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Research Article

Alphabetic points and records in inversion sequences

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Abstract

An alphabetic point in an inversion sequence is a value j where all the values l to its left satisfy $l \leq j$ and all the values r to its right satisfy $r \geq j$. We study alphabetic points and records in inversion sequences of permutations and obtain formulae for the total numbers of alphabetic points, weak records, and strict records.

Keywords: alphabetic points; records; inversion sequences.

2020 Mathematics Subject Classification: 05A05, 05A15.

1. Introduction

In this paper, we study combinatorial statistics in inversion sequences. An inversion sequence $b(1)b(2)\cdots b(n)$ is defined as follows (for example, see [4, 7]). Given a permutation $a(1)a(2)\ldots a(n)$ of $[n]$, let $b(i) = |\{a(j) : j < i \text{ and } a(j) > a(i)\}|$. We place $b(i)$, $(0 \leq b(i) \leq n-1)$ in position i in the inversion sequence. This procedure defines a mapping from permutations of n to inversion sequences with n parts, with the alphabet starting at 0. Thus, an inversion sequence with n parts is a word $c(1)c(2)\ldots c(n)$ where for each i, $c(i)$ is an integer such that $0 \leq c(i) \leq i-1$.

For the convenience of obtaining a nice geometrical bargraph representation, we increase the size of each image point by 1 so that now and for the rest of the paper, an inversion sequence with n parts is a word $c(1)c(2)\ldots c(n)$ where for each i, $c(i)$ is an integer such that $1 \leq c(i) \leq i$. For example, Table 1 shows the inversion sequences of the six permutations of $\{1, 2, 3\}$.

Permutation	Corresponding inversion sequence
123	111
132	112
213	121
231	113
312	122
321	123

Table 1: Inversion sequences of the six permutations of {1,2,3}.

The inverse map from inversion sequences with n parts to permutations of n is just the reverse of the above procedure. So, for example, the inversion sequence 121 is mapped to 213 because in 121 the part 3 must be positioned in the $1st$ position from the right i.e., $\_\,\_3$. Now, part 2 is positioned in the 2nd (second) empty position from the right, i.e., this yields 2_3. Finally, 1 is positioned in the $1st$ empty position from the right, yielding 213. The map and its inverse defines a bijection between permutations of n and inversion sequences of length n.

A strict (respectively, weak) *record* or *left-to-right maximum* is a value j where all the values l to its left satisfy $l < j$, (respectively $l \leq j$). For the definitions of these terms, see [13] and for research papers on records, see [5, 6, 8–11]. An *alphabetic point* in an inversion sequence is a value j where all the values l to its left satisfy $l \leq j$ and all the values r to its right satisfy $r \geq j$. Therefore, an alphabetic point is any point that is both a weak left-to-right maximum and a weak right-to-left minimum. For recent work on alphabetic points in other combinatorial structures, see [2] concerning compositions and words, and [3] concerning restricted growth functions.

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Remark 1.1. *Alphabetic points are analogous to "strong fixed points" in permutations. See p. 49 in [13, Volume 1] as well as sequences A006932 and A052186 in [12]. However, in permutations these are automatically also fixed points. See [1] for the current authors' study of fixed points in inversion sequences where these are not necessarily alphabetic.*

The structure of this paper is as follows: in Section 2 we study alphabetic points, in Section 3 we deal with weak records, and finally in Section 4 we study strict records in inversion sequences.

2. The total number of alphabetic points in inversion sequences

For the approach here, we do not take an inversion sequence, count its alphabetic points and add all these numbers (which is not easy), instead we take any (potential) alphabetic point in a fixed position and create and count all inversion sequences that fit to the alphabetic point in the fixed position.

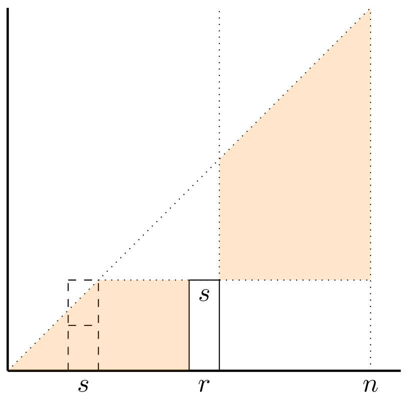

Figure 1: Sketch indicating an alphabetic point s in position r in an inversion sequence.

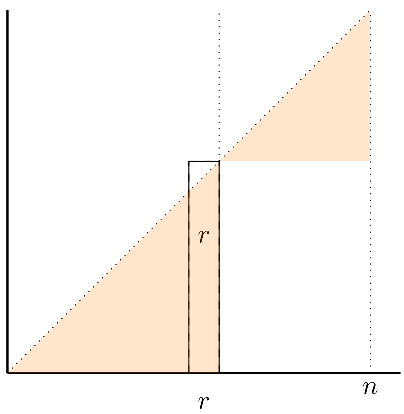

Figure 2: Sketch indicating an alphabetic point s in position $r = s$ in an inversion sequence.

Theorem 2.1. *The total number of alphabetic points in inversion sequences of length $n \geq 2$ is given by*

$$T(n) := n! + (n-1)! + \frac{n!}{2} + \sum_{r=3}^{n}\left((r-1)!(n-r+1)! + \frac{(r-1)!(n-r+2)!}{2!} + \sum_{s=1}^{r-2}\frac{s!(n-s+1)!s^{r-1-s}}{(r-s+1)!}\right) \quad (1)$$

and $T(1) = 1$.

Proof. We refer to Figures 1 and 2. Let part of size s be an alphabetic point in position r of the inversion sequence of length n. Then $1 \leq s \leq r$ and $1 \leq r \leq n$.

Case 1: $r = 1$**.** There are $n!$ such instances since the first position is always alphabetic.

Case 2: $r = 2$**.** By counting all the cases for $s = 1$ and $s = 2$, there are $(n-1)! + \frac{n!}{2}$ such instances.

Case 3: $r \geq 3$**.**

Case 3a: See Figure 2. Here $r = s$, the solid and dashed columns coincide. All possible parts in the shaded triangle to the left of r are less than r and therefore must be counted. The total number of such instances is $(r-1)!$. On the right of r, all parts whose highest point are in the shaded triangle are greater than or equal to r. The number of such instances is $(n-r+1)!$. Together this yields a total for this case of

$$(r-1)!(n-r+1)!.$$

We refer to Figures 1 for the cases below.

Case 3b: Consider $r = s+1$, the solid column is adjacent to the dashed column. All possible parts in the leftmost shaded triangle of Figure 1 are less than or equal to s and therefore must be counted. The total number of such instances is $s! = (r-1)!$. Similarly, on the right of r, all parts whose highest points are in the shaded trapezium are greater than or equal to $s = r-1$. The number of such instances is $\frac{(n-r+2)!}{2!}$. So the total number of such instances is

$$\frac{(r-1)!(n-r+2)!}{2!}$$

.

Case 3c: Here $r > s+1$; column r solid is to the right of the dashed column, but not touching. Again, all possible parts in the leftmost shaded triangle of Figures 1 are less than or equal to s and therefore must be counted. The total number of such instances is $s!$. Again, on the right of r, all parts whose highest points are in the shaded trapezium are greater than or equal to s. The number of such instances is $\frac{(n-s+1)!}{(r-s+1)!}$. Finally the number of instances formed by all possible parts lying in the rectangle between s and r is the number of words on the alphabet $[s]$ with number of parts equal to the rectangle width, i.e., s^{r-1-s}. The product of all the instances for this case is then summed over all possible s values yielding

$$\sum_{s=1}^{r-2} \frac{s!(n-s+1)!s^{r-1-s}}{(r-s+1)!}.$$

Summing all the cases together, we obtain the formula as stated in the theorem. □

3. Number of weak records

Let $a_{n,m,r}$ be the number of inversion sequences $e = e_1e_2\cdots e_n$ having r weak records such that $\max_{1\le i\le n} e_i = m+1$. By considering the last letter e_n in any inversion sequence $e_1e_2\cdots e_n$, we obtain for $0 \le m \le n-1$ and $0 \le r \le n$,

$$a_{n,m,r} = ma_{n-1,m,r} + \sum_{j=0}^{m} a_{n-1,j,r-1}$$

with initial condition $a_{0,0,0} = 1$. Define $A_{n,m}(v) = \sum_{r=0}^{n} a_{n,m,r}v^r$. Thus, by multiplying the recurrence by v^r and summing over $r = 0, 1, \ldots, n$, we obtain

$$A_{n,m}(v) = mA_{n-1,m}(v) + v\sum_{j=0}^{m} A_{n-1,j}(v), \quad 0 \le m \le n, \tag{2}$$

with $A_{0,0}(v) = 1$ (see Table 2).

$n\backslash m$	0	1	2	3	4
1	v				
2	v^2	v^2			
3	v^3	v^2+2v^3	$2v^3$		
4	v^4	$v^2+3v^3+3v^4$	$5v^3+5v^4$	v^3+5v^4	
5	v^5	$v^2+4v^3+6v^4+4v^5$	$11v^3+18v^4+9v^5$	$4v^3+24v^4+14v^5$	$v^3+9v^4+14v^5$

Table 2: Values of $A_{n,m}(v)$ for $0 \le m < n < 6$.

Note that by definitions, we see that any inversion sequence $e = e_1e_2\cdots e_n$ with maximal letter n and n weak records satisfies $1 = e_1 \le e_2 \le \cdots \le e_n \le n$. Thus, the number such inversion sequences are given by n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ (this set is listed in [13, Volume 2] among those discrete objects enumerated by C_n).

Define $A_n(u,v)=\sum_{m=0}^{n-1}A_{n,m}(v)u^m$. Thus, by multiplying (2) by u^m and summing over $m=0,1,\dots,n-1$, we obtain

$$\begin{aligned}A_n(u,v)&=\sum_{m-0}^{m-1}mA_{n-1,m}(v)u^m+v\sum_{m=0}^{n-1}\sum_{j=0}^{m}A_{n-1,j}(v)u^m\\&=u\frac{\partial}{\partial u}A_{n-1}(u,v)+v\sum_{j=0}^{n-1}\sum_{m=j}^{n-1}A_{n-1,j}(v)u^m\\&=u\frac{\partial}{\partial u}A_{n-1}(u,v)+v\sum_{j=0}^{n-1}A_{n-1,j}(v)\frac{u^j-u^n}{1-u}\\&=u\frac{\partial}{\partial u}A_{n-1}(u,v)+\frac{v}{1-u}(A_{n-1}(u,v)-u^nA_{n-1}(1,v)),\end{aligned}$$

where $A_0(u,v)=1$. Define $A(x,u,v)=\sum_{n\geq 0}A_n(x,u,v)x^n$. Hence,

$$A(x,u,v)=1+\frac{vx}{1-u}(A(x,u,v)-uA(ux,1,v))+ux\frac{\partial}{\partial u}A(x,u,v). \tag{3}$$

By Applying (3) several times, we get that the coefficient of x^n, $n=0,1,\dots,4$, in $A(x,u,v)$ are 1, v, $(u+1)v^2$, $(2u^2v+2uv+u+v)v^2$, and $(5u^3v^2+u^3v+5u^2v^2+5u^2v+3uv^2+3uv+v^2+u)v^2$. Let $A_r(x,u)$ be the coefficient of v^r in $A(x,u,v)$. Then (3) gives

$$A_r(x,u)=\delta_{r=0}+\frac{x}{1-u}(A_{r-1}(x,u)-uA_{r-1}(ux,1))+ux\frac{\partial}{\partial u}A_r(x,u).$$

Define $\tilde{A}_r(x,u)=\sum_{n\geq 0}[x^n](A_r(x,u))\frac{x^n}{n!}$. Then, we have the following result.

Proposition 3.1. *For all $r\geq 0$,*

$$\frac{\partial}{\partial x}\tilde{A}_r(x,u)=\frac{1}{1-u}(\tilde{A}_{r-1}(x,u)-u\tilde{A}_{r-1}(ux,1))+u\frac{\partial}{\partial u}\tilde{A}_r(x,u).$$

For instance, by using the fact that $\tilde{A}_0(x,u)=1$, then $\frac{\partial}{\partial x}\tilde{A}_1(x,u)=1+u\frac{\partial}{\partial u}\tilde{A}_1(x,u)$, which shows that $\tilde{A}_1(x,u)=x$. Similarly, $\tilde{A}_2(x,u)=ue^x-ux-u+\frac{1}{2}x^2$.

Next, we consider the total number of weak records in inversion sequences. Here, we use the approach developed in the previous section.

Theorem 3.1. *The total number of weak records in inversion sequences of length $n\geq 2$ is given by*

$$W(n):=2n!+n!\sum_{r=3}^{n}\left(\frac{2}{r}+\frac{1}{r!}\sum_{s=1}^{r-2}s!s^{r-1-s}\right) \tag{4}$$

with $W(1)=1$.

Proof. Let a part of size s be a weak record in position r of the inversion sequence of length n. Then $1\leq s\leq r$ and $1\leq r\leq n$.

Case 1: $r=1$. There are $n!$ such instances.

Case 2: $r=2$. There are $n!$ such instances again. These come from $s=1$ and $s=2$.

Case 3: $r\geq 3$.

Case 3a: $r=s$, the solid and dashed columns coincide. See Figure 2. All possible parts in the shaded triangle to the left of r are less than r and therefore must be counted. The total number of such instances is $(r-1)!$. On the right of r, we count all possible entries in an inversion sequence, i.e., $n(n-1)(n-2)\dots(r+1)$. The number of such instances is $\frac{n!}{r!}$. Together this yields a total for this case of

$$\frac{n!}{r}.$$

Case 3b: $r=s+1$, the solid column is adjacent to the dashed column. All possible parts in the leftmost shaded triangle of Figures 1 are less than or equal to s and therefore must be counted. The total number of such instances is $s!=(r-1)!$. Similarly, on the right of r, all parts must be counted yielding again $\frac{n!}{r!}$. So the total number of such instances is also as before

$$\frac{n!}{r}.$$

Case 3c: $r > s+1$, column r solid is to the right of the dashed column, but not touching. Again, all possible parts in the leftmost shaded triangle of Figures 1 are less than or equal to s and therefore must be counted. The total number of such instances is $s!$. Again, on the right of r, all parts must be counted as before. The number of such instances is

$$\frac{n!}{r!}.$$

Finally the number of instances formed by all possible parts lying in the rectangle between s and r is the number of words on the alphabet $[s]$ with number of parts equal to the rectangle width, i.e., s^{r-1-s}. The product of all the instances for this case is then summed over all possible s values yielding

$$\frac{n!}{r!}\sum_{s=1}^{r-2} s!s^{r-1-s}.$$

Adding all the cases together, we obtain the formula as stated in the theorem. □

4. Number of strict records

Let $s_{n,m,r}$ be the number of inversion sequences $e = e_1e_2\cdots e_n$ having r strong records such that $\max_{1\le i\le n} e_i = m+1$. By considering the last letter e_n in any inversion sequence $e_1e_2\cdots e_n$, we obtain for $0 \le m \le n-1$ and $0 \le r \le n$,

$$s_{n,m,r} = (m+1)s_{n-1,m,r} + \sum_{j=0}^{m-1} a_{n-1,j,r-1}$$

with initial condition $s_{1,0,1} = 1$. Define $S_{n,m}(v) = \sum_{r=0}^{n} a_{n,m,r}v^r$. Thus, by multiplying the recurrence by v^r and summing over $r = 0, 1, \ldots, n$, we obtain

$$S_{n,m}(v) = (m+1)S_{n-1,m}(v) + v\sum_{j=0}^{m-1} S_{n-1,j}(v), \quad 0 \le m \le n-1, \tag{5}$$

with $S_{1,0}(v) = v$ (see Table 3).

$n\backslash m$	0	1	2	3	4
1	v				
2	v	v^2			
3	v	$3v^2$	v^2+v^3		
4	v	$7v^2$	$4v^2+6v^3$	$v^2+4v^3+v^4$	
5	v	$15v^2$	$13v^2+15v^3$	$5v^2+27v^3+10v^4$	$v^2+12v^3+10v^4+v^5$

Table 3: Values of $S_{n,m}(v)$ for $0 \le m < n < 6$.

Define $S_n(u,v) = \sum_{m=0}^{n-1} S_{n,m}(v)u^m$. Thus, by multiplying (5) by u^m and summing over $m = 0, 1, \ldots, n-1$, we obtain

$$S_n(u,v) = u\frac{\partial}{\partial u}S_{n-1}(u,v) + S_{n-1}(u,v) + \frac{uv}{1-u}(S_{n-1}(u,v) - u^{n-1}S_{n-1}(1,v)). \tag{6}$$

with $S_1(u,v) = v$.
Define $S(x,u,v) = \sum_{n\ge1} S_n(u,v)x^n$. Thus, by multiplying (6) by x^n and summing over $n \ge 2$, we obtain

$$S(x,u,v) = vx + ux\frac{\partial}{\partial u}S(x,u,v) + xS(x,u,v) + \frac{uvx}{1-u}(S(x,u,v) - S(ux,1,v)).$$

Let $S_r(x,u)$ be the coefficient of v^r in $S(x,u,v)$. Then

$$S_r(x,u) = x\delta_{r=1} + ux\frac{\partial}{\partial u}S_r(x,u) + xS_r(x,u) + \frac{ux}{1-u}(S_{r-1}(x,u) - S_{r-1}(ux,1)).$$

Define $\tilde{S}_r(x,u) = \sum_{n\ge1}[x^n](S_r(x,u))\frac{x^n}{n!}$. Then, we have the following result.

Proposition 4.1. *For all* $r \ge 0$,

$$\frac{\partial}{\partial x}\tilde{S}_r(x,u) = \delta_{r=1} + u\frac{\partial}{\partial u}\tilde{S}_r(x,u) + \tilde{S}_r(x,u) + \frac{u}{1-u}(\tilde{S}_{r-1}(x,u) - \tilde{S}_{r-1}(ux,1)).$$

For instance, by using the fact that $\tilde{S}_0(x,u)=0$, then $\frac{\partial}{\partial u}\tilde{S}_1(x,u)=1+u\frac{\partial}{\partial u}\tilde{S}_1(x,u)+\tilde{S}_1(x,u)$, which leads to $\tilde{S}_1(x,u)=e^x-1$. Moreover,

$$\frac{\partial}{\partial u}\tilde{S}_2(x,u)=1+u\frac{\partial}{\partial u}\tilde{S}_2(x,u)+\tilde{S}_2(x,u)+\frac{u}{1-u}(e^x-e^{ux}),$$

which leads to

$$S_2(x,u)=u\int_0^x\frac{e^{ute^{x-t}+2x-2t}+e^{x-t}}{ue^{x-t}-1}dt.$$

To study the total of the number of strict records, as before, we extend the approach of the previous two sections.

Theorem 4.1. *The total number of strict records in inversion sequences of length $n\geq 2$ is given by*

$$S(n):=\frac{3}{2}n!+n!\sum_{r=3}^{n}\left(\frac{1}{r}+\frac{r-2}{r-1}\frac{1}{r}+\frac{1}{r!}\sum_{s=2}^{r-2}(s-1)!(s-1)^{r-s}\right) \tag{7}$$

and $S(1)=1$.

Proof. Let a part of size s be a weak record in position r of the inversion sequence of length n. Then $1\leq s\leq r$ and $1\leq r\leq n$.

Case 1: $r=1$**.** There are $n!$ such instances.

Case 2: $r=2$**.** There are $\frac{n!}{2}$ such instances again. These come from $s=2$ in position 2.

Case 3: $r\geq 3$**.**

Case 3a: $r=s$, the solid and dashed columns coincide. See Figure 2. All possible parts in the shaded triangle to the left of r are strictly less than r and therefore must be counted. The total number of such instances is $(r-1)!$. On the right of r, we count all possible entries in an inversion sequence, i.e., $n(n-1)(n-2)\ldots(r+1)$. The number of such instances is $\frac{n!}{r!}$. Together this yields a total for this case of

$$\frac{n!}{r}.$$

Case 3b: $r=s+1$, the solid column is adjacent to the dashed column. All possible parts in the shaded triangle to the left of the dashed column of Figures 1 are less than s and therefore must be counted. The s-th column can have a maximum height of $s-1$. The total number of such instances is $(s-1)!(s-1)=(r-2)!(r-2)$. Similarly, on the right of r, all parts must be counted yielding again $\frac{n!}{r!}$. So the total number of such instances is therefore

$$\frac{r-2}{r-1}\frac{n!}{r}.$$

Case 3c: $r>s+1$, column r solid is to the right of the dashed column, but not touching. Again, all parts to the left of s in Figures 1 that are strictly less than s must be counted. Including the $s-1$ possibilities for the s-th column, the total number of such instances is $(s-1)!(s-1)$. Again, on the right of r, all parts must be counted as before. The number of such instances is

$$\frac{n!}{r!}.$$

Finally, the number of instances formed by all possible parts lying in the rectangle between s and r and less than s, is the number of words on the alphabet $[s-1]$ with number of parts equal to the rectangle width, i.e., $(s-1)^{r-1-s}$. The product of all the instances for this case is then summed over all possible s values yielding

$$\frac{n!}{r!}\sum_{s=2}^{r-2}(s-1)(s-1)!(s-1)^{r-1-s}.$$

Adding all the cases together, we obtain the formula as stated in the theorem. □

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