Research Article
Skew Dyck paths with catastrophes

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Abstract

Skew Dyck paths are like Dyck paths, but an additional south-west step $(-1,-1)$ is allowed, provided that the path does not intersect itself. Lattice paths with catastrophes can drop from any level to the origin in just one step. These two ideas are combined. The analysis is strictly based on generating functions, and the kernel method is used.

Keywords: skew Dyck path; catastrophe; kernel method.

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1. Introduction

The standard random walk on non-negative integers may be visualized by the graph depicted in Figure 1 (only the first 8 states are shown). These walks are often called Dyck paths (up-steps $(1,1)$ and down-steps $(1,-1)$, not going below the x-axis) and appear in many texts (for example, see [8]).

Figure 1: Standard symmetric random walk on the non-negative integers.

Most of the time, Dyck paths need to come back to the x-axis (end at state 0), but one also considers open-ended paths where the level of the end of the path is not specified.[†]

The standard reference for lattice paths with catastrophes is the paper [2]; the very recent papers [3,4] contain some bijective aspects. Dyck paths with catastrophes are characterized by additional steps ('catastrophes') that bring the path back to the x-axis in just one step from any level $j \geq 2$.
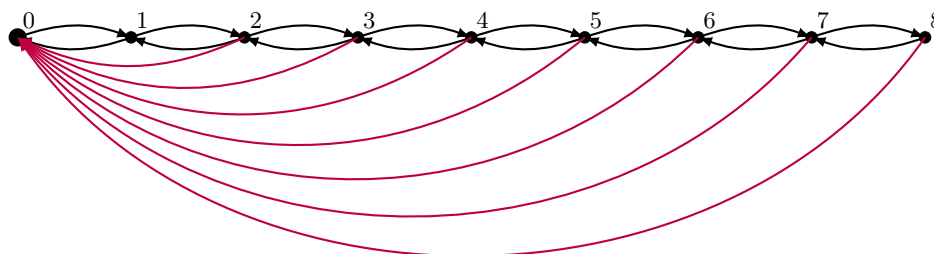

Figure 2: Catastrophes can lead from a state $i \geq 2$ back to the x-axis in one step.

The Figure 2 shows the graph underlying the catastrophes, drawn in purple.

The present note combines skew Dyck paths (with an extra step $(-1,-1)$), as analyzed in [6] and the references given therein, with the concept of catastrophes.

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[†]Philippe Flajolet [1] liked the names 'excursion' respectively 'meander', which we will not use.

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2. A warm-up: Dyck paths and catastrophes

To offer a gentle path for our readers, we analyze the paths as in Fig. 2 (which is also contained in the earlier papers [2,3]), since we will approach the skew Dyck paths with catastrophes in a similar style in the next section.

We introduce generating functions $f_i = f_i(z)$, where the coefficient of z^n counts the number of paths starting at the origin (=the big circle) and ending after n steps at state i (=level i). The following recursions are easy to see:

$$f_0 = 1 + z(f_1 + f_2 + f_3 + f_4 + \cdots),$$
$$f_i = zf_{i-1} + zf_{i+1}, \ i \geq 1.$$

Since f_0 is somewhat special, we leave it out for the moment and compute the other ones, f_i, $i \geq 1$. Eventually we will solve the equation for f_0, which will turn out to be just linear. Therefore we introduce the bivariate generating function

$$F(u) = F(u, z) = \sum_{i \geq 1} u^{i-1} f_i$$

and we treat f_0 as a parameter. Summing the recursions,

$$F(u) = zf_0 + zuF(u) + \frac{z}{u}[F(u) - f_1],$$

and thus

$$F(u) = \frac{zf_0 - \frac{z}{u}f_1}{1 - zu - \frac{z}{u}} = \frac{zf_1 - zuf_0}{zu^2 - u + z} = \frac{zf_1 - zuf_0}{z(u - r_1)(u - r_2)},$$

with

$$r_1 = \frac{1 + \sqrt{1 - 4z^2}}{2z}, \quad r_2 = \frac{1 - \sqrt{1 - 4z^2}}{2z}.$$

An essential step of the *kernel method* [5] is that the 'bad' factor $(u - r_2)$ must be canceled. The reciprocal of this factor does not have a power series expansion in (u, z). Dividing this factor out in both the denominator and numerator, leads to

$$F(u) = \frac{-zf_0}{z(u - r_1)} = \frac{zf_0}{zr_1(1 - u/r_1)},$$

and from this

$$f_i = [u^{i-1}]F(u) = \frac{zf_0}{zr_1^i} = f_0 r_2^i.$$

This formula holds for all $i \geq 0$. It is interesting to note that all these functions are just a multiple of the parameter f_0. Now we can go to the first recursion:

$$f_0 = 1 + z(f_0 r_2^1 + f_0 r_2^2 + f_0 r_2^3 + \cdots) = 1 + zf_0 \frac{r_2}{1 - r_2}$$

and solve the equation,

$$f_0 = \frac{2 - 3z - 2z^2 + z\sqrt{1 - 4z^2}}{2(1 - z - 2z^2 - z^3)}.$$

It has the power series expansion

$$f_0 = 1 + z^2 + z^3 + 3z^4 + 5z^5 + 12z^6 + 23z^7 + 52z^8 + 105z^9 + 232z^{10} + 480z^{11} + 1049z^{12} + \cdots,$$

and the list of coefficients corresponds to sequence A224747 in the OEIS [7]. The papers [2,3] have this already. We can also compute the number of Dyck paths with catastrophes and open (unspecified) end, via the generating function

$$f_0 + f_1 + f_2 + f_3 + f_4 + \ldots$$

$$= \frac{1 - z + (1 + z)\sqrt{1 - 4z^2}}{2(1 - z - 2z^2 - z^3)}$$

$$= 1 + z + 2z^2 + 4z^3 + 8z^4 + 17z^5 + 35z^6 + 75z^7 + 157z^8 + 337z^9 + 712z^{10} + \cdots.$$

This is almost sequence A274115 in the OEIS [7]; $1 + z(f_0 + f_1 + f_2 + \cdots)$ has this sequence as coefficients. Thanks go to Michel Marcus to point out the connection with the sequences in the OEIS.
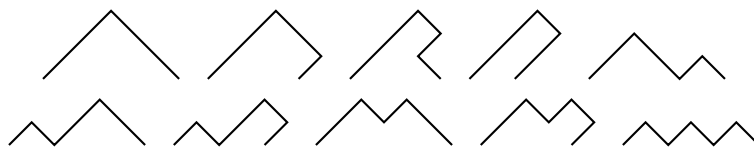
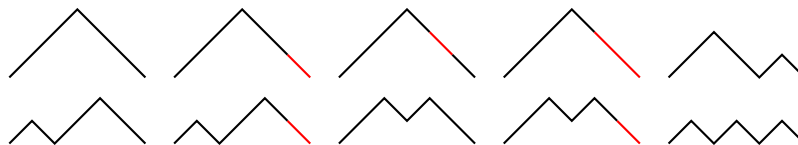Figure 3: All 10 skew Dyck paths of length 6 (consisting of 6 steps).

Figure 4: The 10 paths redrawn, with red south-east edges instead of south-west edges.

3. Skew Dyck paths with catastrophes

Skew Dyck are a variation of Dyck paths, where additionally to steps $(1,1)$ and $(1,-1)$ a south-west step $(-1,-1)$ is also allowed, provided that the path does not intersect itself.

Here is a list of the 10 skew paths consisting of 6 steps:

We prefer to work with the equivalent model (resembling more traditional Dyck paths) where we replace each step $(-1,-1)$ by $(1,-1)$ but label it red. Here is the list of the 10 paths again (Figure 4):

The rules to generate such decorated Dyck paths are: each edge $(1,-1)$ may be black or red, but $\wedge$ and $\vee$ are forbidden. These figures are taken from our earlier paper [6].
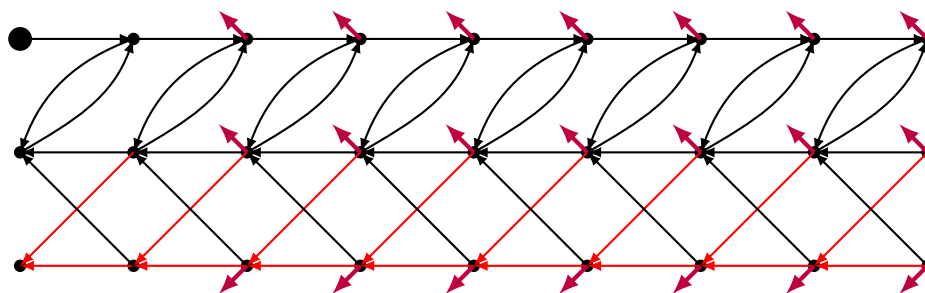

Figure 5: Three layers of states according to the type of steps leading to them (up, down-black, down-red). Purple arrows all end in the origin.

As in the motivating example from the previous section, we introduce generating functions. But since we need 3 layers to control everything, we need to introduce f_j, g_j, h_j. The purple edges all end in the origin and represent the catastrophes. If they are ignored, one models skew Dyck paths in this way, in particular $f_0 + g_0 + h_0$ is the generating function of skew Dyck paths coming back to level 0. However, now, we will deal with the purple arrows as well.

The following recursions can be read off immediately from the diagram drawn in Figure 5:

$$f_0 = 1 + z \sum_{i \geq 2} f_i + z \sum_{i \geq 2} g_i + z \sum_{i \geq 2} h_i,$$
$$f_{i+1} = z f_i + z g_i, \quad i \geq 0,$$
$$g_i = z f_{i+1} + z g_{i+1} + z h_{i+1}, \quad i \geq 0,$$
$$h_i = z g_{i+1} + z h_{i+1}, \quad i \geq 0.$$

As before, we first ignore f_0 and treat is as a parameter. We introduce

$$F(u) = \sum_{i \geq 1} u^{i-1} f_i, \quad G(u) = \sum_{i \geq 0} u^i g_i, \quad H(u) = \sum_{i \geq 0} u^i h_i.$$

Now we sum the recursions and get

$$\sum_{i \geq 0} u^i f_{i+1} = \sum_{i \geq 0} u^i z f_i + \sum_{i \geq 0} u^i z g_i$$

and thus

$$F(u) = f_0 + uzF(u) + zG(u).$$

Further,

$$\sum_{i \geq 0} u^i g_i = \sum_{i \geq 0} u^i z f_{i+1} + \sum_{i \geq 0} u^i z g_{i+1} + \sum_{i \geq 0} u^i z h_{i+1}$$

which translates into

$$G(u) = zF(u) + \frac{z}{u}[G(u) - G(0)] + \frac{z}{u}[H(u) - H(0)];$$

similarly

$$H(u) = \frac{z}{u}[G(u) - G(0)] + \frac{z}{u}[H(u) - H(0)].$$

We will eliminate the functions G and H:

$$G = -\frac{zf_0 + zuF - F}{z}$$

and

$$H = G - zF.$$

Therefore we end up with just one equation for F

$$-uzf_0 - zu^2 F + uF = -z^2 Fu - 2z^2 f_0 + 2zF - 2g_0 z^2 - z^3 F + z^3 f_1$$

which can be solved,

$$F(u) = -\frac{\left(-2zf_0 - 2zg_0 + z^2 f_1 + f_0 u\right) z}{-z^2 u + 2z - z^3 + zu^2 - u} = -\frac{\left(-2zf_0 - 2zg_0 + z^2 f_1 + f_0 u\right) z}{z(u - r_1)(u - r_2)}$$

with

$$r_1 = \frac{1 + z^2 + \sqrt{1 - 6z^2 + 5z^4}}{2z}, \quad r_2 = \frac{1 + z^2 - \sqrt{1 - 6z^2 + 5z^4}}{2z}. \tag{1}$$

The fact that $u - r_2$ must divide the numerator of F leads to

$$g_0 = \frac{-2zf_0 + z^2 f_1 + r_2 f_0}{2z},$$

so that g_0 can be eliminated and only f_1 remains as unknown;

$$F = \frac{zf_0}{1 - zu + z^2 - zr_2}$$

plugging in $u = 0$ allows to compute it:

$$f_1 = \frac{zf_0}{1 + z^2 - zr_2} = \frac{f_0}{r_1}.$$

Therefore the function F is known:

$$F(u) = \frac{f_0}{r_1 - u}$$

and thus

$$f_j = [u^{j-1}]F(u) = \frac{f_0}{r_1^j}.$$

From this, the other functions can be computed as well:

$$G(u) = \frac{1 - zr_1}{z}\frac{f_0}{r_1 - u} = (r_2 - z)\frac{f_0}{r_1 - u}$$

and

$$g_j = [u^j]G(u) = (r_2 - z)\frac{f_0}{r_1^{j+1}},$$

as well as

$$H(u) = (r_2 - 2z)\frac{f_0}{r_1 - u}$$

and

$$h_j = [u^j]H(u) = (r_2 - 2z)\frac{f_0}{r_1^{j+1}}.$$

Now we can finally deal with f_0. The evaluations that are needed are simple:

$$\sum_{j\geq 2} f_j = \frac{f_0/r_1^2}{1 - 1/r_1}, \quad \sum_{j\geq 2} g_j = (r_2 - z)\frac{f_0/r_1^3}{1 - 1/r_1}, \quad \sum_{j\geq 2} h_j = (r_2 - 2z)\frac{f_0/r_1^3}{1 - 1/r_1};$$

of course this is better done with a computer. From this, the equation (just linear) for f_0 can be solved:

$$f_0 = \frac{1 - 9z^3 - 5z^4 + 3z^5 + 2z^6 - (1 - z^2 - z^3)\sqrt{1 - 6z^2 + 5z^4}}{2(1 - 2z^2 - 6z^3 - 3z^4 + z^5 + z^6)}. \tag{2}$$

The series expansion

$$1 + z^3 + z^4 + 4z^5 + 6z^6 + 18z^7 + 31z^8 + 85z^9 + 157z^{10} + 410z^{11} + 792z^{12} + 2004z^{13} + \cdots$$

is not in OEIS [7]. To compute the number of all skew Dyck paths coming back to the x-axis, we need

$$f_0 + g_0 + h_0 = \frac{(1 - z^2)(3 - z)(1 - z - 2z^2 - z^3) - (1 - 4z - 3z^2 + z^3 + z^4)\sqrt{1 - 6z^2 + 5z^4}}{2(1 - 2z^2 - 6z^3 - 3z^4 + z^5 + z^6)}. \tag{3}$$

The series expansion is

$$1 + z^2 + z^3 + 4z^4 + 5z^5 + 17z^6 + 25z^7 + 76z^8 + 125z^9 + 353z^{10} + 625z^{11} + 1681z^{12} + \cdots.$$

One can also sum everything to enumerate the open ended paths in this model:

$$\sum_{j\geq 0} f_j + \sum_{j\geq 0} g_j + \sum_{j\geq 0} h_j = \frac{(1 + z)(1 - 4z + 4z^2 + 4z^3 - z^5) + (1 + 5z + 3z^2 - z^3 - z^4)\sqrt{1 - 6z^2 + 5z^4}}{2(1 - 2z^2 - 6z^3 - 3z^4 + z^5 + z^6)}; \tag{4}$$

the series expansion is

$$1 + z + 2z^2 + 4z^3 + 9z^4 + 18z^5 + 41z^6 + 85z^7 + 193z^8 + 410z^9 + 929z^{10} + 2004z^{11} + \cdots.$$

The results obtained in this section are summarized in a theorem.

Theorem 3.1. *The generating functions related to the model of skew Dyck paths with catastrophes are*

$$f_j = \frac{f_0}{r_1^j},$$

$$g_j = (r_2 - z)\frac{f_0}{r_1^{j+1}},$$

$$h_j = (r_2 - 2z)\frac{f_0}{r_1^{j+1}},$$

with f_0 being given in (2), r_1, r_2 being given in (1) and $f_0 + g_0 + h_0$ being given in (3). The enumeration of all paths, regardless of the final level, is given in (4).

4.   Conclusion

From our analysis, other types of catastrophes (example: only from an even-numbered level one can drop to the ground) can be computed without much further effort. This changes only the equation for f_0; the other quantities stay as they are.

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