices u and v. For $a \geq 0$ and $b > 0$, we have*

$$EDS_{a,b}(G+uv) < EDS_{a,b}(G).$$

For $a \leq 0$ and $b < 0$, we have

$$EDS_{a,b}(G+uv) > EDS_{a,b}(G).$$

Any graph has the matching number ν at most $\lfloor \frac{n}{2} \rfloor$. Stars are the unique connected bipartite graphs with matching number 1. Thus, let us consider bipartite graphs for $2 \leq \nu \leq \lfloor \frac{n}{2} \rfloor$. For $a = b = 1$, Theorem 2.3 was presented in [15].

Theorem 2.3. *Let $a \geq 0$ and $0 < b \leq 1$. For a connected bipartite graph G of order n and matching number ν, where $2 \leq \nu \leq \lfloor \frac{n}{2} \rfloor$, we have*

$$EDS_{a,b}(G) \geq \nu 2^a (n+\nu-2)^b + (n-\nu)2^a(2n-\nu-2)^b,$$

with equality if and only if $G \cong K_{\nu,n-\nu}$.

Proof. Among bipartite graphs of order n and matching number ν, let us denote a graph with the minimum $EDS_{a,b}$ by G'. Without loss of generality, suppose that $|U_1| \leq |U_2|$, where U_1 and U_2 are the partite sets of G'. Let us show by contradiction that $G' \cong K_{\nu,n-\nu}$.

Suppose that $G' \ncong K_{\nu,n-\nu}$. Clearly, $|U_1| \geq \nu$ (otherwise if $|U_1| \leq \nu - 1$, the matching number of G' is at most $\nu - 1$). Note that G' cannot be a subgraph of $K_{\nu,n-\nu}$, (if G' would be a subgraph of $K_{\nu,n-\nu}$, by Lemma 2.2, we get $EDS_{a,b}(G') > EDS_{a,b}(K_{\nu,n-\nu})$). Therefore $\nu < |U_1| \leq |U_2|$.

Let us denote any matching in G' having ν edges by M'. For $j = 1, 2$, let U_j^ν be the subset of U_j containing ν vertices incident with the edges in M'. We have $|U_j| = \nu + l_j$ with $l_j > 0$ (and $2\nu + l_1 + l_2 = n$). Obviously, a vertex $u_1 \in U_1 \setminus U_1^\nu$ is not adjacent to a vertex $u_2 \in U_2 \setminus U_2^\nu$, otherwise there would be the matching $M' \cup \{u_1u_2\}$ in G' with $\nu + 1$ edges.

Let us define the graph H' having the same vertices as G', and containing all the edges between U_1^ν and U_2^ν, between U_1^ν and $U_2 \setminus U_2^\nu$, and between $U_1 \setminus U_1^\nu$ and U_2^ν. The graph G' is a subgraph of H', thus from Lemma 2.2, we obtain $EDS_{a,b}(H') < EDS_{a,b}(G')$. The matching number of H' is at least $\nu + 1$.

Now we construct a graph H'' from H' by deleting all the edges between $U_1 \setminus U_1^\nu$ and U_2^ν, and by adding all the edges between $U_1 \setminus U_1^\nu$ and U_1^ν. The graph H'' is bipartite, having the matching number ν and H'' is a subgraph of $K_{\nu,n-\nu}$. Thus, by Lemma 2.2, we have $EDS_{a,b}(K_{\nu,n-\nu}) < EDS_{a,b}(H'')$.

Let us compare the $EDS_{a,b}$ of the graphs H' and H''. For any $u_1 \in U_1^\nu$ and any $u_2 \in U_2^\nu$, we have $ecc_{H'}(u_1) = ecc_{H''}(u_1) = ecc_{H'}(u_2) = ecc_{H''}(u_2) = 2$ and

$$\begin{aligned}
D_{H'}(u_1) &= \nu + l_2 + 2l_1 + 2(\nu-1) = 3\nu + 2l_1 + l_2 - 2, \\
D_{H''}(u_1) &= \nu + l_2 + l_1 + 2(\nu-1) = 3\nu + l_1 + l_2 - 2, \\
D_{H'}(u_2) &= \nu + l_1 + 2l_2 + 2(\nu-1) = 3\nu + l_1 + 2l_2 - 2, \\
D_{H''}(u_2) &= \nu + 2l_1 + 2l_2 + 2(\nu-1) = 3\nu + 2l_1 + 2l_2 - 2,
\end{aligned}$$

so

$$D_{H'}(u_1) = D_{H''}(u_1) + l_1,\; D_{H'}(u_2) = D_{H''}(u_1) + l_2,\; D_{H''}(u_2) = D_{H''}(u_1) + l_1 + l_2.$$

For $u_1' \in U_1 \setminus U_1^\nu$, we have $ecc_{H'}(u_1') = 3$ and $ecc_{H''}(u_1') = 2$ and

$$\begin{aligned}
D_{H'}(u_1') &= \nu + 2\nu + 3l_2 + 2(l_1-1) = 3\nu + 3l_2 + 2l_1 - 2, \\
D_{H''}(u_1') &= \nu + 2\nu + 2l_2 + 2(l_1-1) = 3\nu + 2l_2 + 2l_1 - 2.
\end{aligned}$$

For $u_2' \in U_2 \setminus U_2^\nu$, we have $ecc_{H'}(u_2') = 3$ and $ecc_{H''}(u_2') = 2$ and

$$\begin{aligned}
D_{H'}(u_2') &= \nu + 2\nu + 3l_1 + 2(l_2-1) = 3\nu + 3l_1 + 2l_2 - 2, \\
D_{H''}(u_2') &= \nu + 2\nu + 2l_1 + 2(l_2-1) = 3\nu + 2l_1 + 2l_2 - 2.
\end{aligned}$$

So, for $j = 1, 2$, we have $D_{H'}(u_j') > D_{H''}(u_j')$ and $ecc_{H'}(u_j') > ecc_{H''}(u_j')$, thus

$$[ecc_{H'}(u_j')]^a [D_{H'}(u_j')]^b > [ecc_{H''}(u_j')]^a [D_{H''}(u_j')]^b$$

for $a \geq 0$ and $b > 0$.

Consequently,

$$
\begin{aligned}
&EDS_{a,b}(H') - EDS_{a,b}(H'')\\
&= \sum_{u_1 \in U_1^\nu} ([ecc_{H'}(u_1)]^a[D_{H'}(u_1)]^b - [ecc_{H''}(u_1)]^a[D_{H''}(u_1)]^b) + \sum_{u_2 \in U_2^\nu} ([ecc_{H'}(u_2)]^a[D_{H'}(u_2)]^b - [ecc_{H''}(u_2)]^a[D_{H''}(u_2)]^b)\\
&\quad + \sum_{u_1' \in U_1 \setminus U_1^\nu} ([ecc_{H'}(u_1')]^a[D_{H'}(u_1')]^b - [ecc_{H''}(u_1')]^a[D_{H''}(u_1')]^b) + \sum_{u_2' \in U_2 \setminus U_2^\nu} ([ecc_{H'}(u_2')]^a[D_{H'}(u_2')]^b - [ecc_{H''}(u_2')]^a[D_{H''}(u_2')]^b)\\
&> \sum_{u_1 \in U_1^\nu} ([ecc_{H'}(u_1)]^a[D_{H'}(u_1)]^b - [ecc_{H''}(u_1)]^a[D_{H''}(u_1)]^b) + \sum_{u_2 \in U_2^\nu} ([ecc_{H'}(u_2)]^a[D_{H'}(u_2)]^b - [ecc_{H''}(u_2)]^a[D_{H''}(u_2)]^b)\\
&= \nu 2^a([D_{H''}(u_1) + l_1]^b - [D_{H''}(u_1)]^b + [D_{H''}(u_1) + l_2]^b - [D_{H''}(u_1) + l_1 + l_2]^b)\\
&\geq 0,
\end{aligned}
$$

since

$$[D_{H''}(u_1) + l_1]^b - [D_{H''}(u_1)]^b + [D_{H''}(u_1) + l_2]^b - [D_{H''}(u_1) + l_1 + l_2]^b = 0$$

if $b = 1$, and by Lemma 2.1,

$$[D_{H''}(u_1) + l_1]^b - [D_{H''}(u_1)]^b > [D_{H''}(u_1) + l_1 + l_2]^b - [D_{H''}(u_1) + l_2]^b.$$

for $0 < b < 1$. We obtain $EDS_{a,b}(H'') < EDS_{a,b}(H')$, so

$$EDS_{a,b}(K_{\nu,n-\nu}) < EDS_{a,b}(H'') < EDS_{a,b}(H') < EDS_{a,b}(G').$$

We have a contradiction. Thus, G' is $K_{\nu,n-\nu}$. For ν vertices of $K_{\nu,n-\nu}$, say v_i, $i = 1, 2, \dots, \nu$, we have $D_{K_{\nu,n-\nu}}(v_i) = n - \nu + 2(\nu - 1) = n + \nu - 2$, and for the other $n - \nu$ vertices, say v_j', $j = 1, 2, \dots, n - \nu$, we get $D_{K_{\nu,n-\nu}}(v_i) = \nu + 2(n - \nu - 1) = 2n - \nu - 2$. Since the eccentricity of every vertex is 2, we obtain

$$EDS_{a,b}(K_{\nu,n-\nu}) = \nu 2^a(n + \nu - 2)^b + (n - \nu)2^a(2n - \nu - 2)^b.$$

□

Theorem 2.4. *Let $a \leq 0$ and $b < 0$. For a connected bipartite graph G of order n and matching number ν, where $2 \leq \nu \leq \lfloor \frac{n}{2} \rfloor$, we have*

$$EDS_{a,b}(G) \leq \nu 2^a(n + \nu - 2)^b + (n - \nu)2^a(2n - \nu - 2)^b,$$

with equality if and only if $G \cong K_{\nu,n-\nu}$.

Proof. Only those parts of the proof are presented which differ from the proof of Theorem 2.3. Among bipartite graphs of order n and matching number ν, let us denote a graph with the maximum $EDS_{a,b}$ by G'. The graph G' is a subgraph of H', thus by Lemma 2.2, we get $EDS_{a,b}(H') > EDS_{a,b}(G')$. Similarly, by Lemma 2.2, we have $EDS_{a,b}(K_{\nu,n-\nu}) > EDS_{a,b}(H'')$. For $j = 1, 2$, we have $D_{H'}(u_j') > D_{H''}(u_j')$ and $ecc_{H'}(u_j') > ecc_{H''}(u_j')$, thus for $a \leq 0$ and $b < 0$, we obtain

$$[ecc_{H'}(u_j')]^a \leq [ecc_{H''}(u_j')]^a \quad \text{and} \quad [D_{H'}(u_j')]^b < [D_{H''}(u_j')]^b.$$

Then

$$[ecc_{H'}(u_j')]^a[D_{H'}(u_j')]^b < [ecc_{H''}(u_j')]^a[D_{H''}(u_j')]^b.$$

Consequently,

$$
\begin{aligned}
&EDS_{a,b}(H') - EDS_{a,b}(H'')\\
&< \sum_{u_1 \in U_1^\nu} ([ecc_{H'}(u_1)]^a[D_{H'}(u_1)]^b - [ecc_{H''}(u_1)]^a[D_{H''}(u_1)]^b) + \sum_{u_2 \in U_2^\nu} ([ecc_{H'}(u_2)]^a[D_{H'}(u_2)]^b - [ecc_{H''}(u_2)]^a[D_{H''}(u_2)]^b)\\
&= \nu 2^a([D_{H''}(u_1) + l_1]^b - [D_{H''}(u_1)]^b + [D_{H''}(u_1) + l_2]^b - [D_{H''}(u_1) + l_1 + l_2]^b)\\
&< 0,
\end{aligned}
$$

since by Lemma 2.1,

$$[D_{H''}(u_1) + l_1]^b - [D_{H''}(u_1)]^b < [D_{H''}(u_1) + l_1 + l_2]^b - [D_{H''}(u_1) + l_2]^b \quad \text{for } b < 0.$$

We obtain $EDS_{a,b}(H'') > EDS_{a,b}(H')$, so

$$EDS_{a,b}(K_{\nu,n-\nu}) > EDS_{a,b}(H'') > EDS_{a,b}(H') > EDS_{a,b}(G').$$

□

Let us denote the independence number, vertex cover number and edge cover number by α, β and β', respectively. It is known that for a graph of order n,

$$\alpha + \beta = n;$$

see [22]. If G has no isolated vertices, then

$$\nu + \beta' = n.$$

If G is bipartite with no isolated vertices,

$$\alpha = \beta', \text{ thus } \nu = \beta;$$

see [22]. So, from Theorems 2.3 and 2.4, we obtain the following corollary.

Corollary 2.1. *For a connected bipartite graph G of order n and vertex cover number β, where $2 \le \beta \le \lfloor \frac{n}{2} \rfloor$, we have*

$$EDS_{a,b}(G) \ge \beta 2^a (n + \beta - 2)^b + (n - \beta) 2^a (2n - \beta - 2)^b$$

if $a \ge 0$ and $0 < b \le 1$, and

$$EDS_{a,b}(G) \le \beta 2^a (n + \beta - 2)^b + (n - \beta) 2^a (2n - \beta - 2)^b$$

if $a \le 0$ and $b < 0$. The equalities hold if and only if $G \cong K_{\beta, n-\beta}$.

In Theorems 2.3 and 2.4, $2 \le \nu \le \lfloor \frac{n}{2} \rfloor$, thus

$$\left\lceil \frac{n}{2} \right\rceil \le \beta' \le n - 2.$$

Since $\nu + \beta' = n$, Theorem 2.3 says that if $a \ge 0$ and $0 < b \le 1$, then for a connected bipartite graph G of order n and matching number $n - \beta'$, we get

$$EDS_{a,b}(G) \ge (n - \beta') 2^a (2n - \beta' - 2)^b + \beta' 2^a (n + \beta' - 2)^b.$$

Similarly, an upper bound can be obtained for $a \le 0$ and $b < 0$. Thus, we obtain Corollary 2.2.

Corollary 2.2. *For a connected bipartite graph G of order n and edge cover number/independence number β', where $\lceil \frac{n}{2} \rceil \le \beta' \le n - 2$, we have*

$$EDS_{a,b}(G) \ge \beta' 2^a (n + \beta' - 2)^b + (n - \beta') 2^a (2n - \beta' - 2)^b$$

if $a \ge 0$ and $0 < b \le 1$, and

$$EDS_{a,b}(G) \le \beta' 2^a (n + \beta' - 2)^b + (n - \beta') 2^a (2n - \beta' - 2)^b$$

if $a \le 0$ and $b < 0$. The equalities hold if and only if $G \cong K_{\beta', n-\beta'}$.

The smallest number of colors needed to color the vertices of a graph G such that no two adjacent vertices have the same color is the chromatic number of G. Clearly, any nonempty graph has chromatic number at least 2. Graphs with given order and chromatic number are investigated in Theorem 2.5. For $a = b = 1$, Theorem 2.5 was presented in [14].

Theorem 2.5. *Let $a \ge 0$ and $b \ge 1$. For any connected graph G with n vertices and chromatic number χ, where each of the χ colors is used for at least two vertices, we have*

$$EDS_{a,b}(G) \ge EDS_{a,b}(K_{p_1, p_2, \dots, p_\chi}),$$

with equality if only if $G \cong K_{p_1, p_2, \dots, p_\chi}$, where $|p_j - p_l| \le 1$ for any $1 \le j < l \le \chi$.

Proof. For the considered set of graphs, let us denote a graph with the minimum $EDS_{a,b}$ by G'. Since the graph G' does not contain edges between vertices colored by the same color, G' is a χ-partite graph. Each of the χ colors is used for at least two vertices, thus each partite set contains at least two vertices. By Lemma 2.2, any two vertices from different partite sets are adjacent, therefore $G' \cong K_{p_1, p_2, \dots, p_\chi}$, where $p_j \ge 2$, $j = 1, 2, \dots, \chi$. Let us show that $|p_j - p_l| \le 1$ for any $1 \le j < l \le \chi$.

Suppose to the contrary that there exist j and l, where $1 \le j < l \le \chi$, such that $|p_j - p_l| \ge 2$. We can suppose that $p_1 \ge p_2 + 2$. We compare $EDS_{a,b}(G')$ and $EDS_{a,b}(G'')$, where $G' = K_{p_1, p_2, \dots, p_\chi}$ and $G'' = K_{p_1 - 1, p_2 + 1, \dots, p_\chi}$. Every vertex in G' and G'' has eccentricity 2. For any u_1' from the first partite set and u_2' from the second partite set of G', we have

$$D_{G'}(u_1') = (n - p_1) + 2(p_1 - 1) = n + p_1 - 2$$

and

$$D_{G'}(u_2') = (n - p_2) + 2(p_2 - 1) = n + p_2 - 2.$$

For any u_1'' from the first partite set and u_2'' from the second partite set of G'', we get

$$D_{G''}(u_1'') = [n-(p_1-1)] + 2(p_1-2) = n+p_1-3 \quad \text{and} \quad D_{G''}(u_2'') = [n-(p_2+1)] + 2p_2 = n+p_2-1.$$

For any other vertex x, we get $D_{G'}(x) = D_{G''}(x)$. Since $p_1 - 1 \geq p_2 + 1$, we obtain

$$\begin{aligned}
&EDS_{a,b}(G') - EDS_{a,b}(G'') \\
&= p_1 2^a(n+p_1-2)^b + p_2 2^a(n+p_2-2)^b - (p_1-1)2^a(n+p_1-3)^b - (p_2+1)2^a(n+p_2-1)^b \\
&= (p_1-1)2^a[(n+p_1-2)^b - (n+p_1-3)^b] - (p_2+1)2^a[(n+p_2-1)^b - (n+p_2-2)^b] + 2^a(n+p_1-2)^b - 2^a(n+p_2-2)^b \\
&> (p_1-1)2^a[(n+p_1-2)^b - (n+p_1-3)^b] - (p_2+1)2^a[(n+p_2-1)^b - (n+p_2-2)^b] \\
&\geq (p_2+1)2^a[(n+p_1-2)^b - (n+p_1-3)^b] - (p_2+1)2^a[(n+p_2-1)^b - (n+p_2-2)^b]
\end{aligned}$$

which equals 0 if $b = 1$, and it is positive for $b > 1$, because from Lemma 2.1,

$$(n+p_1-2)^b - (n+p_1-3)^b > (n+p_2-1)^b - (n+p_2-2)^b.$$

Therefore $EDS_{a,b}(G') - EDS_{a,b}(G'') > 0$ and $EDS_{a,b}(G') > EDS_{a,b}(G'')$. We have a contradiction. Thus $|p_j - p_l| \leq 1$ for any $1 \leq j < l \leq \chi$, hence $G' \cong K_{p_1,p_2,\ldots,p_\chi}$. □

In Theorems 2.6 and 2.7, we obtain bounds on $EDS_{a,b}$ for $b = 1$. For $a = 1$, the bound given in Theorem 2.6 was presented in [13].

Theorem 2.6. *Let $a \geq 0$. For any connected graph G of order n and size m not containing a vertex of degree $n-1$,*

$$EDS_{a,1}(G) \geq [n(n-1) - m]2^{a+1},$$

with equality if and only if the diameter of G is 2.

Proof. Since no vertex of G is adjacent to all the other vertices, we have $ecc_G(u) \geq 2$ for every $u \in V(G)$. Thus,

$$[ecc_G(u)]^a \geq 2^a.$$

For every $u \in V(G)$, $D_G(u) \geq deg_G(u) + 2[n-1-deg_G(u)] = 2(n-1) - deg_G(u)$ since the distance between u and any of the $n-1-deg_G(u)$ vertices not adjacent to u is at least 2. Since

$$\sum_{u \in V(G)} deg_G(u) = 2m,$$

we obtain

$$EDS_{a,1}(G) \geq \sum_{u \in V(G)} 2^a[2(n-1) - deg_G(u)] = 2^{a+1}[n(n-1) - m],$$

with equality if and only if the diameter of G is 2. □

For any graph G, we have

$$n-1 \leq m \leq \binom{n}{2} = \frac{n(n-1)}{2}.$$

We show that there exist graphs with a small size (size close to $n-1$) as well as some graphs with a large size (size close to $\frac{n(n-1)}{2}$) which attain the bound presented in Theorem 2.6.

Note that C_n is the cycle of order n and $\frac{n}{2}K_2$ (for even n) is the set of $\frac{n}{2}$ independent edges. The graphs $K_{c,n-c}$ for $2 \leq c \leq \lfloor \frac{n}{2} \rfloor$, $\overline{\frac{n}{2}K_2}$ for even n and $\overline{C_n}$ have diameter 2 and they do not contain a vertex of degree $n-1$, therefore they belong to the extremal graphs for Theorem 2.6. The graphs $K_{c,n-c}$ have a small size if c is small. We have $|E(K_{c,n-c})| = c(n-c)$. Thus, for $c = 2$, $|E(K_{2,n-2})| = 2n-4$. The graphs $\overline{\frac{n}{2}K_2}$ for even n and $\overline{C_n}$ have a large size. We have

$$\left| E\left(\overline{\frac{n}{2}K_2}\right) \right| = \frac{n(n-2)}{2} \quad \text{and} \quad \left| E\left(\overline{C_n}\right) \right| = \frac{n(n-3)}{2}.$$

For $a = 1$, the bound given in Theorem 2.7 was presented in [12, 13].

Theorem 2.7. *Let $a \geq 0$. If G and $\overline{G}$ are connected graphs of order $n \geq 5$, then*

$$EDS_{a,1}(G) + EDS_{a,1}(\overline{G}) \geq 3n(n-1)2^a,$$

with equality if and only if $d(G) = d(\overline{G}) = 2$.

Proof. Since G is connected, $\overline{G}$ does not contain vertices of degree $n-1$. Analogously, since $\overline{G}$ is connected, G does not contain vertices of degree $n-1$. So, from Theorem 2.6, we have

$$EDS_{a,1}(G) \geq [n(n-1)-m]2^{a+1} \text{ and } EDS_{a,1}(\overline{G}) \geq [n(n-1)-\overline{m}]2^{a+1},$$

where m and $\overline{m}$ are the sizes of G and $\overline{G}$, respectively. By Theorem 2.6, the equalities hold if and only if G and $\overline{G}$ have diameter 2, respectively. Since $m+\overline{m}=\binom{n}{2}=\frac{n(n-1)}{2}$, we obtain

$$EDS_{a,1}(G)+EDS_{a,1}(\overline{G}) \geq 2n(n-1)2^{a+1}-(m+\overline{m})2^{a+1}=4n(n-1)2^a-n(n-1)2^a=3n(n-1)2^a,$$

with equality if and only if $d(G)=d(\overline{G})=2$. □

We show that there exist graphs G such that $d(G)=d(\overline{G})=2$ for every $n\geq 5$, which implies that the bound presented in Theorem 2.7 is sharp for every $n \geq 5$.

Let G_1, G_2, G_3, G_4, G_5 be any (possibly disconnected) graphs of orders n_1, n_2, $n_3, n_4, n_5 \geq 1$, respectively. Let G be the graph obtained from G_1, G_2, G_3, G_4, G_5 by joining any vertex of G_1 and any vertex of G_5, and by joining any vertex of G_i and any vertex of G_{i+1} for $i=1,2,3,4$.

Note that $\overline{G}$ is the graph obtained from $\overline{G_1}$, $\overline{G_2}$, $\overline{G_3}$, $\overline{G_4}$, $\overline{G_5}$ by joining any vertex of G_i and any vertex of G_{i+2} for $i=1,2,3$, and by joining any vertex of G_i and any vertex of G_{i+3} for $i=1,2$. Clearly, $|V(G)|=|V(\overline{G})|=n_1+n_2+n_3+n_4+n_5$.

3. Open problems

Let us present several problems which are open for further research.

Problem 3.1. *Find lower and upper bounds on $EDS_{a,b}$ for trees with given order and diameter.*

Problem 3.2. *Find a graph with the largest $EDS_{a,b}$ among graphs/trees of given order for positive a and b.*

Problem 3.3. *Find bounds on $EDS_{a,b}$ for planar graphs and outerplanar graphs of given order.*

Problem 3.4. *Find bounds on $EDS_{a,b}$ for graphs with given order and number of bridges.*

We suggest to study Problems 3.1, 3.2, 3.3, and 3.4 for general a and b, or for one general parameter and the other parameter being 1.

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