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Research Article

Optimal radio labelings of graphs

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Abstract

Let $\mathbb{N}$ be the set of positive integers. A radio labeling of a graph G is a mapping $\varphi : V(G) \to \mathbb{N} \cup \{0\}$ such that the inequality $|\varphi(u) - \varphi(v)| \geq \mathrm{diam}(G) + 1 - d(u,v)$ holds for every pair of distinct vertices u, v of G, where $\mathrm{diam}(G)$ and $d(u,v)$ are the diameter of G and distance between u and v in G, respectively. The radio number $\mathrm{rn}(G)$ of G is the smallest number k such that G has a radio labeling φ with $\max\{\varphi(v) : v \in V(G)\} = k$. Das et al. [*Discrete Math.* **340** (2017) 855–861] gave a technique to find a lower bound for the radio number of graphs. In [*Algorithms and Discrete Applied Mathematics: CALDAM 2019*, Lecture Notes in Computer Science **11394**, Springer, Cham, 2019, 161–173], Bantva modified this technique for finding an improved lower bound on the radio number of graphs and gave a necessary and sufficient condition to achieve the improved lower bound. In this paper, one more useful necessary and sufficient condition to achieve the improved lower bound for the radio number of graphs is given. Using this result, the radio number of the Cartesian product of the path and wheel graphs is determined.

Keywords: radio labeling; radio number; wheel graph; Cartesian product of graphs.

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1. Introduction

The *channel assignment problem* is the problem to assign channels to each TV or radio transmitters such that the interference constraints are satisfied and the use of spectrum is minimized. The problem was first introduced by Hale [11] in 1980. The interference between transmitters is closely related to geographic location of the transmitters. The closer the transmitters are, the higher the interference is and vice-versa. Hence, the frequency difference between two radio channels assigned to radio transmitters is in the inverse proportion to the distance between two transmitters. Initially only two level interference, namely *high* and *low*, was considered and accordingly, two transmitters are called *very close* and *close*, respectively. In a private communication with Griggs during 1988, Robert proposed a variation of the channel assignment problem in which close transmitters must receive different channels and very close transmitters must receive channels that are at least two apart. This problem is studied by mathematicians using graphs labeling approach.

In a graph, the transmitters are represented by vertices and two vertices are adjacent if two transmitters are very close and distance two apart if they are close. The problem of assignment of channels to transmitters is associated with graph labeling problem. Motivated through this problem, Griggs and Yeh introduced $L(2,1)$-labeling (or distance two labeling) in [9] as follows: An $L(2,1)$-labeling of a graph $G = (V(G), E(G))$ is a function φ from the vertex set $V(G)$ to the set of non-negative integers such that $|\varphi(u) - \varphi(v)| \geq 2$ if $d(u,v) = 1$ and $|\varphi(u) - \varphi(v)| \geq 1$ if $d(u,v) = 2$, where $d(u,v)$ is the distance between u and v in G. The *span of* φ is defined as $\mathrm{span}(\varphi) = \max\{|\varphi(u) - \varphi(v)| : u, v \in V(G)\}$. The λ-number, denoted by $\lambda(G)$, is defined as the minimum span over all $L(2,1)$-labelings of G. The $L(2,1)$-labeling and other distance two labeling problems have been studied by many researchers in the past two and half decades; for example, see the survey articles [4, 20].

In [5, 6], Chartrand et al. extended the constraint on distance from two to the largest possible distance and introduced the concept of radio labeling as follows.

Definition 1.1. *A* radio labeling *of a graph G is a mapping $\varphi : V(G) \to \mathbb{N} \cup \{0\}$ ($\mathbb{N}$ is the set of positive integers) such that the following is satisfied for every pair of distinct vertices $u, v \in V(G)$,*

$$|\varphi(u) - \varphi(v)| \geq \mathrm{diam}(G) + 1 - d(u,v). \tag{1}$$

The assigned integer $\varphi(u)$ is called the label *of u under φ and the* span *of φ is defined as*

$$\mathrm{span}(\varphi) = \max\left\{|\varphi(u) - \varphi(v)| : u, v \in V(G)\right\}.$$

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The radio number *of G, denoted by* $\mathrm{rn}(G)$*, is defined as*

$$\mathrm{rn}(G) := \min_{\varphi}\{\mathrm{span}(\varphi)\}$$

with minimum taken over all radio labelings φ of G. A radio labeling φ is optimal *if* $\mathrm{span}(\varphi) = \mathrm{rn}(G)$.

It is clear that an optimal radio labeling φ always assign 0 to some vertex and hence the span of φ is the maximum integer assigned by φ. A radio labeling is a one-to-one integral function from $V(G)$ to the set of non-negative integers. Therefore, any radio labeling φ induces an ordering $O_\varphi(V(G)) := (x_0, x_1, \ldots, x_{p-1})$ of $V(G)$ such that $0 = \varphi(x_0) < \varphi(x_1) < \ldots < \varphi(x_{p-1}) = \mathrm{span}(\varphi)$, where $p = |V(G)|$. It is clear that if φ is an optimal radio labeling then $\mathrm{span}(\varphi) \leq \mathrm{span}(\psi)$ for any other radio labeling ψ of G.

A radio labeling problem is recognized as one of the tough graph labeling problems. In [5,6], Chartrand et al. gave an upper bound for the radio number of paths and cycles. Liu and Zhu determined the exact radio number for paths and cycles in [15]. Even determining the radio number for basic graph families like paths and cycles was challenging. In [16–18], Vaidya and Bantva determined the radio number for the total graph of paths, strong product $P_2 \boxtimes P_n$ and linear cacti. The radio number of trees remain the focus of many researchers in recent years. In [10], Halász and Tuza determined the radio number of level-wise regular trees. In [13], Li et al. determined the radio number of complete m-ary trees. In [14], Liu gave a lower bound for the radio number of trees and, a necessary and sufficient condition to achieve the lower bound; the author presented a class of trees, namely spiders, achieving this lower bound. In [3], Bantva et al. gave a different necessary and sufficient condition to achieve this lower bound and presented banana trees, firecrackers trees and a special class of caterpillars achieving this lower bound. Recently, Bantva and Liu gave a lower bound for the radio number of block graphs and three necessary and sufficient condition to achieve the lower bound in [2]. They also discussed the radio number of line graph of trees and block graphs. Liu et al. [7] also studied the radio k-labeling of trees. In [8], Das et al. gave a technique to find a lower bound for the radio number of graphs. In [1], Bantva improved this technique to find a lower bound for the radio number of graphs and gave a necessary and sufficient condition to achieve the improved lower bound. Using these results, the author determined the radio number of the Cartesian product of paths and Peterson graph.

In this paper, one more useful necessary and sufficient condition to achieve the improved lower bound for the radio number of graphs given in [1] is established. Some subgraphs of a given graph G are characterized such that if the radio number of G achieves the lower bound given in [1] then these subgraphs also achieve the lower bound. Using these results, the radio numbers of the Cartesian product of the path graphs with wheel, star and friendship graphs are determined.

2. Preliminaries

The book [19] is followed for standard graph-theoretic terms and notation. Only simple finite connected graphs are considered throughout this paper. The distance $d_G(u, v)$ between two vertices u and v is the least length of a path joining u and v in a graph G. The suffix is dropped whenever the graph G is clear in the context. The diameter of a graph G, denoted by $\mathrm{diam}(G)$, is $\max\{d_G(u, v) : u, v \in V(G)\}$. The neighborhood of $v \in G$, written as $N(v)$, is the set of vertices adjacent to v. Let $S \subseteq V(G)$. Define $N(S) = \{u \in V(G) \setminus S : u \text{ is adjacent to } v \in S\}$. The subgraph induced by S, denoted by $G(S)$, is a subgraph of G whose vertex set is S and edge set is $E(G(S)) = \{e = (u, v) \in E(G) : u, v \in S\}$. For any $u \in V(G)$, let $d_G(u, S) = \min\{d_G(u, v) : v \in S\}$ and $\mathrm{diam}(S) = \max\{d_G(u, v) : u, v \in S\}$. It is clear that if $|S| = 1$ then $\mathrm{diam}(S) = 0$.

Let H be an induced connected subgraph of G with $\mathrm{diam}(H) = k$. Define layers L_i of graph G with respect to subgraph H as follows: Set $L_0 = V(H)$ and $L_1 = N(L_0)$. Recursively define $L_{i+1} = N(L_i)$ for $1 \leq i \leq h - 1$, where

$$h = \max\{d_G(u, H) : u \in V(G)\},$$

which is known as the maximal level in a graph G. Since G is a connected graph, $L_i \neq \emptyset$ for $0 \leq i \leq h$. Define the total distance of layers of graph G, denoted by $L(G)$, as

$$L(G) := \sum_{i=1}^{h} |L_i| i.$$

For a graph G, define

$$\delta(G) = \begin{cases} 1, & \text{if } |L_0| = 1; \\ 0, & \text{if } |L_0| \geq 2. \end{cases}$$

Let G be any connected graph then for any $u, v \in V(G)$, note that the distance between u and v in a graph G satisfies the following inequality:

$$d(u, v) \leq d(u, L_0) + d(v, L_0) + \mathrm{diam}(L_0). \tag{2}$$

In [8], Das et al. gave a technique to find a lower bound for the radio number of graphs. In [1], Bantva improved this technique and gave a lower bound for the radio number of graphs which is given in the following theorem.

Theorem 2.1. [1] *Let G be a simple connected graph of order p, diameter d and $L_0 \subseteq V(G)$. Take $k = \mathrm{diam}(L_0)$ and $\delta = \delta(G)$. Then*

$$\mathrm{rn}(G) \geq (p-1)(d-k+1) + \delta - 2L(G). \tag{3}$$

Although, both the lower bounds given in [8] and [1] seem to be identical in notation but the difference lies in fixing the L_0. In [8], Das et al. set a vertex or a clique of graph G as L_0 while Bantva set all vertices of an induced subgraph H of G as L_0 with the property that two non-adjacent vertices of $V(H)$ have distance equal to $\mathrm{diam}(L_0)$. The readers may notice that this improved technique gives a better lower bound for the radio number of graphs, which is sharp for some classes of graph. The author of [1] presented one such class of graphs, which consists of the Cartesian product of the path graph and Peterson graph. In this paper, the condition to fix L_0 is further relaxed as follows. Set $L_0 = V(H)$, where H is a connected induced subgraph of G with the property that the vertices of G can be ordered as $x_0, x_1, \ldots, x_{p-1}$ such that $d(x_i, x_{i+1}) = d(x_i, L_0) + d(x_{i+1}, L_0) + \mathrm{diam}(L_0)$ for $0 \leq i \leq p-2$.

Bantva [1] also gave a necessary and sufficient condition (given in the next theorem) to achieve the lower bound (3) for the radio number of graphs.

Theorem 2.2. [1] *Let G be a simple connected graph of order p, diameter d and L_0 is as described earlier. Denote $k = \mathrm{diam}(L_0)$ and $\delta = \delta(G)$. Then*

$$\mathrm{rn}(G) = (p-1)(d-k+1) + \delta - 2L(G) \tag{4}$$

holds if and only if there exists a radio labeling φ with $0 = \varphi(x_0) < \varphi(x_1) < \ldots < \varphi(x_{p-1}) = \mathrm{span}(\varphi) = \mathrm{rn}(G)$ such that all the following hold for $0 \leq i \leq p-1:$

(a) $d(x_i, x_{i+1}) = d(x_i, L_0) + d(x_{i+1}, L_0) + k$,

(b) $x_0, x_{p-1} \in L_0$ *if* $|L_0| \geq 2$ *and* $x_0 \in L_0$, $x_{p-1} \in L_1$ *if* $|L_0| = 1$,

(c) $\varphi(x_0) = 0$ *and* $\varphi(x_{i+1}) = \varphi(x_i) + d + 1 - d(x_i, L_0) - d(x_{i+1}, L_0) - k$.

3. Main result

In this section, one more useful necessary and sufficient condition to achieve the improved lower bound for the radio number of graphs given in [1] is established, which rely only on the ordering of vertices of a graph.

Theorem 3.1. *Let G be a simple connected graph of order p, diameter $d \geq 2$ and L_0 is fixed in G as described earlier. Denote $k = \mathrm{diam}(L_0)$ and $\delta = \delta(G)$. Then*

$$\mathrm{rn}(G) = (p-1)(d-k+1) + \delta - 2L(G) \tag{5}$$

holds if and only if there exists an ordering $O(V(G)) := (x_0, x_1, \ldots, x_{p-1})$ of $V(G)$ such that the following conditions are satisfy.

(a) $d(x_0, L_0) + d(x_{p-1}, L_0) = 1$ *if* $|L_0| = 1$ *and* $d(x_0, L_0) + d(x_{p-1}, L_0) = 0$ *if* $|L_0| \geq 2$*;*

(b) *the distance between two vertices x_i and x_j $(0 \leq i < j \leq p-1)$ satisfy*

$$d(x_i, x_j) \geq \sum_{t=i}^{j-1} (d(x_t, L_0) + d(x_{t+1}, L_0) + k - d - 1) + d + 1. \tag{6}$$

Moreover, under the conditions (a) and (b), the mapping φ defined by

$$\varphi(x_0) = 0, \tag{7}$$

$$\varphi(x_{i+1}) = \varphi(x_i) + d + 1 - d(x_i, L_0) - d(x_{i+1}, L_0) - k, \; 0 \leq i \leq p-2 \tag{8}$$

is an optimal radio labeling of G.

Proof. Necessity: Suppose that (5) holds then there exists an optimal radio labeling φ of G which induces an ordering $O_\varphi(V(G)) := (x_0, x_1, \ldots, x_{p-1})$ of $V(G)$ with $0 = \varphi(x_0) < \varphi(x_1) < \ldots < \varphi(x_{p-1}) = \text{span}(\varphi) = \text{rn}(G)$ such that the conditions (a)-(c) of Theorem 2.2 hold. By Theorem 2.2(b), it is clear that $d(x_0, L_0) + d(x_{p-1}, L_0) = 1$ when $|L_0| = 1$ and

$$d(x_0, L_0) + d(x_{p-1}, L_0) = 0 \quad \text{when } |L_0| \geq 2.$$

By Theorem 2.2(c), for any two vertices x_i and x_j $(j > i)$ in ordering $O_\varphi(V(G)) := (x_0, x_1, \ldots, x_{p-1})$ of $V(G)$, we obtain

$$\varphi(x_j) - \varphi(x_i) = \sum_{t=i}^{j-1}(d + 1 - d(x_t, L_0) - d(x_{t+1}, L_0) - k).$$

Note that φ is a radio labeling of G and so $\varphi(x_j) - \varphi(x_i) \geq d + 1 - d(x_i, x_j)$. Substituting this in the above equation, we obtain

$$d(x_i, x_j) \geq \sum_{t=i}^{j-1}(d(x_t, L_0) + d(x_{t+1}, L_0) + k - d - 1) + d + 1. \tag{9}$$

Sufficiency: Suppose that an ordering $O(V(G)) := (x_0, x_1, \ldots, x_{p-1})$ of $V(G)$ satisfies conditions (a)-(b) of hypothesis and φ is defined by (7) and (8). Note that it is enough to prove that φ is a radio labeling with span equal to the right-hand side of (5). Let x_i and x_j $(0 \leq i < j \leq p-1)$ be two arbitrary vertices then by (8) and using (6), we have

$$\begin{aligned}\varphi(x_j) - \varphi(x_i) &= (j-i)(d+1) - \sum_{t=i}^{j-1}(d(x_t, L_0) + d(x_{t+1}, L_0) + k) \\ &\geq d + 1 - d(x_i, x_j)\end{aligned}$$

and hence φ is a radio labeling. The span of φ is given by

$$\begin{aligned}\text{span}(\varphi) &= \varphi(x_{p-1}) - \varphi(x_0) \\ &= \sum_{i=0}^{p-2}(\varphi(x_{i+1}) - \varphi(x_i)) \\ &= \sum_{i=0}^{p-2}(d + 1 - d(x_i, L_0) - d(x_{i+1}, L_0) - k) \\ &= (p-1)(d+1) - \sum_{i=0}^{p-2}(d(x_i, L_0) + d(x_{i+1}, L_0) + k) \\ &= (p-1)(d-k+1) - 2L(G) + d(x_0, L_0) + d(x_{p-1}, L_0) \\ &= (p-1)(d-k+1) + \delta - 2L(G).\end{aligned}$$

Therefore, $\text{rn}(G) \leq (p-1)(d-k+1) + \delta - 2L(G)$. This together with (3) implies (5).

□

A graph with no cycle is called *acyclic* graph. A *forest* is an acyclic graph. A *tree* is a connected acyclic graph. A *spanning subgraph* of a graph G is a subgraph with vertex set $V(G)$. Let H be a connected proper induced subgraph of a graph G. A *spanning subgraph rooted at* H of a graph G is a subgraph G_H of G with vertex set $V(G_H) = V(G)$ and $G_H(V(H)) \cong H$. A *spanning tree rooted at* H of a graph G, denoted by T_H, is a spanning subgraph rooted at H of G such that $T_H \setminus H$ is a forest. A spanning tree T_H rooted at H is called *minimum distance spanning tree rooted at* H if $L(T_H) = L(G)$, denoted by T^m_H.

Observation 3.1. *Let G be a simple connected graph of order p, diameter $d \geq 2$ and L_0 is fixed in G as described earlier. Let $T^m_{L_0}$ be a minimum distance spanning tree rooted at L_0 of G. Then*

(a) $\text{diam}(T^m_{L_0}) = \text{diam}(G)$,

(b) $d_{T^m_{L_0}}(u, L_0) = d_G(u, L_0)$ *for all* $u \in V(T^m_{L_0})$,

(c) $L(T^m_{L_0}) = L(G)$,

(d) $d_{T^m_{L_0}}(u, v) \geq d_G(u, v)$ *for all* $u, v \in V(G)$.

Theorem 3.2. *Let G be a simple connected graph of order p, diameter $d \geq 2$ and L_0 is fixed in G as described earlier. Denote* $\mathrm{diam}(L_0) = k$. *If* $\mathrm{rn}(G)$ *attains a lower bound given in* (3) *then* $\mathrm{rn}(T^m_{L_0})$ *attains a lower bound given in* (3) *and* $\mathrm{rn}(T^m_{L_0}) = \mathrm{rn}(G)$.

Proof. Since $\mathrm{rn}(G)$ attains a lower bound given in (3), there exists an ordering $O(V(G))$:=(x_0, x_1, …, x_{p-1}) of $V(G)$ which satisfies conditions (a)-(b) of Theorem 3.1. Then by Observation 3.1, the same ordering of $V(T^m_{L_0}) = V(G)$ satisfies conditions (a)-(b) of Theorem 3.1. Hence, $\mathrm{rn}(T^m_{L_0})$ attains a lower bound given in (3). Since $V(T^m_{L_0}) = V(G)$, $\mathrm{diam}(T^m_{L_0}) = \mathrm{diam}(G)$ and $L(T^m_{L_0}) = L(G)$, it is clear that $\mathrm{rn}(T^m_{L_0}) = \mathrm{rn}(G)$.. □

Theorem 3.3. *Let G be a simple connected graph of order p, diameter $d \geq 2$ and L_0 is fixed in G as described earlier. Denote* $\mathrm{diam}(L_0) = k$. *Let $G = G_1 \supseteq G_2 \supseteq \ldots \supseteq G_t = T^m_{L_0}$ be a sequence of subgraphs obtained by deleting edges in G to obtain $T^m_{L_0}$. If* $\mathrm{rn}(G)$ *attains a lower bound given in* (3) *then for $1 \leq i \leq t$,* $\mathrm{rn}(G_i)$ *attains a lower bound given in* (3) *and* $\mathrm{rn}(G_i) = \mathrm{rn}(G)$.

Proof. The proof is similar to the proof of Theorem 3.2. □

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. The *Cartesian product* of G and H, denoted by $G \Box H$, is a graph with vertex set $V(G \Box H) = V(G) \times V(H)$, where two vertices (g_1, h_1) and (g_2, h_2) are adjacent if $g_1 = g_2$ and $h_1h_2 \in E(H)$, or $h_1 = h_2$ and $g_1g_2 \in E(G)$. A *path* P_m on m vertices is a tree in which each vertex has degree at most 2. Denote the vertex set of P_m by $V(P_m) = \{u_1, u_2, \ldots, u_m\}$ with $E(P_m) = \{u_iu_{i+1} : 1 \leq i \leq m-1\}$. A *wheel graph* W_n is a graph obtained by joining each vertex of a cycle C_n to a new vertex v_0. Denote the vertex set of W_n by $V(W_n) = \{v_0, v_1, \ldots, v_n\}$ with $E(W_n) = \{v_0v_i, v_0v_n, v_iv_{i+1}, v_1v_n : 1 \leq i \leq n-1\}$. Observe that the diameter of $P_m \Box W_n$ is $m + 1$.

Theorem 3.4. *Let $m \geq 3$ and $n \geq 7$ be any integers. Then*

$$\mathrm{rn}(P_m \Box W_n) = \begin{cases} \frac{1}{2}(m^2n + m^2 + 2m - 2), & \text{if } m \text{ is even;} \\ \frac{1}{2}(m^2n + m^2 + 2m + n - 1), & \text{if } m \text{ is odd.} \end{cases} \tag{10}$$

Proof. We consider the following two cases.

Case-1: m is even.
In this case, set $\{(u_{m/2}, v_0), (u_{m/2+1}, v_0)\}$ of $P_m \Box W_n$ as L_0 then $\mathrm{diam}(L_0) = k = 1$ and the maximum level in $P_m \Box W_n$ is $h = m/2$. The order of $P_m \Box W_n$ and $L(P_m \Box W_n)$ are given by

$$p := m(n+1), \tag{11}$$

$$L(P_m \Box W_n) := \frac{m}{4}(mn + 2n + m - 2). \tag{12}$$

Substituting (11) and (12) into (3) we obtain the right-hand side of (10) which is a lower bound for the radio number of $\mathrm{rn}(P_m \Box W_n)$. We prove that this lower bound is tight. For this purpose, we give an ordering $O(V(P_m \Box W_n)) := (x_0, x_1, \ldots, x_{p-1})$ of $V(P_m \Box W_n)$ which satisfies conditions (a) and (b) of Theorem 3.1. Let τ and σ be two permutations defined on $\{1, 2, \ldots, n\}$ as follows:

$$\tau(j) = \begin{cases} n-1, & \text{if } j = 1; \\ n, & \text{if } j = 2; \\ j-2, & \text{if } 3 \leq j \leq n. \end{cases}$$

$$\sigma(j) = \begin{cases} \lceil j/4 \rceil, & \text{if } j \equiv 1 \pmod 4; \\ \sum_{t=0}^{k-2} \lceil (n-t)/4 \rceil + \lceil j/4 \rceil, & \text{if } j \equiv k \pmod 4,\ k = 2, 3, 4. \end{cases}$$

Using these two permutations we first rename $(u_i, v_j)(1 \leq i \leq m, 0 \leq j \leq n)$ as (a_r, b_s) as follows.

$$(a_r, b_s) = \begin{cases} (u_i, v_j), & \text{if } 1 \leq i \leq m \text{ and } j = 0; \\ (u_i, v_{\sigma\tau(j)}), & \text{if } 1 \leq i \leq m/2 \text{ and } 1 \leq j \leq n; \\ (u_i, v_{\sigma(j)}), & \text{if } m/2 < i \leq m \text{ and } 1 \leq j \leq n. \end{cases}$$

We now define an ordering $O(V(P_m \Box W_n)) := (x_0, x_1, \ldots, x_{p-1})$ as follows: Let $x_t := (a_r, b_s)$, where

$$t := \begin{cases} 2(m/2 - r)(n+1) + 2s, & \text{if } 1 \leq r \leq m/2 \text{ and } 1 \leq s \leq n; \\ 2(m - r)(n+1) + 2s - 1, & \text{if } m/2 < r \leq m \text{ and } 1 \leq s \leq n; \\ 2(m/2 - r)(n+1), & \text{if } 1 \leq r \leq m/2 \text{ and } s = 0; \\ 2(m - r + 1)(n+1) - 1, & \text{if } m/2 < r \leq m \text{ and } s = 0. \end{cases}$$

Note that $d(x_0, L_0) + d(x_{p-1}, L_0) = 0$. Hence, the condition (a) in Theorem 3.1 is satisfied.

Claim-1: The above defined ordering $O(V(P_m \Box W_n)) := (x_0, x_1, \dots, x_{p-1})$ satisfies (6).
Let x_i and x_j $(0 \le i < j \le p-1)$ be any two arbitrary vertices. Denote the right-hand side of (6) by $E(i,j)$. Let $O(V(P_m \Box W_n)) := (x_0, x_1, \dots, x_{p-1}) = U_0 \cup U_1 \cup \ldots \cup U_{m/2-1}$, where $U_t := (x_{2t(n+1)}, x_{(2t(n+1)+1)}, \dots, x_{2t(n+1)+2n+1})$ for $0 \le t \le m/2-1$. It is clear that $d(x_i, L_0) + d(x_{i+1}, L_0) \le (d+1)/2$ for all $0 \le i \le p-2$. Now if $x_i \in U_a, x_j \in U_b$. If $b > a+1$ then $E(i,j) < 0 < d(x_i, x_j)$. If $b = a+1$ then we consider the following two cases: (i) $j = i+2n$ and (ii) $j \ne i+2n$. If $j = i+2n$ then $d(x_i, x_j) = 1$ and in this case, $E(i,j) < 0 < d(x_i, x_j)$ and if $j \ne i+2n$ then $d(x_i, x_j) = 2$ and in this case, $E(i,j) \le 2 \le d(x_i, x_j)$. If $x_i, x_j \in U_a$ then if $x_i = x_{2t(n+1)}$ or $x_j = x_{2t(n+1)+2n+1}$ $(0 \le t \le m/2-1)$ then $E(i,j) \le 1 \le d(x_i, x_j)$. If $x_{2t(n+1)} < x_i < x_j < x_{2t(n+1)+2n+1}$ then if $d(x_i, x_j) = 1$ then $j - i \ge \lceil n/2 \rceil + 1 \ge 3$ and hence $E(i,j) < 0 \le d(x_i, x_j)$ and $d(x_i, x_j) \ge 2$ then $E(i,j) \le 2 \le d(x_i, x_j)$ which completes the proof of Claim-1.

Case-2: m is odd.
In this case, set $\{(u_{(m+1)/2}, v_0)\}$ of $P_m \Box W_n$ as L_0 then $\mathrm{diam}(L_0) = k = 0$ and the maximum level in $P_m \Box W_n$ is $h = (m+1)/2$. The order of $P_m \Box W_n$ and $L(P_m \Box W_n)$ are given by

$$p := m(n+1) \tag{13}$$

$$L(P_m \Box W_n) := \frac{1}{4}(m^2 n + m^2 + 4mn - n - 1). \tag{14}$$

Substituting (13) and (14) into (3) we obtain the right-hand side of (10) which is a lower bound for the radio number of $\mathrm{rn}(P_m \Box W_n)$. We prove that this lower bound is tight. For this purpose, we give an ordering $O(V(P_m \Box W_n)) := (x_0, x_1, \dots, x_{p-1})$ of $V(P_m \Box W_n)$ which satisfies conditions (a) and (b) of Theorem 3.1. Let τ and σ are as defined earlier in Case-1. Let α be a permutation defined on $\{1, 2, \dots, n\}$ as follows:

$$\alpha(j) = \begin{cases} n-3, & \text{if } j = 1; \\ n-2, & \text{if } j = 2; \\ n-1, & \text{if } j = 3; \\ n, & \text{if } j = 4; \\ j-4, & \text{if } 5 \le j \le n. \end{cases}$$

Using permutations α, τ and σ, we first rename $(u_i, v_j)(1 \le i \le m, 0 \le j \le n)$ as (a_r, b_s) as follows:

$$(a_r, b_s) = \begin{cases} (u_i, v_j), & \text{if } 1 \le i \le m \text{ and } j = 0; \text{ or } i = m \text{ and } 1 \le j \le n; \\ (u_i, v_{\tau(j)}), & \text{if } i = 1 \text{ and } 1 \le j \le n; \\ (u_i, v_{\alpha(j)}), & \text{if } i = (m+1)/2 \text{ and } 1 \le j \le n; \\ (u_i, v_{\sigma\tau(j)}), & \text{if } 2 \le i \le (m-1)/2 \text{ and } 1 \le j \le n; \\ (u_i, v_{\sigma(j)}), & \text{if } (m+3)/2 \le i \le m-1 \text{ and } 1 \le j \le n. \end{cases}$$

We now define an ordering $O(V(P_m \Box W_n)) := (x_0, x_1, \dots, x_{p-1})$ as follows: Let $x_t := (a_r, b_s)$, where

$$t := \begin{cases} 3s-1, & \text{if } r = 1 \text{ and } 1 \le s \le n; \\ 3s, & \text{if } r = (m+1)/2 \text{ and } 1 \le s \le n; \\ 3s-2, & \text{if } r = m \text{ and } 1 \le s \le n; \\ 3n+2, & \text{if } r = 1 \text{ and } s = 0; \\ 0, & \text{if } r = (m+1)/2 \text{ and } s = 0; \\ 3n+1, & \text{if } r = m \text{ and } s = 0; \\ 3n+2+2(r-2)(n+1)+2s, & \text{if } 1 < r < (m+1)/2 \text{ and } 1 \le s \le n; \\ 3n+2+2(r-(m+1)/2-1)(n+1)+2s-1, & \text{if } (m+1)/2 < r < m \text{ and } 1 \le s \le n; \\ 3n+2+2(r-1)(n+1), & \text{if } 1 < r < (m+1)/2 \text{ and } s = 0; \\ 3n+2+2(r-(m+1)/2)(n+1)-1, & \text{if } (m+1)/2 < r < m \text{ and } s = 0. \end{cases}$$

Note that $d(x_0, L_0) + d(x_{p-1}, L_0) = 1$. Hence, the condition (a) in Theorem 3.1 is satisfied.

Claim-2: The above defined ordering $O(V(P_m \Box W_n)) := (x_0, x_1, \dots, x_{p-1})$ satisfies (6).
Let x_i and x_j $(0 \le i < j \le p-1)$ be any two arbitrary vertices. Denote the right-hand side of (6) by $E(i,j)$. Let $O(P_m \Box W_n) :=$

$U_1 \cup \ldots \cup U_{(m-1)/2}$, where $U_1 := (x_0, x_1, \ldots, x_{3(n+1)-1})$ and $U_{t+2} := (x_{2t(n+1)+3(n+1)}, \ldots, x_{2t(n+1)+5(n+1)-1})$ for $0 \le t \le (m-5)/2$. Let $x_i \in U_a$ and $x_j \in U_b$. Assume $a = b = 1$. In this case, if $j \ge i+3$ then $E(i,j) \le 0 < d(x_i, x_j)$. If $j = i+2$ then note that $d(x_i, x_j) \ge (d+2)/2$ and hence $E(i,j) \le d/2 \le d(x_i, x_j)$. Let $a = 1$ and $b > 1$. If $j \ge i+3$ then $E(i,j) \le 0 < d(x_i, x_j)$. If $j = i+2$ then if $x_i = x_{3(n+1)-2}$ then note that $d(x_i, x_j) = (d-1)/2$ and $E(i,j) = (d-4)/2 < d(x_i, x_j)$. If $x_i = x_{3(n+1)-1}$ then note that $d(x_i, x_j) = 2$ and $E(i,j) = 1 < d(x_i, x_j)$. Let $a, b \ge 2$. If $b > a+1$ then $E(i,j) < 0 < d(x_i, x_j)$. If $b = a+1$ then we consider the following two cases: (i) $j = i+2n$ and (ii) $j \ne i+2n$. If $j = i+2n$ then $d(x_i, x_j) = 1$ and in this case, $E(i,j) < 0 < d(x_i, x_j)$ and if $j \ne i+2n$ then $d(x_i, x_j) = 2$ and in this case, $E(i,j) \le 2 \le d(x_i, x_j)$. If $x_i, x_j \in U_a$. If $x_j = x_{2t(n+1)+5(n+1)-2}$ or $x_j = x_{2t(n+1)+5(n+1)-1}$ $(0 \le t \le (m-5)/2)$ then $E(i,j) \le 1 \le d(x_i, x_j)$; otherwise $E(i,j) \le 2 \le d(x_i, x_j)$ which completes the proof of Claim-2.

□

An *n-star*, denoted by $K_{1,n}$, is a tree consisting of n leaves and another vertex joined to all leaves by edges. Denote the vertex set of $K_{1,n}$ by $V(K_{1,n}) = \{v_0, v_1, \ldots, v_n\}$ and take $E(K_{1,n}) = \{v_0 v_i : 1 \le i \le n\}$. A *friendship graph* F_n is a graph obtained by identifying one vertex of n copies of cycle C_3 with a common vertex. Denote the vertex set of F_n by $V(F_n) = \{v_0, v_1, \ldots, v_{2n}\}$ with $E(F_n) = \{v_0 v_i, v_0 v_{n+i}, v_{2i-1} v_{2i} : 1 \le i \le n\}$.

Corollary 3.1. *Let $m \ge 3$ and $n \ge 7$ be any integers. Then*

$$\mathrm{rn}(P_m \square K_{1,n}) = \begin{cases} \frac{1}{2}(m^2 n + m^2 + 2m - 2), & \text{if } m \text{ is even;} \\ \frac{1}{2}(m^2 n + m^2 + 2m + n - 1), & \text{if } m \text{ is odd.} \end{cases}$$

Proof. Observe that $P_m \square K_{1,n}$ can be regarded as a subgraph of $P_m \square W_n$ with identical $L_0 = \{(u_{m/2}, v_0), (u_{m/2+1}, v_0)\}$ when m is even and $L_0 = \{(u_{(m+1)/2}, v_0)\}$ when m is odd and hence by Theorem 3.3, the radio number of $P_m \square W_n$ and $P_m \square K_{1,n}$ are identical. □

Corollary 3.2. *Let $m \ge 3$ and $n \ge 4$ be any integers. Then*

$$\mathrm{rn}(P_m \square F_n) = \begin{cases} \frac{1}{2}(2m^2 n + m^2 + 2m - 2), & \text{if } m \text{ is even;} \\ \frac{1}{2}(2m^2 n + m^2 + 2m + 2n - 1), & \text{if } m \text{ is odd.} \end{cases}$$

Proof. Observe that $P_m \square F_n$ can be regarded as a subgraph of $P_m \square W_{2n}$ with identical $L_0 = \{(u_{m/2}, v_0), (u_{m/2+1}, v_0)\}$ when m is even and $L_0 = \{(u_{(m+1)/2}, v_0)\}$ when m is odd and hence by Theorem 3.3, the radio number of $P_m \square W_{2n}$ and $P_m \square F_n$ are identical. □

Example 3.1. In Table 1, an ordering of vertices and the corresponding optimal radio labeling of $P_7 \square W_7$ is shown.

Table 1: An ordering and optimal radio labeling for vertices of $P_7 \square W_7$.

$(u_i, v_j)\frac{i\to}{j\downarrow}$	1		2		3		4		5		6		7	
0	x_{23}	72	x_{39}	139	x_{55}	206	$\underline{x_0}$	**<u>0</u>**	x_{38}	133	x_{54}	200	x_{22}	69
1	x_{17}	51	x_{31}	104	x_{47}	171	x_{12}	37	x_{24}	76	x_{40}	143	x_1	5
2	x_{20}	60	x_{35}	120	x_{51}	187	x_{15}	46	x_{28}	92	x_{44}	159	x_4	14
3	x_2	6	x_{25}	80	x_{41}	147	x_{18}	55	x_{32}	108	x_{48}	175	x_7	23
4	x_5	15	x_{29}	96	x_{45}	163	x_{21}	64	x_{36}	124	x_{52}	191	x_{10}	32
5	x_8	24	x_{33}	112	x_{49}	179	x_3	10	x_{26}	84	x_{42}	151	x_{13}	41
6	x_{11}	33	x_{37}	128	x_{53}	195	x_6	19	x_{30}	100	x_{46}	167	x_{16}	50
7	x_{14}	42	x_{27}	88	x_{43}	155	x_9	28	x_{34}	116	x_{50}	183	x_{19}	59

Example 3.2. In Table 2, an ordering of vertices and the corresponding optimal radio labeling of $P_8 \square W_7$ is shown.

4. Concluding remarks

In [12], Kim et al. determined the radio number of Cartesian product of paths and complete graph $P_m \square K_n$.

Theorem 4.1. [12] *Let $m \ge 4$ and $n \ge 3$ be integers. Then*

$$\mathrm{rn}(P_m \square K_n) = \begin{cases} \frac{1}{2}(m^2 n - 2m + 2), & \text{if } m \text{ is even;} \\ \frac{1}{2}(m^2 n - 2m + n + 2), & \text{if } m \text{ is odd.} \end{cases} \quad (15)$$

Table 2: An ordering and optimal radio labeling for vertices of $P_8 \Box W_7$.

$(u_i, v_j)\ {}^{i\rightarrow}_{j\downarrow}$	1		2		3		4		5		6		7		8	
0	x_{48}	201	x_{32}	134	x_{16}	67	$\underline{x_0}$	$\underline{\mathbf{0}}$	x_{63}	263	x_{47}	196	x_{31}	129	x_{15}	62
1	x_{56}	234	x_{40}	167	x_{24}	100	x_8	33	x_{49}	206	x_{33}	139	x_{17}	72	x_1	5
2	x_{60}	250	x_{44}	183	x_{28}	116	x_{12}	49	x_{53}	222	x_{37}	155	x_{21}	88	x_5	21
3	x_{50}	210	x_{34}	143	x_{18}	76	x_2	9	x_{57}	238	x_{41}	171	x_{25}	104	x_9	37
4	x_{54}	226	x_{38}	159	x_{22}	92	x_6	25	x_{61}	254	x_{45}	187	x_{29}	120	x_{13}	53
5	x_{58}	242	x_{42}	175	x_{26}	108	x_{10}	41	x_{51}	214	x_{35}	147	x_{19}	80	x_3	13
6	x_{62}	258	x_{46}	191	x_{30}	124	x_{14}	57	x_{55}	230	x_{39}	163	x_{23}	96	x_7	29
7	x_{52}	218	x_{36}	151	x_{20}	84	x_4	17	x_{59}	246	x_{43}	179	x_{27}	112	x_{11}	45

Theorem 4.1 can also be proved using Theorem 3.1. The order and total level of $P_m \Box K_n$ are given by

$$p := mn \tag{16}$$

$$L(P_m \Box K_n) := \begin{cases} \dfrac{1}{2}(mn(m-2)), & \text{if } m \text{ is even;} \\[2ex] \dfrac{1}{4}((m^2-1)n), & \text{if } m \text{ is odd.} \end{cases} \tag{17}$$

Substituting (16) and (17) in (3), we obtained the right-hand side of (15), which is a lower bound for the radio number of $P_m \Box K_n$. Now, it is easy to prove that the radio labeling given in [12] satisfies conditions (a) and (b) of Theorem 3.1 and hence the right-hand side of (3) is exactly the radio number of $P_m \Box K_n$, which is (15) in the present case.

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