The automorphism group of the Andrásfai graph

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Abstract

For \( n = 3k - 1 \), denote by \( Z_n \) the additive group of integers modulo \( n \), where \( k \) is an integer greater than 1. Let \( C \) be the subset of \( Z_n \) consisting of the elements congruent to 1 modulo 3. The Andrásfai graph \( \text{And}(k) \) is the Cayley graph \( \text{Cay}(Z_n; C) \). In this note, it is shown that the automorphism group of the graph \( \text{And}(k) \) is isomorphic to the dihedral group of order \( 2n \).

Keywords: Cayley graph; Andrásfai graph; automorphism group.

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1. Introduction

In this paper, a graph \( \Gamma = (V, E) \) is considered as an undirected simple graph, where \( V = V(\Gamma) \) is the vertex set and \( E = E(\Gamma) \) is the edge set. The terminology and notation used in this paper are the same as given in [4, 15].

Let \( m \) be a positive integer. Denote by \( Z_m \) the additive group of integers modulo \( m \). Let \( k > 1 \) be an integer and \( n = 3k - 1 \). Let \( C = \{3t + 1 : 0 \leq t \leq k - 1\} \) be the subset of \( Z_n \) consisting of the elements congruent to 1 modulo 3. It is easy to see that \( C \) is a symmetric set; that is, \( C \) is an inverse closed subset of the group \( Z_n \). The Cayley graph \( \text{Cay}(Z_n; C) \) is known as the Andrásfai graph \( \text{And}(k) \). It is easy to check that the graph \( \text{And}(2) \) is isomorphic to the 5-cycle and the graph \( \text{And}(3) \) is the Möbius ladder of order 8. Also, the graph \( \text{And}(4) \) is depicted in Figure 1.

![Figure 1: The Andrásfai graph And(4).](image1.png)

The Cayley graph \( \text{And}(k) \) was first used by Andrásfai in [1] (such graphs were also appeared in his book [2]). It is not hard to show that the graph \( \text{And}(k) \) has diameter 2 and girth 4. The Andrásfai graph \( \text{And}(k) \) has some interesting properties and it is a classic example in the subject of graph homomorphism [4]. For a given graph \( G \), one of the problems concerning \( G \) is the determination of its automorphism group. To the best of the author’s knowledge, the automorphism group of the graph \( \text{And}(k) \) is still unknown. The main aim of the present paper is to determine the automorphism group \( \text{Aut}(\text{And}(k)) \) of \( \text{And}(k) \). It is shown that \( \text{Aut}(\text{And}(k)) \cong D_{2n} \), where \( D_{2n} \) denotes the dihedral group of order \( 2n \) and \( n = 3k - 1 \).

2. Preliminaries

The graphs \( \Gamma_1 = (V_1, E_1) \) and \( \Gamma_2 = (V_2, E_2) \) are said to be isomorphic if there is a bijection \( \alpha : V_1 \rightarrow V_2 \) such that \( \{a, b\} \in E_1 \) if and only if \( \{\alpha(a), \alpha(b)\} \in E_2 \) for all \( a, b \in V_1 \). In such a case, the bijection \( \alpha \) is called an isomorphism. An automorphism of a graph \( \Gamma \) is an isomorphism of \( \Gamma \) with itself. The set of automorphisms of \( \Gamma \) with the operation of composition of functions is a group, known as the automorphism group of \( \Gamma \) and denoted by \( \text{Aut}(\Gamma) \).

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The group of all permutations of a set \( V \) is denoted by \( \text{Sym}(V) \) or just by \( \text{Sym}(n) \) when \( |V| = n \). A permutation group \( G \) on \( V \) is a subgroup of \( \text{Sym}(V) \). In this case, it is said that \( G \) acts on \( V \). If \( G \) acts on \( V \), one says that \( G \) is transitive on \( V \) (or \( G \) acts transitively on \( V \)), when there is just one orbit. It means that for any two given elements \( u \) and \( v \) of \( V \), there is an element \( \beta \) of \( G \) such that \( \beta(u) = v \). If \( \Gamma \) is a graph with vertex set \( V \) then one can view each automorphism of \( \Gamma \) as a permutation on \( V \) and so \( \text{Aut}(\Gamma) = G \) is a permutation group on \( V \).

A graph \( \Gamma \) is said to be vertex-transitive if \( \text{Aut}(\Gamma) \) acts transitively on \( V(\Gamma) \). For \( v \in V(\Gamma) \) and \( G = \text{Aut}(\Gamma) \) the stabilizer subgroup \( G_v \) is the subgroup of \( G \) consisting of all automorphisms fixing \( v \). In the vertex-transitive case, all stabilizer subgroups \( G_v \) are conjugate in \( G \), and consequently, they are isomorphic.

In this case, the index of \( G_v \) in \( G \) is given by the equation

\[
|G : G_v| = \frac{|G|}{|G_v|} = |V(\Gamma)|.
\]

This fact is known as the Orbit-Stabilizer theorem, which is a useful tool in finding the automorphism group of vertex-transitive graphs. Let \( G \) be any abstract finite group with identity 1 and suppose that \( \Omega \) is a set of \( G \) with the following two properties.

(i). \( x \in \Omega \implies x^{-1} \in \Omega \);

(ii). \( 1 \notin \Omega \).

The Cayley graph \( \Gamma = \Gamma(G; \Omega) \) is the (simple) graph whose edge set and vertex set are \( E(\Gamma) = \{(g, h) : g^{-1}h \in \Omega\} \) and \( V(\Gamma) = G \), respectively.

Although, in most situations, it is difficult to determine the automorphism group of a graph \( G \), and see how it acts on its vertex set or edge set, there are various papers in the literature on this topic; for example [3, 5–14, 16, 17].

A group \( G \) is said to be a semidirect product of \( N \) by \( Q \), denoted by \( G = N \rtimes Q \), if \( G \) contains subgroups \( N \) and \( Q \) such that:

(i). \( N \trianglelefteq G \) (\( N \) is a normal subgroup of \( G \));

(ii). \( NQ = G \);

(iii). \( N \cap Q = 1 \).

3. Main result

**Definition 3.1.** Let \( k > 1 \) be an integer and \( n = 3k - 1 \). Let \( C = \{3t + 1 : 0 \leq t \leq k - 1\} \) be the subset of \( \mathbb{Z}_n \) consisting of the elements congruent to 1 modulo 3. It is easy to see that \( C \) is a symmetric set; that is, \( C \) is an inverse closed subset of the group \( \mathbb{Z}_n \). The Cayley graph \( \text{Cay}(\mathbb{Z}_n; C) \) is known as the Andrásfai graph, denoted by \( \text{And}(k) \).

It follows from Definition 3.1 that \( \text{And}(k) \) is a regular graph of valency \( k \) and the vertex set of \( \text{And}(k) \) is the set \( V = V_0 \cup V_1 \cup V_2 \), where \( V_0 = \{3t : 0 \leq t \leq k - 1\}, V_1 = \{3t + 1 : 0 \leq t \leq k - 1\} \), and \( V_2 = \{3t + 2 : 0 \leq t \leq k - 2\} \). Thus, one has \( |V_0| = |V_1| = k \) and \( |V_2| = k - 1 \). If \( v \in V_0 \), then \( v = 3j \) for some \( j \) with \( 0 \leq j \leq k - 1 \). Now, it is easy to see that

\[
N(v) = \{3i + 1 : j \leq i \leq k - 1\} \cup \{3l + 2 : 0 \leq l \leq j - 1\}.
\]

(1)

Also, if \( w \in V_2 \), then \( w = 3j + 2, 0 \leq j \leq k - 2 \), and thus we have

\[
N(w) = \{3i + 1 : 0 \leq i \leq j\} \cup \{3l : j + 1 \leq l \leq k - 1\}.
\]

(2)

Now, from (1) and (2), it follows that the graph induced by the set \( V_0 \cup V_2 \) in \( \text{And}(k) \) is a bipartite graph such that the vertex \( 3j = v \in V_0 \) has \( j \) neighbors in \( V_2 \) and the vertex \( 3j + 2 = w \in V_2 \) has \( k - j - 1 \) neighbors in \( V_0 \). Note that all the neighbors of the vertex \( v = 0 \) are in \( V_1 \). Let \( H = \{(V_0 - \{0\}) \cup V_2\} \) be the subgraph induced by the set \( (V_0 - \{0\}) \cup V_2 \) in \( \text{And}(k) \). Thus, \( H \) is a connected bipartite graph such that if \( v, w \) are distinct vertices in \( H \), then we have \( N(v) \neq N(w) \) (note that the vertex \( v = 3(k - 1) \) is adjacent to every vertex in \( V_2 \) and the vertex \( w = 2 \) is adjacent to any vertex in \( V_0 \)).

In the sequel, the following fact is needed.

**Lemma 3.1.** Let \( \Gamma = (U \cup W, E) \) be a connected bipartite graph, where \( U \cap W = \emptyset \). If \( f \) is an automorphism of the graph \( \Gamma \), then \( f(U) = U \) and \( f(W) = W \), or \( f(U) = W \) and \( f(W) = U \).
Proof. Since automorphisms of $\Gamma$ preserve distance between vertices and because two vertices are in the same part if and only if they are at even distance from each other, the result follows.

\[ \text{Theorem 3.1. Let } k > 1 \text{ be an integer and } n = 3k - 1. \text{ For the automorphism group } \text{Aut}(\text{And}(k)) \text{ of the graph } \text{And}(k), \text{ it holds that } \text{Aut}(\text{And}(k)) \cong \mathbb{D}_{2n}, \text{ where } \mathbb{D}_{2n} \text{ denotes the dihedral group of order } 2n. \]

Proof. Let $\Gamma = (V,E) = \text{And}(k)$. Let $A = \text{Aut}(\Gamma)$ be the automorphism group of $\Gamma$. Consider the vertex $v = 0$ and let $A_0$ be its stabilizer subgroup; that is, $A_0 = \{ a \in A : a(0) = 0 \}$. The graph $\Gamma$, being a Cayley graph, is a vertex-transitive graph. From the well-known Orbit-Stabilizer theorem, it follows that

$$|V| = \frac{|A|}{|A_v|},$$

and hence $|A| = |V||A_v|$, where $v$ is a vertex in $\Gamma$.

In the first step of the proof, $|A_0|$ is determined. Let $f \in A_0$. Let $V_0, V_1$, and $V_2$ be the subsets of $V$ which are defined preceding (1) and take $W_0 = V_0 - \{0\}$. Thus, for the restriction of $f$ to $N(0) = V_1$, one has $f(V_1) = V_1$ and hence $f(W_0 \cup V_2) = W_0 \cup V_2$. Let $H$ be the subgraph induced by the set $W_0 \cup V_2$ in the graph $\Gamma = \text{And}(k)$ and $g = f|_{W_0 \cup V_2}$. Hence, $g$ is an automorphism of $H$. It is clear that $H$ is a connected bipartite graph with parts $W_0$ and $V_2$ such that $|W_0| = |V_2| = k - 1$. In each part of the graph $H$ there is exactly one vertex $x_j$ of degree $j$, $1 \leq j \leq k - 1$. In other words, the vertex $v_j = 3j$ is the unique vertex in $W_0$ of degree $j$. Also, the vertex $w_j = 3k - 1 - 3j = (3(k - j) - 1 = 3(k - j - 1) + 2$ is the unique vertex in $V_2$ of degree $j$. Note that $w_j = 3k - 1 - 3j$ is the inverse of $v_j = 3j$ in the cyclic group $\mathbb{Z}_{3k - 1}$ and hence it is denoted by $-v_j$.

Note that the mapping $g$ is an automorphism of the connected bipartite graph $H$. Thus, from Lemma 3.1, it follows that

(i) $g(W_0) = W_0$ or
(ii) $g(W_0) = V_2$.

(i) Assume that $g(W_0) = W_0$. Since the vertex $v = 3j$ is the unique vertex of $W_0$ of degree $j$, for every $w \in W_0$ one has $g(w) = w$. Similarly, for every $v \in V_2$ one has $g(v) = v$. In other words, the restriction of the automorphism $f$ to the set $W_0 \cup V_2$ is the identity mapping. We show that if $x \in V_1$, then $f(x) = x$. Note that we have $f(V_1) = V_1$. If $v = 3j + 1$ is a vertex in $V_j$, then the set of neighbors of $v$ in $W_0$ is $V_j = \{ 3t : 1 \leq t \leq j \}$. Hence, $v$ has exactly $j$ neighbors in $W_0$. Since the number of neighbors of $v$ and $f(v)$ in $W_0$ are equal, hence we must have $v = f(v)$. From our discussion it follows that if $g(W_0) = 0$, then the automorphism $f$ is the identity automorphism of the graph $\text{And}(k)$, that is, $f = 1$.

(ii) Suppose that $g(W_0) = V_2$. Let $v \in W_0$. We saw that the vertex $-v$ is the unique vertex in $V_2$ such that its degree in the graph $H$ is equal to the degree of $v$; that is, $\text{deg}_H(-v) = \text{deg}_H(v)$. It follows that for every vertex $x$ of $H$ we have $g(x) = -x$. Since $\Gamma = \text{And}(k)$ is an Abelian Cayley graph, the mapping $a : V(\Gamma) \to V(\Gamma)$ defined by the rule $a(v) = -v$ is an automorphism of the graph $\Gamma$. Let $b = af$. Thus, $b$ is an automorphism of $\Gamma$ such that its restriction to $W_0$ is the identity automorphism. Now, by what is proved in (i), it follows that $b = 1$. Since $a$ has order 2 in $\text{Aut}(\Gamma)$, one has $f = a$.

We now conclude that if $A = \text{Aut}(\Gamma)$ and $A_0$ is the stabilizer subgroup of the vertex $v = 0$, then $A_0 = \{ 1, a \}$, and hence we have $|A_0| = 2$. Now, from Orbit-Stabilizer theorem it follows that $|A| = |A_0||V(\Gamma)| = 2(3k - 1)$. On the other hand, we know that $\text{Aut}(\Gamma)$ has a subgroup isomorphic to the cyclic group $\mathbb{Z}_{3k - 1}$; that is, $S = \{ f_v \mid v \in \mathbb{Z}_{3k - 1} \}$, where $f_v : V(\Gamma) \to V(\Gamma)$, $f_v(x) = x + v$ for every $x \in \mathbb{Z}_{3k - 1}$. It is easy to check that $A_0 \cap S = \{ 1 \}$. Hence,

$$|A_0S| = \frac{|A_0||S|}{|A_0 \cap S|} = 2(3k - 1) = |A|.$$

Thus, $A = SA_0$. Since the index of $S$ in $A$ is 2, $S$ is a normal subgroup of $A$. Now, one has $A \cong S \times A_0 \cong \mathbb{Z}_{3k - 1} \times \mathbb{Z}_2 \cong \mathbb{D}_{2n}$, where $n = 3k - 1$.

\[ \square \]

References