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Research Article

On chromatic vertex stability of 3-chromatic graphs with maximum degree 4

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Abstract

The (independent) chromatic vertex stability ($\mathrm{ivs}_\chi(G)$) $\mathrm{vs}_\chi(G)$ is the minimum size of (independent) set $S \subseteq V(G)$ such that $\chi(G - S) = \chi(G) - 1$. The question of how large must the chromatic number $\chi(G)$ of a graph G be, in terms of the maximum degree $\Delta(G)$, to ensure the equality $\mathrm{ivs}_\chi(G) = \mathrm{vs}_\chi(G)$ was raised by Akbari et al. [*European J. Combin.* **102** (2022) #103504]; the authors showed that $\mathrm{ivs}_\chi(G) = \mathrm{vs}_\chi(G)$ if $\chi(G) \in \{\Delta(G), \Delta(G)+1\}$, and also pointed out to graphs with $\chi(G) \le (\Delta(G)+1)/2$ for which $\mathrm{ivs}_\chi(G) > \mathrm{vs}_\chi(G)$. In the light of their findings, they raised the following problem: Is it true that $\chi(G) \ge \Delta(G)/2 + 1$ always implies $\mathrm{ivs}_\chi(G) = \mathrm{vs}_\chi(G)$? This threshold question was recently answered in the negative by Cambrie et al. [*arXiv*: 2203.13833v1, (2022)]. In this paper, we show that the smallest instance for counterexamples is the case $(\chi(G), \Delta(G)) = (3, 4)$, with the smallest possible order being 9 (and there are 30 such graphs). We construct exponentially many graphs G having $\Delta(G) = 4$, $\chi(G) = 3$, $\mathrm{ivs}_\chi(G) = 3$, and $\mathrm{vs}_\chi(G) = 2$.

Keywords: chromatic vertex stability; independent chromatic vertex stability; chromatic number; maximum degree.

2020 Mathematics Subject Classification: 05C15.

1. Introduction

Let G be a graph. Its edge stability number, $\mathrm{es}_\chi(\mathrm{G})$, is the minimum number of edges whose deletion results in a graph H with $\chi(H) = \chi(G) - 1$. The edge stability number was introduced in 1980 by Staton [9], and rediscovered in 2008 by Arumugam, Hamid, and Muthukamatchi [4]. For recent results on this invariant, see e.g. [1,3,6,8].

General concept of stability number appeared in [5], but the first paper on chromatic vertex stability number was written by Akbari, Beikmohammadi, Klavžar, and Movarraei in 2021, see [2]. The chromatic vertex stability $\mathrm{vs}_\chi(G)$ of G is the minimum number of vertices of G such that their deletion results in a graph H with $\chi(H) = \chi(G) - 1$. Analogously, the independent chromatic vertex stability $\mathrm{ivs}_\chi(G)$ of G is the minimum number of independent vertices of G such that their deletion results in a graph H with $\chi(H) = \chi(G) - 1$. Obviously, $\mathrm{vs}_\chi(G) \le \mathrm{ivs}_\chi(G)$. The main result of [2] is the following.

Theorem 1.1. *If G is a graph with $\chi(G) \in \{\Delta(G), \Delta(G)+1\}$ then $\mathrm{vs}_\chi(G) = \mathrm{ivs}_\chi(G)$.*

The authors defined the threshold function $f(\Delta)$ as the smallest quantity such that, for any graph G of maximum degree Δ, it must hold that $\mathrm{vs}_\chi(G) = \mathrm{ivs}_\chi(G)$ provided that $\chi(G) \ge f(\Delta)$. Notice that Theorem 1.1 asserts $f(\Delta) \le \Delta$. They also showed that as soon as $\chi(G) \le (\Delta(G)+1)/2$ the equality $\mathrm{vs}_\chi(G) = \mathrm{ivs}_\chi(G)$ need no longer be true, and consequently asked the following question (see Problem 3.2 in [2]).

Problem 1.1. *Is it true that $f(\Delta) \le \frac{\Delta(G)}{2} + 1$, that is, if G is a graph with $\chi(G) \ge \frac{\Delta(G)}{2} + 1$ does it then always hold that $\mathrm{vs}_\chi(G) = \mathrm{ivs}_\chi(G)$?*

The question of Problem 1.1 was recently answered in the negative by Cambie et al. [7], who proved that $f(\Delta) = \Delta$ for $3 \le \Delta \le 10$. Additionally, they determined the threshold $f(\Delta)$ to within two values (and indeed sometimes a unique value) for graphs of sufficiently large maximum degree.

In this paper, we focus on the smallest value $\chi(G) = 3$ for which counterexamples to Problem 1.1 exist, i.e. we consider the case when $\chi(G) = 3$ and $\Delta(G) = 4$. A simple 'ladder-like' counterexample on 9 vertices is depicted in Figure 1. Notice that the ladder part of the counterexample can be of any length $4k$ for $k \ge 2$.

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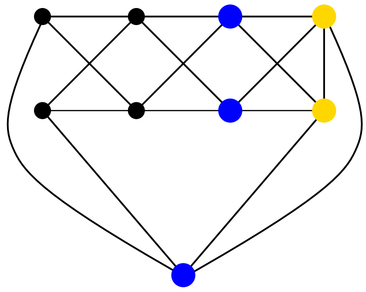

Figure 1: A graph G with $\Delta(G) = 4$ and $\chi(G) = 3$. The blue independent 3-set realizes $\mathrm{ivs}_\chi(G) = 3$ and the yellow 2-set realizes $\mathrm{vs}_\chi(G) = 2$.

Our main result is the following.

Theorem 1.2. *For each $n \geq 9$ there are at least $\max\left\{1, 2^{\lfloor \frac{n-11}{2} \rfloor}\right\}$ planar graphs with $\chi(G) = 3$, $\Delta(G) = 4$, $\mathrm{ivs}_\chi(G) = 3$ and $\mathrm{vs}_\chi(G) = 2$.*

Observe that if $\chi(G) = 3$ and $\Delta(G) = 4$, then $\chi(G) = \frac{\Delta(G)}{2} + 1 = \Delta(G) - 1$. Hence, the bound on $\chi(G)$ in Theorem 1.1 cannot be relaxed when $\chi(G) = 3$.

2. Proofs

We start with a pair of simple observations followed by a couple of lemmas.

Observation 2.1. *For every graph G, $\mathrm{ivs}_\chi(G)$ equals the minimum size of a colour class over all proper $\chi(G)$-colourings of G. Hence $|V(G)| \geq \mathrm{ivs}_\chi(G) \cdot \chi(G)$.*

Proof. Let us first notice that there exists a proper $\chi(G)$-colouring of G with a colour class of size $\mathrm{ivs}_\chi(G)$. Indeed, take $S \subseteq V(G)$ to be an independent set of size $|S| = \mathrm{ivs}_\chi(G)$ such that $\chi(G - S) = \chi(G) - 1$. Use a proper $(\chi(G) - 1)$-colouring of $G - S$ and assign to all vertices of S a new colour. So $\mathrm{ivs}_\chi(G)$ is not less than the minimum size of a colour class over all proper $\chi(G)$-colourings of G.

Contrarily, consider a proper $\chi(G)$-colouring of G which minimizes the size of a colour class, and let S be such a minimum colour class. Then S is an independent subset of $V(G)$ and $\chi(G - S) \leq \chi(G) - 1$. In fact, we must have equality here for otherwise G would admit a proper $(\chi(G) - 1)$-colouring. So $\mathrm{ivs}_\chi(G)$ is also not more than the minimum size of a colour class over all proper $\chi(G)$-colourings of G, which proves our point.

The inequality $|V(G)| \geq \mathrm{ivs}_\chi(G) \cdot \chi(G)$ is now an immediate consequence. □

Observation 2.2. *If $\Delta(G) \leq 2$ then $\mathrm{vs}_\chi(G) = \mathrm{ivs}_\chi(G)$.*

Proof. We may assume that $\mathrm{vs}_\chi(G) \geq 2$. Indeed, if $\mathrm{vs}_\chi(G) = 1$ then obviously $\mathrm{ivs}_\chi(G) = 1$ as well. We may also assume that G is connected. Then G is either a path or an even cycle. In either case

$$\mathrm{vs}_\chi(G) = \mathrm{ivs}_\chi(G) = \left\lfloor \frac{|V(G)|}{2} \right\rfloor.$$

□

As already mentioned, we are interested in finding graphs G for which $\chi(G) \geq \frac{\Delta(G)}{2} + 1$ and $\mathrm{ivs}_\chi(G) > \mathrm{vs}_\chi(G)$. Our first lemma establishes some implications for the order and the considered stability parameters.

Lemma 2.1. *If $\mathrm{ivs}_\chi(G) > \mathrm{vs}_\chi(G)$ and $\chi(G) \geq \frac{\Delta(G)}{2} + 1$ then $|V(G)| \geq 9$, $\mathrm{ivs}_\chi(G) \geq 3$, $\mathrm{vs}_\chi(G) \geq 2$ and $\chi(G) \geq 3$. Moreover, if $|V(G)| = 9$ then $\mathrm{ivs}_\chi(G) = 3$, $\mathrm{vs}_\chi(G) = 2$ and $\chi(G) = 3$.*

Proof. Since $\mathrm{ivs}_\chi(G) > \mathrm{vs}_\chi(G)$, we must have $\mathrm{vs}_\chi(G) \geq 2$ and consequently $\mathrm{ivs}_\chi(G) \geq 3$.

If $\chi(G) \leq 2$ then from $\chi(G) \geq \frac{\Delta(G)}{2} + 1$ we get $\Delta(G) \leq 2(\chi(G) - 1) \leq 2$, which in view of Observation 2.2 contradicts $\mathrm{ivs}_\chi(G) > \mathrm{vs}_\chi(G)$. Hence $\chi(G) \geq 3$.

From the inequality stated in Observation 2.1, it follows that $|V(G)| \geq \mathrm{ivs}_\chi(G) \cdot \chi(G) \geq 3 \cdot 3$, that is, $|V(G)| \geq 9$. And if $|V(G)| = 9$ then $\chi(G) = \mathrm{ivs}_\chi(G) = 3$ and $\mathrm{vs}_\chi(G) = 2$. □

Figure 2 depicts two graphs, respectively denoted by G_9 and G_{10} in regard to their orders. The former one can be obtained from the octahedron graph by subdividing the edges of a triangle. It has $\Delta(G_9) = 4$, $\chi(G_9) = \mathrm{ivs}_\chi(G_9) = 3$ and $\mathrm{vs}_\chi(G_9) = 2$. Observe that $\chi(G_9 - \{x, y\}) = 2$ if and only if $\{x, y\} = \{v_i, v_j\}$, where $1 \le i < j \le 3$, and for an independent set of vertices $\{x, y, z\}$ we have $\chi(G_9 - \{x, y, z\}) = 2$ if and only if $\{x, y, z\} = \{u_i, v_i, w_i\}$, where $1 \le i \le 3$. The graph G_{10} is obtained from G_9 by adding the vertex q and connecting it to w_2 and w_3. It also has $\Delta(G_{10}) = 4$, $\chi(G_{10}) = \mathrm{ivs}_\chi(G_{10}) = 3$ and $\mathrm{vs}_\chi(G_{10}) = 2$. Again $\chi(G_{10} - \{x, y\}) = 2$ if and only if $\{x, y\} = \{v_i, v_j\}$, where $1 \le i < j \le 3$, and for an independent set of vertices $\{x, y, z\}$ we have $\chi(G_{10} - \{x, y, z\}) = 2$ if and only if $\{x, y, z\} = \{u_i, v_i, w_i\}$ where $2 \le i \le 3$.

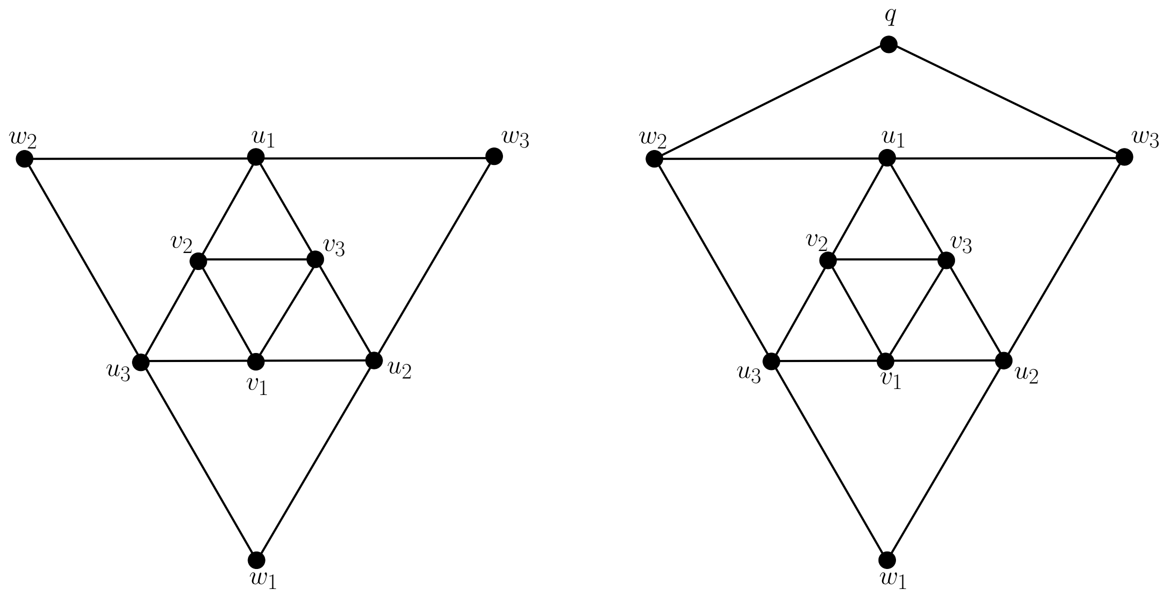

Figure 2: The graph G_9 (left) and the graph G_{10} (right).

Our second lemma concerns the case $\chi(G) = \mathrm{ivs}_\chi(G) = 3$ and $\mathrm{vs}_\chi(G) = 2$. Under the assumption $\chi(G) \ge \frac{\Delta(G)}{2} + 1$ we establish the maximum degree of G and the vertex degrees of every 2-set which realizes $\mathrm{vs}_\chi(G)$.

Lemma 2.2. *Let G be a graph with $\chi(G) = \mathrm{ivs}_\chi(G) = 3$, $\mathrm{vs}_\chi(G) = 2$ and $\chi(G) \ge \frac{\Delta(G)}{2} + 1$. Then $\Delta(G) = 4$ and for every $v_1, v_2 \in V(G)$ such that $\chi(G - \{v_1, v_2\}) = 2$ we have $\deg_G(v_1) = \deg_G(v_2) = 4$.*

Proof. Let G satisfy the assumptions of the lemma and let $v_1, v_2 \in V(G)$ be such that $\chi(G - \{v_1, v_2\}) = 2$. Then

$$\Delta(G) \le 2(\chi(G) - 1) = 4.$$

For argument's sake, suppose that $\deg_G(v_1) \le 3$. Since $\mathrm{ivs}_\chi(G) = 3 > 2$, we have $v_1v_2 \in E(G)$. Further, since $\chi(G - v_2) = 3$, there must be an odd cycle in $G - v_2$; moreover, every such cycle passes through v_1 since $G - \{v_1, v_2\}$ is bipartite. Consequently $\deg_G(v_1) = 3$.

Let u_1, u_2 be the neighbours of v_1 in $G - v_2$. Since every odd cycle in $G - v_2$ passes through both u_1, u_2, we conclude that $G - \{u_i, v_2\}$ is bipartite as well, $i \in \{1, 2\}$. From $\mathrm{ivs}_\chi(G) = 3 > 2$ it follows that $u_1v_2, u_2v_2 \in E(G)$. Moreover, $u_1u_2 \notin E(G)$, for otherwise v_1, v_2, u_1, u_2 induces a K_4, implying $\chi(G) \ge 4$.

Since $\mathrm{ivs}_\chi(G) = 3$, there must be an odd cycle in $G - \{u_1, u_2\}$. This cycle cannot pass through v_1 since $\deg_{G-\{u_1,u_2\}}(v_1) = 1$. If this cycle does not pass through v_2 as well then it is in $G - \{v_1, v_2\}$ which means that $\chi(G - \{v_1, v_2\}) = 3$, a contradiction. Hence, there is an odd cycle passing through v_2 in $G - \{u_1, u_2, v_1\}$, which means that $\deg_G(v_2) \ge 5$. This contradiction settles the lemma. □

A computer search shows that there are precisely 30 graphs G of order 9 and having $\Delta(G) = 4$, $\chi(G) = 3$, $\mathrm{ivs}_\chi(G) = 3$ and $\mathrm{vs}_\chi(G) = 2$. Several of them (including G_9) are planar and four are obtained by adding an edge to another graph from the same collection.

For every $n \ge 9$ let S_n be the set of graphs G on n vertices such that $\Delta(G) = 4$, $\chi(G) = 3$, $\mathrm{ivs}_\chi(G) = 3$ and $\mathrm{vs}_\chi(G) = 2$. Thus $G_9 \in S_9$ and $G_{10} \in S_{10}$. By $C_\chi(G)$ we denote the set of vertices $x \in V(G)$ such that there is some $y \in V(G)$ for which $\chi(G - \{x, y\}) = 2$; note that every such y is a neighbour of x. For example, $C_\chi(G_9) = C_\chi(G_{10}) = \{v_1, v_2, v_3\}$. In view of our next result, for every $n \ge 9$ there is a planar graph in S_n which is topologically equivalent to G_9 or G_{10}.

Proposition 2.1. *Let $G \in S_n$ and $e_1, e_2, \dots, e_t \in E(G)\backslash E([C_\chi(G)])$, i.e., each e_i has at most one endvertex in $C_\chi(G)$. Let $n_1, n_2, \dots, n_t$ be positive even integers. For every i, $1 \le i \le t$, subdivide e_i with n_i new vertices, and denote the resulting graph by H. Then $\Delta(H) = 4$, $\chi(H) = 3$, $\mathrm{ivs}_\chi(H) = 3$ and $\mathrm{vs}_\chi(H) = 2$. In other words, $H \in S_{n+(n_1+\cdots+n_t)}$.*

Proof. Obviously $\Delta(H) = \Delta(G) = 4$. Since $\chi(G) = 3$, the graph G has an odd cycle. Since to any edge of this cycle we added an even number (possibly zero) of vertices, H also has an odd cycle; thus $\chi(H) \ge 3$. Moreover, if $S \subseteq V(G)$ is such that $G - S$ is bipartite then $H - S$ is bipartite as well. Hence, $\chi(H) = 3$, $\mathrm{ivs}_\chi(H) \le 3$ and $\mathrm{vs}_\chi(H) \le 2$.

If there is some $v \in V(H)$ such that $\chi(H - v) = 2$, then $v \notin V(G)$. So v is obtained by subdividing an edge, say xy, of G. However, as $H - v$ is bipartite, both $G - x$ and $G - y$ are bipartite, a contradiction. Hence, $\mathrm{vs}_\chi(H) = 2$.

Finally, let us show that $\mathrm{ivs}_\chi(H) = 3$. Supposing the opposite, there are $u, v \in V(H)$ such that $\chi(H - \{u, v\}) = 2$ and $uv \notin E(H)$. It cannot be that both u and v are in $V(G)$, because we did not subdivide edges connecting vertices of $C_\chi(G)$. So we may assume that v is obtained by subdividing an edge xy of G, where $y \notin C_\chi(G)$. Since $H - \{u, v\}$ is bipartite, so is $H - \{u, y\}$. But then u cannot be a vertex of G as well. Hence, u is obtained by subdividing an edge wz of G. As $H - \{u, y\}$ is bipartite, so is $H - \{z, y\}$. However, this contradicts the fact that $y \notin C_\chi(G)$. □

From Proposition 2.1 we deduce that $S_n \neq \emptyset$ for every $n \ge 9$. Indeed, if n is odd then take G_9, subdivide the edge u_2w_1 with $n - 9$ new vertices and denote the resulting graph by G_n. Analogously if n is even then take G_{10}, subdivide the edge u_2w_1 with $n - 10$ new vertices and denote the resulting graph by G_n. Then G_n is a connected planar graph and $G_n \in S_n$, by Proposition 2.1. Our next result shows that S_n contains exponentially many planar graphs.

Theorem 2.1. *For each $n \ge 11$ there are at least $2^{\lfloor \frac{n-11}{2} \rfloor}$ 2-connected planar graphs in S_n.*

Proof. In view of G_{11} and G_{12}, we assume $n \ge 13$. Take G_n and relabel the vertices of the $u_2 - u_3$ path that passes through w_1 by $u_2 = a_0, a_1, a_2, \dots, a_{\ell-1} = w_1, a_\ell = u_3$; here $\ell = n - 7$ if n is odd and $\ell = n - 8$ if n is even. Note that ℓ is even. Let $E_n = \{a_1a_{\ell-2}, a_2a_{\ell-3}, \dots, a_{\ell/2-2}a_{\ell/2+1}\}$. For every $E' \subseteq E_n$, denote by $H_{n,E'}$ the graph obtained from G_n by adding the edges of E'. Obviously $H_{n,E'}$ is planar, $\Delta(H_{n,E'}) = 4$ and $\chi(H_{n,E'}) = 3$. Moreover, $H_{n,E'} - \{x, y\}$ is bipartite if $\{x, y\} = \{v_i, v_j\}$ where $1 \le i < j \le 3$, which implies that $\mathrm{vs}_\chi(H_{n,E'}) = 2$. Also, $H_{n,E'} - \{u_2, v_2, w_2\}$ is bipartite which gives $\mathrm{ivs}_\chi(H_{n,E'}) \le 3$. On the other hand, since G_n is a subgraph of $H_{n,E'}$ of certain chromaticity and $\mathrm{ivs}_\chi(G_n) = 3$, we have $\mathrm{ivs}_\chi(H_{n,E'}) = 3$ as well. Thus $H_{n,E'} \in S_n$.

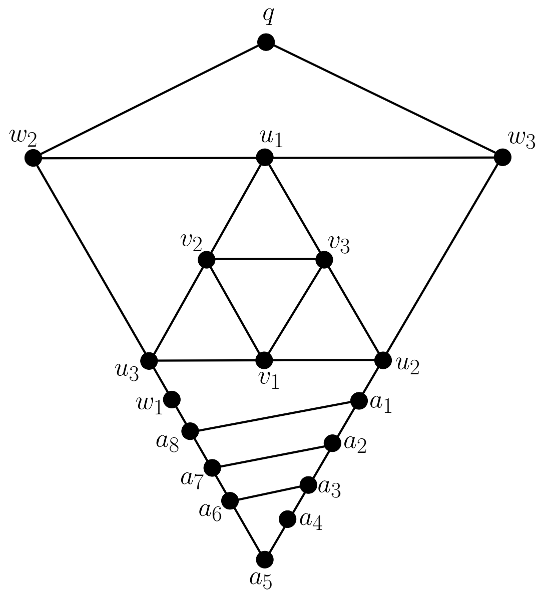

Figure 3: The graph $H_{18,E_{18}}$.

Let $E', E^* \subseteq E_n$, where $E' \neq E^*$. We show that the graphs $H_{n,E'}$ and H_{n,E^*} are not isomorphic. This is obvious if $|E'| \neq |E^*|$. So assume that $|E'| = |E^*| \ge 1$. We show that $\mathrm{Aut}(H_{n,E'})$, the group of automorphisms of $H_{n,E'}$, (and also $\mathrm{Aut}(H_{n,E^*})$) is trivial. That is, every automorphism of $H_{n,E'}$ fixes all the vertices of $H_{n,E'}$.

There are exactly 6 vertices of degree 4 in $H_{n,E'}$, namely $u_1, u_2, u_3, v_1, v_2, v_3$. Since each of u_1, u_2, u_3 is in only one triangle in $H_{n,E'}$ whereas each of v_1, v_2, v_3 is in three such triangles, every automorphism must preserve the sets $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$. The vertices u_1 and u_2 are both adjacent to the same vertex of $H_{n,E'} - \{v_1, v_2, v_3\}$. Also the vertices u_1 and u_3 are both adjacent to the same vertex of $H_{n,E'} - \{v_1, v_2, v_3\}$. But u_2 and u_3 are not adjacent to the same vertex of $H_{n,E'} - \{v_1, v_2, v_3\}$, because $n \ge 13$. Consequently, every automorphism of $H_{n,E'}$ fixes u_1.

In view of $a_{\ell-1}(= w_1)$, the vertex u_3 has a neighbour of degree 2 which is not connected to u_1. If u_2 does not have such a neighbour, then every automorphism of $H_{n,E'}$ fixes also u_2 and u_3. So assume that also u_2 has a neighbour of degree 2 which is not connected to u_1. Now start at u_2, proceed with the above mentioned neighbour of u_2 and construct a longest path P_2, interior vertices of which have all degree 2. Analogously start at u_3, proceed with the above mentioned neighbour of u_3 and construct a longest path P_3, interior vertices of which have all degree 2. Finally, let i be the smallest index such that $a_i a_{\ell-1-i} \in E'$. Then P_2 has length i while P_3 has length $i+1$. Hence, every automorphism of $H_{n,E'}$ must fix also u_2 and u_3. Consequently, every automorphism of $H_{n,E'}$ fixes all the vertices of $H_{n,E'}$, and so $H_{n,E'}$ and H_{n,E^*} are not isomorphic graphs.

Since E_n has $\frac{\ell}{2} - 2 = \lfloor\frac{n-7}{2}\rfloor - 2 = \lfloor\frac{n-11}{2}\rfloor$ edges and every subset gives different graph, there are exactly $2^{\lfloor\frac{n-11}{2}\rfloor}$ nonisomorphic graphs $H_{n,E'}$. □

Remark. *Observe that considering subsets of a zig-zag path $a_{l-1}, a_1, a_{l-2}, a_2, \ldots$ instead of set of isolated edges, one can obtain 2^{n-11} graphs satisfying the assumptions of Theorem 2.1, since the obtained graphs G have $|\mathrm{Aut}(G)| \le 2$, and still different subsets distinguish them.*

We conclude the paper by presenting another, more general, construction of graphs G with $\Delta(G) = 4$, $\chi(G) = 3$, $\mathrm{ivs}_\chi(G) = 3$ and $\mathrm{vs}_\chi(G) = 2$. Let H be a bipartite graph with $\Delta(H) \le 4$ such that there exists a cycle $C_{2k} \subseteq H$ with $k \ge 3$ and a pair $a, b \in V(C_{2k})$ of non-adjacent vertices in H on odd distance $d_H(a,b)$ and having $\deg_H(a) = \deg_H(b) = 2$. Take the union of H with a disjoint triangle uvw and add the edges av, bv, aw, bw. Denote the resulting graph by G (see Figure 4).

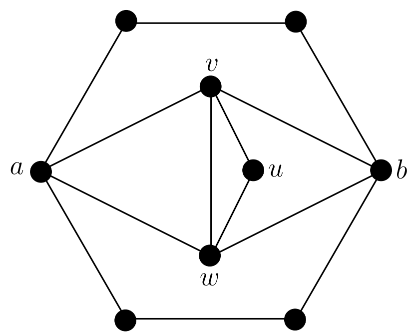

Figure 4: The graph G if $H = C_6$.

Proposition 2.2. *If H is of order m then $G \in S_{m+3}$. Moreover, if H is 2-connected (resp. planar) then G has the same property.*

Proof. Clearly, the order of G is $m+3$. As $\Delta(H) \le 4$ and $\deg_H(a) = \deg_H(b) = 2$ we have $\Delta(G) = 4$. In view of the triangle uvw, the graph G is not bipartite. In order to show $\chi(G) = 3$, note that $G - \{u,v,w\}$ is bipartite. Take a proper 2-colouring φ of $G - \{u,v,w\}$ with colours 1 and 3 such that, without loss of generality, $\varphi(a) = 1$ and $\varphi(b) = 3$ (here we use that $d(a,b)$ is odd). Now change the colour of b to 1 and the colour of every $c \in N_{G-\{u,v,w\}}(b)$ to 2. Note that by assigning the colour 1 to u, the colour 2 to v and the colour 3 to w we obtain a proper 3-colouring of G.

Let us show next that $\mathrm{ivs}_\chi(G) = 3$. Since $G - \{a,b,u\}$ is bipartite, we have $\mathrm{ivs}_\chi(G) \le 3$. Suppose there are non-adjacent vertices x, y such that $G - \{x,y\}$ is bipartite. In view of the triangle uvw, the intersection $\{x,y\} \cap \{u,v,w\}$ is a singleton. We argue that this intersection is not the vertex u due to the triangles avw and bvw. Let P and Q be the two $a-b$ paths in C_{2k}, and recall that both these paths are of odd lengths. Consequently, each of the cycles $C' = P \cup avb$, $C'' = P \cup awb$, $C''' = Q \cup avb$, and $C'''' = Q \cup awb$ is odd. Hence $\{x,y\} \cap \{v,w\} \neq \emptyset$, which further implies that $\{x,y\} \cap \{a,b\} = \emptyset$. However, then at least one of the cycles C', C'', C''', C'''' appears in $G - \{x,y\}$. The obtained contradiction shows $\mathrm{ivs}_\chi(G) = 3$.

Finally, we prove that $\mathrm{vs}_\chi(G) = 2$. Clearly $\mathrm{vs}_\chi(G) \le 2$, because $G - \{v,w\}$ is bipartite. And since $\mathrm{vs}_\chi(G) = 1$ implies $\mathrm{ivs}_\chi(G) = 1$, we have $\mathrm{vs}_\chi(G) = 2$. □

Remark. *Note in passing that the order of the bound $|S_n| \ge 2^{\lfloor\frac{n-11}{2}\rfloor}$ obtained in Theorem 2.1 is not (asymptotically) optimal. Propositions 2.1 and 2.2 enable one to construct connected planar graphs within S_n with considerable ease. However, establishing a more precise asymptotic estimate of $|S_n|$ was not the focus of this short article; instead, the aim was simply to point out to the existence of exponentially many graphs in S_n.*

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