Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **10** (2022) 64–67
DOI: 10.47443/dml.2022.046

Research Article

Involutions containing exactly r pairs of intersecting arcs

Toufik Mansour*

Department of Mathematics, University of Haifa, 3498838 Haifa, Israel

(Received: 30 March 2022. Received in revised form: 28 May 2022. Accepted: 30 May 2022. Published online: 4 June 2022.)

Abstract

The generating function $F_r(x)$ that counts the involutions on n letters containing exactly r pairs of intersecting arcs in their graphical representation is studied. More precisely, an algorithm that computes the generating function $F_r(x)$ for any given $r \geq 0$ is presented. To derive the result for a given r, the algorithm performs certain routine checks on involutions of length $2r+2$ without fixed points. The algorithm is implemented in Maple and yields explicit formulas for $0 \leq r \leq 4$.

Keywords: involution; arc-pattern 3412.

2020 Mathematics Subject Classification: 05A15, 05A05.

1. Introduction

In recent years, much attention has been paid to the problem of counting the permutations of length n containing a given number $r \geq 0$ of occurrences of a certain pattern. Most of the researchers considered only the case $r = 0$; namely, studying permutations avoiding a given pattern. Only a few of them considered the case $r > 0$, usually restricting themselves to the patterns of length 3. For patterns of length 3, there are two different cases $\tau = 123$ and $\tau = 132$ (see Table 1).

r	Number of permutations in S_n containing 123 exactly r times	Reference
0	$\frac{1}{n+1}\binom{2n}{n}$	[6]
1	$\frac{3}{n}\binom{2n}{n-3}$	[9]
2	$\frac{59n^2+117n+100}{2n(2n-1)(n+5)}\binom{2n}{n-4}$	[4]
$r = 3, 4, \ldots, 10$		[7]
r	Number of permutations in S_n containing 132 exactly r times	Reference
0	$\frac{1}{n+1}\binom{2n}{n}$	[6]
1	$\binom{2n-3}{n-3}$	[1]
2	$\frac{n^3+17n^2-80n+80}{2n(n-1)}\binom{2n-6}{n-2}$	[8]
$r \geq 3$		[8]

Table 1: Counting occurrences of 123 (132) in a permutation.

Let I_n denote the set of all involutions in S_n, that is, $I_n = \{\sigma \in S_n \mid \sigma^2 = id\}$. On I_n, the focus of the pattern occurrence counting problem has been on the cases $r = 0, 1$, and patterns of size at most 4 (for instance, see [3, 5] and references therein).

*E-mail address: tmansour@univ.haifa.ac.il

In order to present the main result of this paper, a graphical representation of an involution and the following notation is needed. For $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in I_n$, its graphical representation is a graph with vertices $1, 2, \ldots, n$ on a horizontal line and arcs connecting (i, σ_i) for $\sigma_i \neq i$. Henceforth, the involution is identified with its graphical representation. For example, Figure 1 presents the involution $1462(10)37985$.

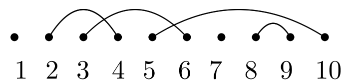
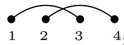


Figure 1: Graphical representation of the involution $1462(10)37985$.

In this paper, we fix the pattern τ to be , or just say τ is the arc-pattern 3412 (where the term *arc-pattern* refers to the fact that each vertex of τ is a termination point of an arc). We say that an involution $\sigma \in I_n$ *contains* τ if there exist two arcs (a, b) (c, d) in σ where the induced subgraph of σ with vertices a, b, c, d equals τ. In other words, σ contains τ if there exists a pair of arcs (a, b) and (c, d) such that $a < c < b < d$ (i.e., (a, b) intersects (c, d)). We define $int(\sigma)$ to be the number of occurrences of τ in σ. We denote the set of involutions σ of I_n having $int(\sigma) = r$ by $I_{n,r}$ (see Table 2). We define the generating function for the cardinality of $I_{n,r}$ for a fixed r by $F_r(x)$, that is, $F_r(x) = \sum_{n\geq 0} |I_{n,r}|x^n$. In this paper, we

$r\backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	1	2	4	9	21	51	127	323	835	2188	5798	15511	41835	113634	310572
1	0	0	0	0	1	5	21	77	266	882	2850	9042	28314	87802	270270	827190
2	0	0	0	0	0	0	3	21	112	504	2070	7986	29502	105534	368368	1261260
3	0	0	0	0	0	0	1	7	48	264	1305	5907	25156	102232	400789	1526835
4	0	0	0	0	0	0	0	0	10	90	625	3575	18270	380594	1610010	6571660

Table 2: Number of involutions in $I_{n,r}$, where $0 \leq n \leq 15$ and $0 \leq r \leq 4$.

study the generating function $F_r(x)$. More precisely, we present an algorithm that computes the generating function $F_r(x)$ for any given $r \geq 0$. To obtain the result for a given r, the algorithm performs certain routine checks on members of I_{2r+2} without fixed points (recall that i is a fixed point of σ if $\sigma_i = i$). The algorithm has been implemented in Maple and yields explicit formulas for $0 \leq r \leq 4$.

2. Main result

To any involution $\sigma \in I_n$, we assign a bipartite graph G_σ as follows. Let $V_1 = [n]$ be the vertices in the first part of G_σ and $V_4 = \{abcd \mid (a, c) \text{ intersects } (b, d) \text{ in } \sigma\}$ be the vertices in the second part. Entry $i \in V_1$ is connected by an edge to occurrence $q \in V_4$ if i is a letter in q. For example, Figure 2 presents the bipartite graph for the involution $1462(10)37985$.

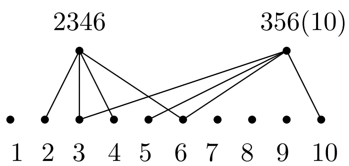
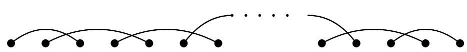


Figure 2: The bipartite graph of the involution $1462(10)37985$.

Let G' be any arbitrary connected component of G_σ with vertices V'. Define $V_1' = V_1 \cap V'$ and $V_4' = V_4 \cap V'$. By the definitions of the arc-pattern τ, we see that the greatest possible number of vertices in $[n]$ of a connected component of G_σ for which there are exactly r occurrences of τ is realized with



This leads to the following basic lemma.

Lemma 2.1. *For any connected component G' of G_σ, we have that $|V_1'| \leq 2|V_4'| + 2$.*

We denote the maximal connected component of G_σ containing the entry 1 by G'_σ. Clearly, there is no fixed point $i \neq 1$ such that i belongs to G'_σ. Thus, the number of vertices $i \in [n]$ belonging to G'_σ is even, or G'_σ consists only of the vertex 1.

Let $\sigma' = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_{2t}}$ be the entries of σ that belong to G'_σ and let $\pi_\sigma \in I_{2t}$ be the corresponding involution. In this context, σ' is called the *kernel* of σ, with π_σ referred to as the *kernel shape* of σ and $2t$ the *size* of the kernel shape. The *capacity* c_{π_σ} of σ is defined as the number of pairs of arcs that intersect in π_σ.

The following statement is implied immediately by Lemma 2.1.

Theorem 2.1. *Let $\sigma \in I_{n,r}$. Then the size of the kernel shape ρ of σ is at most $2r+2$ and it has no fixed points.*

We say that ρ is a *kernel involution* if it is the kernel shape for some involution σ. Clearly, ρ is a kernel involution if and only if $\pi_\rho = \rho$. Let $\rho \in I_n$ be any kernel involution, and we denote the set of involutions of all possible sizes whose kernel shape equals ρ by $I(\rho)$.

For any $\sigma \in I(\rho)$, we decompose it into smaller involutions as follows. Let $\sigma \in I_n \cap I(\rho)$. By the definitions, any edge (i, σ_i) $(\sigma_i \neq i)$ of σ is either an edge of ρ or does not intersect any edge of ρ. Thus, σ satisfies either

- If $\sigma_1 = 1$, then $int(\sigma) = int(\sigma_2\cdots\sigma_n)$. In this case, we say that the kernel shape of σ is $\rho = 1$;
- If $\sigma' = \sigma_{i_1}\cdots\sigma_{i_{2t}}$ is the kernel of σ, then σ can be written as $\sigma_{i_1}\sigma^{(1)}\cdots\sigma_{i_{2t}}\sigma^{(2t)} \in I_n$ such that $int(\sigma) = int(\rho) + int(\sigma^{(1)}\cdots\sigma^{(2t)})$, that is, any edge of σ is either in σ' or in $\sigma^{(1)}\cdots\sigma^{(2t)}$, and any fixed point of σ belongs to $\sigma^{(1)}\cdots\sigma^{(2t)}$. Clearly, any letter of $\sigma^{(i)}$ is smaller than any letter of $\sigma^{(i+1)}$ for $i = 1, 2, \ldots, 2t-1$.

Hence, the generating function for the number of involutions $\sigma \in I_{n,r}$ with kernel shape $\rho \in I_{2t}$ is given by

$$\begin{cases} xF_r(x) & \text{if } \rho = 1, \\ x^{2t} \displaystyle\sum_{i_1+\cdots+i_{2t}=r-c_\rho} \prod_{j=1}^{2t} F_{i_j}(x) & \text{otherwise.} \end{cases}$$

Define K_r to be the set of all kernel shapes $\rho \in I_{2n}$ having $c_\rho = r$, where $1 \leq n \leq r+1$ (see Theorem 2.1). Hence, we can state our main result.

Theorem 2.2. *The generating function $F_r(x)$ for $r \geq 0$ is given by*

$$F_r(x) = \delta_{r=0} + xF_r(x) + \sum_{\rho \in \cup_{j=0}^{r} K_j} x^{2t} \sum_{i_1+\cdots+i_{2t}=r-c_\rho} \prod_{j=1}^{2t} F_{i_j}(x).$$

Next, we apply this theorem for $0 \leq r \leq 4$.

Cases $r = 0, 1, \ldots, 4$

Note that $K_0 = \{21\}$. Thus, Theorem 2.2 gives

$$F_0(x) = 1 + xF_0(x) + x^2F_0^2(x),$$

which leads to

$$F_0(x) = M(x) = \frac{1 - x - \sqrt{1-2x-3x^2}}{2x^2},$$

the generating function for the Mozkin numbers, as shown in [5]. For $r = 1$, we have $K_1 = \{3412\}$. Thus, Theorem 2.2 gives

$$F_1(x) = xF_1(x) + 2x^2F_1(x)F_0(x) + x^4F_0^4(x),$$

which implies

$$F_1(x) = \frac{x^4M^4(x)}{1 - x - 2x^2M(x)}.$$

For $r = 2$, we have $K_2 = \{351624, 465132, 546213\}$. Thus, Theorem 2.2 gives

$$F_2(x) = xF_2(x) + 2x^2F_0(x)F_2(x) + x^2F_1^2(x) + 4x^4F_0^3(x)F_1(x) + 3x^6F_0^6(x),$$

which implies

$$F_2(x) = \frac{x^6M^6(x)(1 - x - x^2M(x))(3 - 3x - 5x^2M(x))}{(1 - x - 2x^2M(x))^3}.$$

Similarly, for $r = 3, 4$, we have

$$F_r(x) = \frac{P_r(x)}{(1 - x - 2x^2M(x))^{2r-1}},$$

where

$$\begin{aligned}P_3(x) &= x^6M^6(x)\Big((x-1)^4 + 8x^2(x-1)^3M(x) + 12x^2(3x^2-2x+1)(x-1)^2M^2(x)\\&+ 2x^4(x-1)(49x^2-66x+33)M^3(x) + 8x^6(19x^2-34x+17)M^4(x)\\&+ 4x^6(x-1)(23x^2+16x-8)M^5(x) - 6x^8(25x^2-64x+32)M^6(x)\\&- 384x^{10}(x-1)M^7(x) - 256x^{12}M^8(x)\Big),\\P_4(x) &= x^8M^8(x)\Big(10(x-1)^6 + 110x^2(x-1)^5M(x) + x^2(567x^2-130x+65)(x-1)^4M^2(x)\\&+ 2x^4(899x^2-582x+291)(x-1)^3M^3(x) + 2x^6(1937x^2-2226x+1113)(x-1)^2M^4(x)\\&+ 2x^6(x-1)(2725x^4-3870x^3+1143x^2+792x-198)M^5(x) + 2x^8(1203x^4+1538x^3\\&- 7953x^2+7184x-1796)M^6(x) - 56x^{10}(x-1)(165x^2-464x+232)M^7(x)\\&- x^{12}(22355x^2-46848x+23424)M^8(x) - 21056x^{14}(x-1)M^9(x) - 7552x^{16}M^{10}(x)\Big).\end{aligned}$$

General r

By induction on r, Theorem 2.2, together with the cases $r = 0, 1, 2$, yields the following result.

Theorem 2.3. *The generating function $F_r(x)$ for $r \geq 0$ is rational in x and $\sqrt{1-2x-3x^2}$. Moreover, it can be written as $Q_r(x)/\sqrt{1-2x-3x^2}^{2r-1}$, where $Q_r(x)$ is a polynomial in $\sqrt{1-2x-3x^2}$ with rational function coefficients.*

3. Further results

Theorem 2.2 can be extended as follows. Let $F_r(x,q)$ be the generating function for the number of involutions in $I_{n,r}$ according to the number of fixed points. Then, by our structure, we have the following result.

Theorem 3.1. *The generating function $F_r(x,q)$ for $r \geq 0$ is given by*

$$F_r(x,q) = \delta_{r=0} + xqF_r(x) + \sum_{\rho\in\cup_{j=0}^{r}K_j} x^{2t} \sum_{i_1+\cdots+i_{2t}=r-c_\rho} \prod_{j=1}^{2t} F_{i_j}(x,q).$$

For example, Theorem 3.1 gives

$$F_0(x,q) = 1 + xqF_0(x,q) + x^2F_0(x,q)$$

and

$$F_1(x,q) = xqF_1(x,q) + 2x^2F_1(x,q)F_0(x,q) + x^4(F_0(x,q))^4.$$

Hence,

$$F_0(x,q) = \frac{1-xq-\sqrt{(1-xq)^2-4x^2}}{2x^2}$$

and

$$F_1(x,q) = \frac{(1-xq-\sqrt{(1-xq)^2-4x^2})^4}{16x^4\sqrt{(1-xq)^2-4x^2}}.$$

We conclude this paper by referring the reader to [2] for the $q = 0$ case of Theorem 3.1.

References

[1] M. Bóna, Permutations with one or two 132-subsequences, *Discrete Math.* **181** (1998) 267–274.
[2] W. Y. C. Chen, E. Y. P. Deng, R. R. X. Du, R. P. Stanley, C. H. Yan, *Crossings and nestings of matchings and partitions*, *Trans. Amer. Math. Soc.* **359** (2007) 1555–1575.
[3] E. Egge, T. Mansour, Bivariate generating functions for involutions restricted by 3412, *Adv. in Appl. Math.* **36** (2006) 118–137.
[4] M. Fulemk, Enumeration of permutations containing a prescribed number of occurrences of a pattern of length three, *Adv. in Appl. Math.* **30** (2003) 607–632
[5] O. Guibert, *Combinatoire Des Permutations á Motifs Exclus en Liaison Avec Mots, Cartes Planaires et Tableaux de Young*, Ph.D. thesis, Université Bordeaux-I, Bordeaux, 1995.
[6] D. E. Knuth, *The Art of Computer Programming*, 2nd Edition, Addison Wesley, Reading, 1973.
[7] T. Mansour, Counting occurrences of 123 in a permutation, Preprint.
[8] T. Mansour, A. Vainshtein, Counting occurrences of 132 in a permutation, *Adv. in Appl. Math.* **28** (2002) 185–195.
[9] J. Noonan, The number of permutations containing exactly one increasing subsequence of length three, *Discrete Math.* **152** (1996) 307–313.