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Research Article

Families of specialized Euler sums

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Abstract

A family of Euler sums is investigated that adds a new important class to the vast literature of existing knowledge of representation of Euler sums in terms of well-known special functions such as the Riemann zeta and Dirichet beta functions. Some examples are given to highlight the obtained theorems.

Keywords: Euler sums; polygamma functions; Riemann zeta function.

2020 Mathematics Subject Classification: 11M06, 11M35, 26B15, 33B15, 42A70.

1. Introduction

The study of Euler sums has its beginnings in the works of Euler [5, 6, 8]. In 1644, Mengoli was among the first to study the sum of the reciprocal of the square of the natural numbers

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots.$$

Euler's brilliance and insight eventually led him to the solution of the Basel problem

$$\zeta(2)=\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots=\sum_{n\geq 1}\frac{1}{n^2}.$$

Euler's enhanced insight and application is clear when he found the surprising result

$$\zeta(2k)=\sum_{n\geq 1}\frac{1}{n^{2k}}=\frac{(-1)^{k+1}\,2^{2k-1}\pi^{2k}}{(2k)!}B_{2k}.$$

Here k is a positive integer and B_j are the Bernoulli numbers defined by

$$\sum_{j\geq 0}\frac{B_j t^j}{j!}=\frac{t}{e^t-1},\quad (|t|<2\pi).$$

Remark 1.1. *Let $f(t)$ be the above generating function of Bernoulli numbers. Since $\lim_{t\to 0} f(t)=1, f(t)$ is analytic at $t=0$. Also*

$$e^t-1=0\Leftrightarrow\ t=2k\pi,\ \ (k=0,\pm 1,\pm 2,\cdots),$$

which implies that $f(t)$ has simple poles at $t=2k\pi,\ \ (k=\pm 1,\pm 2,\cdots)$. Therefore $f(t)$ is analytic in an open disk of radius 2π centered at $t=0$. Hence the above Maclaurin series expansion of $f(t)$ is available and so $B_j=f^{(j)}(0),(j=0,1,2,3,\cdots)$.

Euler [6] gave a list of formula which for $q\in\mathbb{N}\backslash\{1\}$ can be written as

$$S_{1,q}^{++}=\sum_{n\geq 1}\frac{H_n}{n^q}=\frac{q+2}{2}\zeta(q+1)-\frac{1}{2}\sum_{j=1}^{q-2}\zeta(q-j)\,\zeta(j+1)$$

where the harmonic numbers $H_n^{(p)},\ p\in\mathbb{N},\ \ H_n^{(p)}=\sum_{j=1}^{n}\frac{1}{j^p}$ are the finite versions of the Riemann zeta function $\zeta(p)=\sum_{j\geq 1}\frac{1}{j^p}$ and its alternating version $\eta(p)=\sum_{j\geq 1}\frac{(-1)^{n+1}}{j^p}$. Here we define the set of natural numbers $\mathbb{N}:=\{1,2,3,...\},\ \mathbb{N}_0:=\{0,1,2,3,...\}=\mathbb{N}\cup\{0\},\ \mathbb{Z}^-:=\{-1,-2,-3,...\}=\mathbb{Z}_0^-\setminus\{0\}$. The usual notation applies for $\mathbb{C}$,

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the set of complex numbers, $\mathbb{R}$ the set of real numbers and $\mathbb{R}^+$ the set of positive numbers. Significant further developments in Euler sums lay dormant for about a century and eventually Nielsen [9], Bailey et al. [2], Borwein et al. [3], Flajolet and Salvy [7], Sitaramanchandrarao [11] and others supplemented and extended the works of Euler. Nielsen and later Borwein gave closed form expressions of $S^{++}_{p,q} = \sum\limits_{n\geq 1} \frac{H^{(p)}_n}{n^q}$ for $q \in \mathbb{N}\backslash\{1\}, p \in \mathbb{N}$. Sitaramanchandrarao gave an expression for

$$S^{+-}_{1,q} = \sum_{n\geq 1} \frac{(-1)^{n+1} H_n}{n^q}.$$

For $p+q$ an odd weight, Flajolet and Salvy gave an identity for

$$S^{+-}_{p,q} = \sum_{n\geq 1} \frac{(-1)^{n+1} H^{(p)}_n}{n^q}$$

and recently, Alzer and Choi [1] published a result for, $p \in \mathbb{N}$,

$$S^{+-}_{p,1} = \sum_{n\geq 1} \frac{(-1)^{n+1} H^{(p)}_n}{n}.$$

Let us now define $h^{(p)}_n = \sum\limits_{j=1}^{n} \frac{1}{(2j-1)^p}$, $p \in \mathbb{C}, n \in \mathbb{C} \geq 1$, then $h^{(p)}_n = H^{(p)}_{2n} - 2^{-p} H^{(p)}_n$ and we are interested in investigation of the variant Euler sums

$$S^{++}_{p,q}(0,a) = \sum_{n\geq 1} \frac{H^{(p)}_n(0)}{(n+a)^q} \tag{1}$$

where $H^{(p)}_n(0) = H^{(p)}_n$ and $a \in \mathbb{C}\backslash\{-1,-2,-3,....\}$. An identity for $S^{++}_{1,q}(0,a)$ has recently been established by Sofo and Cvijovic [17]. For $q \geq 2$, and $(p+q)$ an odd weight, Nimbran and Sofo [10] gave an identity for

$$\sum_{n\geq 1} \frac{h^{(p)}_n}{\left(n-\frac{1}{2}\right)^q}$$

from which one may extract an identity for

$$S^{++}_{p,q}\left(0,-\frac{1}{2}\right) = \sum_{n\geq 1} \frac{H^{(p)}_n}{\left(n-\frac{1}{2}\right)^q}.$$

Later in [23] Xu and Wang published results for $S^{++}_{1,q}\left(0,-\frac{1}{2}\right)$ and $S^{++}_{p,q}\left(0,-\frac{1}{2}\right)$ and considered the more general sum

$$T_{p_1,p_2,p_3,...p_k,q} = \sum_{n\geq 1} \frac{h^{(p_1)}_n, h^{(p_2)}_n, h^{(p_3)}_n, ..., h^{(p_k)}_n}{\left(n-\frac{1}{2}\right)^q}, q \geq 2.$$

In this paper we give a direct proof to the identity $S^{++}_{p,q}(0,a) = \sum\limits_{n\geq 1} \frac{H^{(p)}_n}{(n+a)^q}$ and then develop a (presumably) new identity for

$$T^{++}_{p,q}(a,t) = \sum_{n\geq 1} \frac{n^t H^{(p)}_n}{(n+a)^q},$$

where $t \in \mathbb{N}_0, p \in \mathbb{N}, q \in \mathbb{N} \geq t+2$ and $a \in \mathbb{C}\backslash\{-1,-2,-3,....\}$. For example, we evaluate the closed form

$$8\sum_{n\geq 1} \frac{nH^{(2)}_n}{(2n+1)^3} = 32\text{Li}_4\left(\frac{1}{2}\right) + \frac{4}{3}\ln^4 2 - 8\zeta(2)\ln^2 2 - \frac{121}{4}\zeta(4) + 28\zeta(3)\ln 2 - \frac{49}{2}\zeta(2)\zeta(3) + \frac{93}{2}\zeta(5). \tag{2}$$

We recall the harmonic numbers

$$H_n = \sum_{j=1}^{n} \frac{1}{j} = \gamma + \psi(n+1)$$

where γ is the familiar Euler Mascheroni constant and for complex values of z, $z \in \mathbb{C}\backslash\{0,-1,-2,\cdots\}$, $\psi(z)$ is the digamma (or psi) function defined by

$$\psi(z) := -\frac{d}{dz}\{\log\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)},$$

where $\Gamma(z)$ is the gamma function. In some examples that follow, we encounter the Clausen function where the generalized Clausen functions are defined for $z \in \mathbb{C}$ with $\Re(z) > 1$ as,

$$S_z(x) = \sum_{k\geq 1} \frac{\sin(kx)}{k^z}, \quad C_z(x) = \sum_{k\geq 1} \frac{\cos(kx)}{k^z}$$

and may be extended to all the complex plane through analytic continuation. When z is replaced by a non negative integer n, the standard Clausen functions are defined by the Fourier series

$$\mathrm{Cl}_n(x) = \begin{cases} \sum_{k\geq 1} \frac{\sin(kx)}{k^n}, & \text{for } n \text{ even}, \\ \sum_{k\geq 1} \frac{\cos(kx)}{k^n}, & \text{for } n \text{ odd}. \end{cases}$$

The polylogarithm function $\mathrm{Li}_p(z)$ is, for $|z| \leq 1$,

$$\mathrm{Li}_p(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^p} \tag{3}$$

and in terms of the Polylogarithm,

$$\mathrm{Cl}_n(\theta) = \begin{cases} \frac{i}{2}\left(\mathrm{Li}_n\left(e^{-i\theta}\right) - \mathrm{Li}_n\left(e^{i\theta}\right)\right), & \text{for even } n, \\ \frac{1}{2}\left(\mathrm{Li}_n\left(e^{-i\theta}\right) + \mathrm{Li}_n\left(e^{i\theta}\right)\right), & \text{for odd } n. \end{cases}$$

The polygamma function

$$\psi^{(k)}(z) = \frac{d^k}{dz^k}\{\psi(z)\} = (-1)^{k+1} k! \sum_{r=0}^{\infty} \frac{1}{(r+z)^{k+1}} \tag{4}$$

has the recurrence

$$\psi^{(k)}(z+1) = \psi^{(k)}(z) + \frac{(-1)^k k!}{z^{k+1}}$$

and can be connected to the Clausen function in the following way. The Clausen function of rational argument and even integer order is

$$\mathrm{Cl}_{2m}\left(\frac{\pi p}{q}\right) = \sum_{k\geq 1} \frac{\sin\left(\frac{k\pi p}{q}\right)}{k^{2m}}$$

and if p is an odd integer, then

$$\mathrm{Cl}_{2m}\left(\frac{\pi p}{q}\right) = \frac{(2q)^{-2m}}{(2m-1)!} \sum_{j=1}^{q} \sin\left(\frac{j\pi p}{q}\right)\left(\psi^{(2m-1)}\left(\frac{j}{2q}\right) - \psi^{(2m-1)}\left(\frac{j+q}{2q}\right)\right), \tag{5}$$

and if p is an even integer, then

$$(2m-1)!\,(q)^{2m}\,\mathrm{Cl}_{2m}\left(\frac{\pi p}{q}\right) = \sum_{j=1}^{q} \sin\left(\frac{j\pi p}{q}\right)\psi^{(2m-1)}\left(\frac{j}{q}\right). \tag{6}$$

There exists a large number of research papers exploring the representation, analysis and specific evaluations of Euler sums, see [16, 20, 21]. Some pertinent papers dealing with Euler sums are [12, 14, 19] and the excellent books [20, 22]. Many specific cases of the type (7) may be represented in terms of special functions such as the Riemann zeta function, the Clausen function and the polygamma functions. The papers [13, 15, 17, 18] also examined some representations of Euler sums.

2. Main results

Theorem 2.1. *If* $a \in \mathbb{C}\backslash\{-1,-2,-3,...\}$, $(p,q) \in \mathbb{N}, t \in \mathbb{N}_0$ *with* $p \geq 2,\ q \geq t+2$, *then*

$$T_{p,q}^{++}(a,t) = \sum_{n\geq 1} \frac{n^t H_n^{(p)}}{(n+a)^q} \tag{7}$$

$$= \sum_{j=0}^{t} (-1)^{j+t+1} \binom{t}{j} a^{t-j} \left(\sum_{k\geq 1} \frac{H_{k+a-1}^{(q-j)}}{k^p} - \zeta(q-j)\,\zeta(p)\right) \tag{8}$$

where $H_n^{(p)}$ *are harmonic numbers of order* p *and* $\zeta(\cdot)$ *are the Riemann zeta functions.*

Proof. We can write

$$T_{p,q}^{++}(a,t)=\sum_{n\geq 1}\frac{n^t H_n^{(p)}}{(n+a)^q}=\sum_{n\geq 1}\frac{n^t}{(n+a)^q}\sum_{k=1}^{n}\frac{1}{k^p} \tag{9}$$

From [4], one notes the manipulation of a double series in the form

$$\sum_{n\geq 0}\sum_{k=0}^{n}\Omega_{k,n}=\sum_{n\geq 0}\sum_{k\geq 0}\Omega_{k,n+k}$$

one can rewrite (9) as

$$\begin{aligned}\sum_{n\geq 1}\frac{n^t H_n^{(p)}}{(n+a)^q}&=\sum_{n\geq 1}\frac{n^t}{(n+a)^q}\sum_{k=1}^{n}\frac{1}{k^p}=\sum_{k\geq 1}\frac{1}{k^p}\sum_{n\geq 0}\frac{(n+k)^t}{(n+a+k)^q}\\&=\sum_{k\geq 1}\frac{(-1)^q}{k^p}\sum_{j=0}^{t}\binom{t}{j}a^{t-j}\frac{\psi^{(q-1-j)}(a+k)}{(q-1-j)!},\end{aligned} \tag{10}$$

where the polygamma functions are defined for $a\in\mathbb{C}\setminus\{-1,-2,-3,...\}$. Using the identity, relating the Polygamma function to the harmonic numbers

$$H_{\rho}^{(q+1)}=\zeta(q+1)+\frac{(-1)^q}{q!}\psi^{(q)}(p+1)$$

we get from (10)

$$\begin{aligned}\sum_{n\geq 1}\frac{n^t H_n^{(p)}}{(n+a)^q}&=\sum_{j=0}^{t}\binom{t}{j}a^{t-j}\sum_{k\geq 1}\frac{(-1)^q\,\psi^{(q-1-j)}(a+k)}{k^p(q-1-j)!}\\&=\sum_{j=0}^{t}(-1)^{j+t+1}\binom{t}{j}a^{t-j}\left(\sum_{k\geq 1}\frac{H_{k+a-1}^{(q-j)}}{k^p}-\zeta(q-j)\zeta(p)\right)\end{aligned}$$

and Theorem 2.1 is proved. □

The next corollary deals with the special case of $t=0$ for the representation of the sum in (7).

Corollary 2.1. *If* $a\in\mathbb{C}\setminus\{-1,-2,-3,...\}$, $(p,q)\in\mathbb{N},t=0$ *with* $p\geq 2,\ q\geq 2$, *then*

$$\begin{aligned}T_{p,q}^{++}(a,0)&=\sum_{n\geq 1}\frac{H_n^{(p)}}{(n+a)^q}\\&=\zeta(q)\zeta(p)-\sum_{k\geq 1}\frac{H_{k+a-1}^{(q)}}{k^p}.\end{aligned} \tag{11}$$

Proof. Follows directly from (2.1). □

Corollary 2.2. *If* $a=\frac{1}{2}$, $(p,q)\in\mathbb{N}$ *with* $p\geq 2,\ q\geq 2$, *then*

$$\begin{aligned}\frac{1}{2^q}T_{p,q}^{++}\left(\frac{1}{2},0\right)&=\frac{1}{2^q}\sum_{n\geq 1}\frac{H_n^{(p)}}{(n+\frac{1}{2})^q}\\&=\zeta(p)\left(\frac{1}{2^q}\zeta(q)+\eta(q)\right)+2^{p-1}S_{q,p}^{+-}+\left(\frac{1}{2^q}-2^{p-1}\right)S_{q,p}^{++}\end{aligned}$$

where $\eta(q)$ *is the Dirichlet eta, or the alternating zeta function and* $S_{q,p}^{++}$, $S_{q,p}^{+-}$ *are defined below in the proof.*

Proof. For $p\geq 1$ and $q\geq 2$, we utilize the following notation

$$S_{p,q}^{++}(\alpha,\beta)=\sum_{n\geq 1}\frac{H_n^{(p)}(\alpha)}{(n+\beta)^q},\ S_{p,q}^{+-}(\alpha,\beta)=\sum_{n\geq 1}\frac{(-1)^{n+1}H_n^{(p)}(\alpha)}{(n+\beta)^q}$$

where

$$\zeta(p,\alpha)=H_n^{(p)}(\alpha)=\sum_{j=1}^{n}\frac{1}{(n+\alpha)^p},n\in\mathbb{N},p\in\mathbb{C},\alpha\in\mathbb{C}\setminus\{-1,-2,-3,....\}.$$

Consider from Corollary 2.1,

$$T^{++}_{p,q}\left(\frac{1}{2},0\right)=S^{++}_{p,q}\left(0,\frac{1}{2}\right)=\zeta(q)\,\zeta(p)-\sum_{n\geq 1}\frac{H^{(q)}_{n-\frac{1}{2}}}{n^p}$$

and from the double argument of the polygamma function [14]

$$H^{(q)}_{2n}=\eta(q)+\frac{1}{2^q}H^{(q)}_n+\frac{1}{2^q}H^{(q)}_{n-\frac{1}{2}}$$

we have

$$\begin{aligned}S^{++}_{p,q}\left(0,\frac{1}{2}\right) &= \zeta(q)\,\zeta(p)-\sum_{n\geq 1}\frac{2^q}{n^p}\left(H^{(q)}_{2n}-\eta(q)-\frac{1}{2^q}H^{(q)}_n\right)\\ &= \zeta(q)\,\zeta(p)+2^q\eta(q)\,\zeta(p)+\sum_{n\geq 1}\frac{H^{(q)}_n}{n^p}-2^q\sum_{n\geq 1}\frac{H^{(q)}_{2n}}{n^p}\\ &= \zeta(q)\,\zeta(p)+2^q\eta(q)\,\zeta(p)+S^{++}_{q,p}-2^{q+p-1}\left(S^{++}_{q,p}-S^{+-}_{q,p}\right),\end{aligned}$$

and therefore

$$S^{++}_{p,q}\left(0,\frac{1}{2}\right)=\zeta(p)\left(\frac{1}{2^q}\zeta(q)+\eta(q)\right)+2^{p-1}S^{+-}_{q,p}+\left(\frac{1}{2^q}-2^{p-1}\right)S^{++}_{q,p}.$$

As noted earlier this result, in a modified form, was proved by Nimbran and Sofo [10] and later by Xu and Wang [23]. □

Remark 2.1. *Similar analysis allows us to evaluate* $T^{++}_{p,q}\left(\frac{1}{2},t\right)$, *so that after some simplification we have*

$$\begin{aligned}T^{++}_{p,q}\left(\frac{1}{2},t\right) &= \sum_{n\geq 1}\frac{n^t H^{(p)}_n}{\left(n+\frac{1}{2}\right)^q}\\ &= \sum_{j=0}^{t}(-1)^{j+t+1}\binom{t}{j}\left(\frac{1}{2}\right)^{t-j}\begin{pmatrix}\left(2^{p+q-j-1}-1\right)S^{++}_{q-j,p}-2^{p+q-j-1}S^{+-}_{q-j,p}\\ -\left(2^{q-j}\eta(q-j)+\zeta(q-j)\right)\zeta(p)\end{pmatrix},\end{aligned}$$

where $\eta(\cdot)$ *is the alternating zeta function.*

The following required Euler sum identity appears in [17].

Corollary 2.3. *Let* x *be a real number* $x\in\mathbb{C}\backslash\{-1,-2,-1,...\}$ *and assume that* $q\in\mathbb{N}\backslash\{1\}$. *Then*

$$\begin{aligned}T^{++}_{1,q}(x,0) &= \sum_{n\geq 1}\frac{H_n}{(n+x)^q}=S^{++}_{1,q}(0,x)\\ &= \frac{(-1)^q}{(q-1)!}\begin{pmatrix}(\psi(x)+\gamma)\,\psi^{(q-1)}(x)\\ -\frac{1}{2}\psi^{(q)}(x)+\sum_{j=1}^{q-2}\binom{q-2}{j}\psi^{(j)}(x)\,\psi^{(q-j-1)}(x)\end{pmatrix}\end{aligned}\qquad(12)$$

where γ *is the Euler Mascheroni constant.*

The following proposition generalizes the result (12).

Proposition 2.1. *Let* x *be a real number,* $x\neq -1,-2,-1,...,$ *and assume that* $q\in\mathbb{N}\backslash\{1\}$. *Then*

$$\begin{aligned}T^{++}_{1,q}(x,1) &= \sum_{n\geq 1}\frac{n\,H_n}{(n+x)^{q+1}}=S^{++}_{1,q}(0,x)\\ &+\frac{x}{q}\left((\gamma+\psi(x))\,\psi^{(q)}(x)+\psi^{(q-1)}(x)\,\psi^{(\prime)}(x)-\frac{1}{2}\psi^{(q+1)}(x)\right)\\ &+\frac{\alpha}{q}\sum_{j=1}^{q-2}\binom{q-2}{j}\left(\psi^{(j+1)}(x)\,\psi^{(q-j-1)}(x)+\psi^{(j)}(x)\,\psi^{(q-j)}(x)\right),\end{aligned}\qquad(13)$$

the sum $S^{++}_{1,q}(0,x)$ *is given by (12) and* $\psi^{(q)}(x)$ *are the polygamma functions.*

Proof. In (12) we put $x=\frac{1}{y}, y\neq 0$, differentiate with respect to y and then rename y as x so that (13) follows. Similar analysis allows us to evaluate $T^{++}_{1,q}(x,t)$ for $t\in\mathbb{N}$ and $q\geq t+2$. □

3. Examples

In what follows, some examples are discussed. From (12) put $x=\frac{2}{3}$ and $q=3$, therefore

$$\begin{aligned}
T_{1,3}^{++}\left(\frac{2}{3},0\right) &= \sum_{n\geq 1}\frac{H_n}{\left(n+\frac{2}{3}\right)^3} = S_{1,3}^{++}\left(0,\frac{2}{3}\right)\\
&= -\frac{1}{2}\left(\psi\left(\frac{2}{3}\right)+\gamma\right)\psi^{(2)}\left(\frac{2}{3}\right)-\frac{1}{4}\psi^{(3)}\left(\frac{2}{3}\right)-\frac{1}{2}\psi^{(1)}\left(\frac{2}{3}\right)\psi^{(1)}\left(\frac{2}{3}\right)\\
&= \frac{\pi^3\sqrt{3}}{3}\ln 3+\frac{13\pi\sqrt{3}}{6}\zeta(3)-\frac{39}{2}\zeta(3)\ln 3-10\zeta(4)-\frac{1}{2}\left(\psi^{(1)}\left(\frac{2}{3}\right)\right)^2+\frac{1}{4}\psi^{(3)}\left(\frac{2}{3}\right),
\end{aligned}$$

since

$$\psi^{(1)}\left(\frac{2}{3}\right)=\frac{2\pi^2}{3}-3\sqrt{3}\mathrm{Cl}_2\left(\frac{2\pi}{3}\right),\psi^{(3)}\left(\frac{2}{3}\right)=\frac{8\pi^4}{3}-162\sqrt{3}\mathrm{Cl}_4\left(\frac{2\pi}{3}\right)$$

we can simplify as

$$\begin{aligned}
\sum_{n\geq 1}\frac{H_n}{\left(n+\frac{2}{3}\right)^3} &= \frac{\pi^4}{3}+\frac{\pi^3\sqrt{3}}{3}\ln 3+\frac{13\pi\sqrt{3}}{6}\zeta(3)-\frac{39}{2}\zeta(3)\ln 3\\
&\quad +2\sqrt{3}\pi^2\mathrm{Cl}_2\left(\frac{2\pi}{3}\right)-\frac{27}{2}\left(\mathrm{Cl}_2\left(\frac{2\pi}{3}\right)\right)^2-\frac{81\sqrt{3}}{2}\mathrm{Cl}_4\left(\frac{2\pi}{3}\right).
\end{aligned}$$

From (12) put $x=\frac{1}{4}$ and $q=3$, therefore

$$\sum_{n\geq 1}\frac{H_n}{\left(n+\frac{1}{4}\right)^3}=192\beta(4)-32G^2-8\pi^2G-14\pi\zeta(3)-84\zeta(3)\ln 2+\pi^4-3\pi^3\ln 2,$$

where the Catalan constant

$$G=\beta(2)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\approx 0.91597$$

is a special case of the Dirichlet beta function

$$\begin{aligned}
\beta(z) &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^z},\ (\text{for } \mathrm{Re}(z)>0)\\
&= \frac{1}{(-2)^{2z}(z-1)!}\left(\psi^{(z-1)}\left(\frac{1}{4}\right)-\psi^{(z-1)}\left(\frac{3}{4}\right)\right)\\
&= \frac{i}{2}\left(\mathrm{Li}_z(-i)-\mathrm{Li}_z(i)\right)
\end{aligned}$$

with functional equation

$$\beta(1-z)=\left(\frac{2}{\pi}\right)^z\sin\left(\frac{\pi z}{2}\right)\Gamma(z)\beta(z)$$

extending the Dirichlet Beta function to the left hand side of the complex plane $\mathrm{Re}(z)\leq 0$.
From (13) put $x=\frac{2}{3}$ and $q=2$, therefore

$$\begin{aligned}
T_{1,3}^{++}\left(\frac{2}{3},1\right) &= \sum_{n\geq 1}\frac{n\ H_n}{\left(n+\frac{2}{3}\right)^3} = S_{1,2}^{++}\left(0,\frac{2}{3}\right)\\
&\quad +\frac{1}{3}\left(\left(\gamma+\psi\left(\frac{2}{3}\right)\right)\psi^{(2)}\left(\frac{2}{3}\right)+\psi^{(1)}\left(\frac{2}{3}\right)\psi^{(1)}\left(\frac{2}{3}\right)-\frac{1}{2}\psi^{(3)}\left(\frac{2}{3}\right)\right)\\
&= 13\zeta(3)\ln 3+13\zeta(3)-\frac{2\pi^3\sqrt{3}}{9}\ln 3-\pi^2\ln 3-\frac{2\pi^4}{9}-\frac{\pi^3\sqrt{3}}{9}-22\sqrt{3}\mathrm{Cl}_4\left(\frac{2\pi}{3}\right)\\
&\quad +\left(\frac{9\sqrt{3}}{2}\ln 3-\frac{4\pi^2\sqrt{3}}{9}-\frac{3\pi}{2}+9\mathrm{Cl}_2\left(\frac{2\pi}{3}\right)-\frac{27}{2}\right)\mathrm{Cl}_2\left(\frac{2\pi}{3}\right)-\frac{13\pi\sqrt{3}}{9}\zeta(3).
\end{aligned}$$

From (7) put $a = \frac{1}{2}$ and $p = 2, q = 3, t = 1$, therefore

$$\begin{aligned} T_{2,3}^{++}\left(\frac{1}{2},1\right) &= 8\sum_{n\geq 1}\frac{nH_n^{(2)}}{(2n+1)^3} = \sum_{n\geq 1}\frac{nH_n^{(2)}}{\left(n+\frac{1}{2}\right)^3} \\ &= \sum_{j=0}^{1}(-1)^j\left(\begin{array}{c}1\\ j\end{array}\right)\left(\frac{1}{2}\right)^{1-j}\left(\sum_{k\geq 1}\frac{H_{k-\frac{1}{2}}^{(3-j)}}{k^2}-\zeta(3-j)\zeta(2)\right) \\ &= 32\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{4}{3}\ln^4 2-8\zeta(2)\ln^2 2-\frac{121}{4}\zeta(4)+28\zeta(3)\ln 2-\frac{49}{2}\zeta(2)\zeta(3)+\frac{93}{2}\zeta(5), \end{aligned}$$

where $\mathrm{Li}_4(\frac{1}{2})$ is the polylogarithm function described by (3).
From (7) put $a = 2$ and $p = 3, q = 5, t = 3$, therefore

$$\begin{aligned} T_{3,5}^{++}(2,3) &= \sum_{n\geq 1}\frac{n^3H_n^{(3)}}{(n+2)^5} \\ &= \sum_{j=0}^{3}(-1)^{j+1}\left(\begin{array}{c}3\\ j\end{array}\right)(2)^{3-j}\left(\sum_{k\geq 1}\frac{H_{k+1}^{(5-j)}}{k^3}-\zeta(5-j)\zeta(3)\right) \\ &= \sum_{j=0}^{3}(-1)^{j+1}\left(\begin{array}{c}3\\ j\end{array}\right)(2)^{3-j}\left(S_{5-j,3}^{++}+\frac{1}{n^3(n+1)^{5-j}}-\zeta(5-j)\zeta(3)\right) \\ &= 96-33\zeta(2)-12\zeta(4)+3\zeta(6)+8S_{5,3}^{++}-17\zeta(3)-2\zeta(2)\zeta(3)-3\zeta^2(3) \\ &\quad +204\zeta(7)-\frac{7}{2}\zeta(5)-120\zeta(2)\zeta(5)-8\zeta(3)\zeta(5), \end{aligned}$$

We know from Borwein [3] that for the Euler sum $S_{p,q}^{++}$ there exists closed form solutions, in terms of Riemann zeta functions, for integers $(p,q), q \geq 2$ and of $p+q$ being an odd weight. Also $S_{p,q}^{++}$ admits a closed form solution for $p = q$ and the pair $(p,q) = (4,2)$ and $(2,4)$.

4. Concluding remarks

We have studied families of Euler sums. We have given a direct proof of the Euler family $S_{p,q}^{++}(0,a)$ and given a closed form representation of the new Euler family $T_{p,q}^{++}(a,t)$. Some examples are highlighted in which we detail the representation of these sums in terms of special functions such as Beta functions, Clausen functions and Zeta functions.

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