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Research Article

The automorphism group of the Andrásfai graph

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Abstract

For $n = 3k-1$, denote by $\mathbb{Z}_n$ the additive group of integers modulo n, where k is an integer greater than 1. Let C be the subset of $\mathbb{Z}_n$ consisting of the elements congruent to 1 modulo 3. The Andrásfai graph $And(k)$ is the Cayley graph $Cay(\mathbb{Z}_n; C)$. In this note, it is shown that the automorphism group of the graph $And(k)$ is isomorphic to the dihedral group of order $2n$.

Keywords: Cayley graph; Andrásfai graph; automorphism group.

2020 Mathematics Subject Classification: 05C25.

1. Introduction

In this paper, a graph $\Gamma = (V, E)$ is considered as an undirected simple graph, where $V = V(\Gamma)$ is the vertex set and $E = E(\Gamma)$ is the edge set. The terminology and notation used in this paper are the same as given in [4, 15].

Let m be a positive integer. Denote by $\mathbb{Z}_m$ the additive group of integers modulo m. Let $k > 1$ be an integer and $n = 3k-1$. Let $C = \{3t+1 \ : \ 0 \leq t \leq k-1\}$ be the subset of $\mathbb{Z}_n$ consisting of the elements congruent to 1 modulo 3. It is easy to see that C is a symmetric set; that is, C is an inverse closed subset of the group $\mathbb{Z}_n$. The Cayley graph $Cay(\mathbb{Z}_n; C)$ is known as the Andrásfai graph $And(k)$. It is easy to check that the graph $And(2)$ is isomorphic to the 5-cycle and the graph $And(3)$ is the Möbius ladder of order 8. Also, the graph $And(4)$ is depicted in Figure 1.

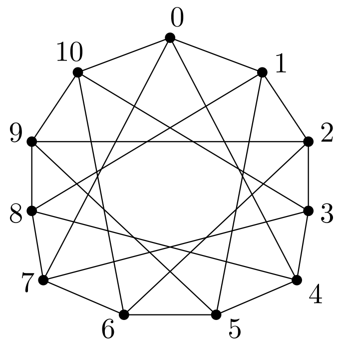

Figure 1: The Andrásfai graph $And(4)$.

The Cayley graph $And(k)$ was first used by Andrásfai in [1] (such graphs were also appeared in his book [2]). It is not hard to show that the graph $And(k)$ has diameter 2 and girth 4. The Andrásfai graph $And(k)$ has some interesting properties and it is a classic example in the subject of graph homomorphism [4]. For a given graph G, one of the problems concerning G is the determination of its automorphism group. To the best of the author's knowledge, the automorphism group of the graph $And(k)$ is still unknown. The main aim of the present paper is to determine the automorphism group $Aut(And(k))$ of $And(k)$. It is shown that $Aut(And(k)) \cong \mathbb{D}_{2n}$, where $\mathbb{D}_{2n}$ denotes the dihedral group of order $2n$ and $n = 3k-1$.

2. Preliminaries

The graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ are said to be *isomorphic* if there is a bijection $\alpha : V_1 \longrightarrow V_2$ such that $\{a, b\} \in E_1$ if and only if $\{\alpha(a), \alpha(b)\} \in E_2$ for all $a, b \in V_1$. In such a case, the bijection α is called an *isomorphism*. An *automorphism* of a graph Γ is an isomorphism of Γ with itself. The set of automorphisms of Γ with the operation of composition of functions is a group, known as the *automorphism group* of Γ and denoted by $Aut(\Gamma)$.

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The group of all permutations of a set V is denoted by $Sym(V)$ or just by $Sym(n)$ when $|V| = n$. A *permutation group* G on V is a subgroup of $Sym(V)$. In this case, it is said that G *acts* on V. If G acts on V, one says that G is *transitive* on V (or G acts *transitively* on V), when there is just one orbit. It means that for any two given elements u and v of V, there is an element β of G such that $\beta(u) = v$. If Γ is a graph with vertex set V then one can view each automorphism of Γ as a permutation on V and so $Aut(\Gamma) = G$ is a permutation group on V.

A graph Γ is said to be *vertex-transitive* if $Aut(\Gamma)$ acts transitively on $V(\Gamma)$. For $v \in V(\Gamma)$ and $G = Aut(\Gamma)$ the *stabilizer subgroup* G_v is the subgroup of G consisting of all automorphisms fixing v. In the vertex-transitive case, all stabilizer subgroups G_v are conjugate in G, and consequently, they are isomorphic. In this case, the index of G_v in G is given by the equation

$$|G : G_v| = \frac{|G|}{|G_v|} = |V(\Gamma)|.$$

This fact is known as the Orbit-Stabilizer theorem, which is a useful tool in finding the automorphism group of vertex-transitive graphs. Let G be any abstract finite group with identity 1 and suppose that Ω is a set of G with the following two properties.

(i). $x \in \Omega \Longrightarrow x^{-1} \in \Omega$;

(ii). $1 \notin \Omega$.

The *Cayley graph* $\Gamma = \Gamma(G;\Omega)$ is the (simple) graph whose edge set and vertex set are $E(\Gamma) = \{\{g, h\} : g^{-1}h \in \Omega\}$ and $V(\Gamma) = G$, respectively.

Although, in most situations, it is difficult to determine the automorphism group of a graph G, and see how it acts on its vertex set or edge set, there are various papers in the literature on this topic; for example [3, 5–14, 16, 17].

A group G is said to be a semidirect product of N by Q, denoted by $G = N \rtimes Q$, if G contains subgroups N and Q such that:

(i). $N \trianglelefteq G$ (N is a normal subgroup of G);

(ii). $NQ = G$;

(iii). $N \cap Q = 1$.

3. Main result

Definition 3.1. *Let $k > 1$ be an integer and $n = 3k - 1$. Let $C = \{3t + 1 :\ 0 \leq t \leq k - 1\}$ be the subset of $\mathbb{Z}_n$ consisting of the elements congruent to 1 modulo 3. It is easy to see that C is a symmetric set; that is, C is an inverse closed subset of the group $\mathbb{Z}_n$. The Cayley graph $Cay(\mathbb{Z}_n; C)$ is known as the Andrásfai graph, denoted by $And(k)$.*

It follows from Definition 3.1 that $And(k)$ is a regular graph of valency k and the vertex set of $And(k)$ is the set $V = V_0 \cup V_1 \cup V_2$, where $V_0 = \{3t :\ 0 \leq t \leq k - 1\}$, $V_1 = \{3t + 1 :\ 0 \leq t \leq k - 1\}$, and $V_2 = \{3t + 2 :\ 0 \leq t \leq k - 2\}$. Thus, one has $|V_0| = |V_1| = k$ and $|V_2| = k - 1$. If $v \in V_0$, then $v = 3j$ for some j with $0 \leq j \leq k - 1$. Now, it is easy to see that

$$N(v) = \{3i + 1 :\ j \leq i \leq k - 1\} \cup \{3l + 2 :\ 0 \leq l \leq j - 1\}. \tag{1}$$

Also, if $w \in V_2$, then $w = 3j + 2$, $0 \leq j \leq k - 2$, and thus we have

$$N(w) = \{3i + 1 :\ 0 \leq i \leq j\} \cup \{3l :\ j + 1 \leq l \leq k - 1\}. \tag{2}$$

Now, from (1) and (2), it follows that the graph induced by the set $V_0 \cup V_2$ in $And(k)$ is a bipartite graph such that the vertex $3j = v \in V_0$ has j neighbors in V_2 and the vertex $3j + 2 = w \in V_2$ has $k - j - 1$ neighbors in V_0. Note that all the neighbors of the vertex $v = 0$ are in V_1. Let $H = \langle (V_0 - \{0\}) \cup V_2 \rangle$ be the subgraph induced by the set $(V_0 - \{0\}) \cup V_2$ in $And(k)$. Thus, H is a connected bipartite graph such that if v, w are distinct vertices in H, then we have $N(v) \neq N(w)$ (note that the vertex $v = 3(k - 1)$ is adjacent to every vertex in V_2 and the vertex $w = 2$ is adjacent to any vertex in V_0).

In the sequel, the following fact is needed.

Lemma 3.1. *Let $\Gamma = (U \cup W, E)$ be a connected bipartite graph, where $U \cap W = \emptyset$. If f is an automorphism of the graph Γ, then $f(U) = U$ and $f(W) = W$, or $f(U) = W$ and $f(W) = U$.*

Proof. Since automorphisms of Γ preserve distance between vertices and because two vertices are in the same part if and only if they are at even distance from each other, the result follows. $\square$

Theorem 3.1. *Let $k > 1$ be an integer and $n = 3k - 1$. For the automorphism group $Aut(And(k))$ of the graph $And(k)$, it holds that $Aut(And(k)) \cong \mathbb{D}_{2n}$, where $\mathbb{D}_{2n}$ denotes the dihedral group of order $2n$.*

Proof. Let $\Gamma = (V, E) = And(k)$. Let $A = Aut(\Gamma)$ be the automorphism group of Γ. Consider the vertex $v = 0$ and let A_0 be its stabilizer subgroup; that is, $A_0 = \{a \in A : \ a(0) = 0\}$. The graph Γ, being a Cayley graph, is a vertex-transitive graph. From the well-known Orbit-Stabilizer theorem, it follows that

$$|V| = \frac{|A|}{|A_v|}$$

and hence $|A| = |V||A_v|$, where v is a vertex in Γ.

In the first step of the proof, $|A_0|$ is determined. Let $f \in A_0$. Let V_0, V_1, and V_2 be the subsets of V which are defined preceding (1) and take $W_0 = V_0 - \{0\}$. Thus, for the restriction of f to $N(0) = V_1$, one has $f(V_1) = V_1$ and hence $f(W_0 \cup V_2) = W_0 \cup V_2$. Let H be the subgraph induced by the set $W_0 \cup V_2$ in the graph $\Gamma = And(k)$ and $g = f|_{W_0 \cup V_2}$. Hence, g is an automorphism of H. It is clear that H is a connected bipartite graph with parts W_0 and V_2 such that $|W_0| = |V_2| = k - 1$. In each part of the graph H there is exactly one vertex x_j of degree j, $1 \leq j \leq k - 1$. In other words, the vertex $v_j = 3j$ is the unique vertex in W_0 of degree j. Also, the vertex $w_j = 3k - 1 - 3j = 3(k - j) - 1 = 3(k - j - 1) + 2$ is the unique vertex in V_2 of degree j. Note that $w_j = 3k - 1 - 3j$ is the inverse of $v_j = 3j$ in the cyclic group $\mathbb{Z}_{3k-1}$ and hence it is denoted by $-v_j$. Note that the mapping g is an automorphism of the connected bipartite graph H. Thus, from Lemma 3.1, it follows that

(i). $g(W_0) = W_0$ or

(ii) $g(W_0) = V_2$.

(i) Assume that $g(W_0) = W_0$. Since the vertex $v = 3j$ is the unique vertex of W_0 of degree j, for every $w \in W_0$ one has $g(w) = w$. Similarly, for every $v \in V_2$ one has $g(v) = v$. In other words, the restriction of the automorphism f to the set $W_0 \cup V_2$ is the identity mapping. We show that if $x \in V_1$, then $f(x) = x$. Note that we have $f(V_1) = V_1$. If $v = 3j + 1$ is a vertex in V_1, then the set of neighbors of v in W_0 is $N_1 = \{3t : \ \ 1 \leq t \leq j\}$. Hence, v has exactly j neighbors in W_0. Since the number of neighbors of v and $f(v)$ in W_0 are equal, hence we must have $v = f(v)$. From our discussion it follows that if $g(W_0) = 0$, then the automorphism f is the identity automorphism of the graph $And(k)$, that is, $f = 1$.

(ii) Suppose that $g(W_0) = V_2$. Let $v \in W_0$. We saw that the vertex $-v$ is the unique vertex in V_2 such that its degree in the graph H is equal to the degree of v; that is, $deg_H(-v) = deg_H(v)$. It follows that for every vertex x of H we have $g(x) = -x$. Since $\Gamma = And(k)$ is an Abelian Cayley graph, the mapping $a : V(\Gamma) \to V(\Gamma)$ defined by the rule $a(v) = -v$ is an automorphism of the graph Γ. Let $b = af$. Thus, b is an automorphism of Γ such that its restriction to W_0 is the identity automorphism. Now, by what is proved in (i), it follows that $b = 1$. Since a has order 2 in $Aut(\Gamma)$, one has $f = a$.

We now conclude that if $A = Aut(\Gamma)$ and A_0 is the stabilizer subgroup of the vertex $v = 0$, then $A_0 = \{1, a\}$, and hence we have $|A_0| = 2$. Now, from Orbit-Stabilizer theorem it follows that $|A| = |A_v||V(\Gamma)| = 2(3k - 1)$. On the other hand, we know that $Aut(\Gamma)$ has a subgroup isomorphic to the cyclic group $\mathbb{Z}_{3k-1}$; that is, $S = \{f_v \mid v \in \mathbb{Z}_{3k-1}\}$, where $f_v : V(\Gamma) \to V(\Gamma)$, $f_v(x) = x + v$ for every $x \in \mathbb{Z}_{3k-1}$. It is easy to check that $A_0 \cap S = \{1\}$. Hence,

$$|A_0 S| = \frac{|A_0||S|}{|A_0 \cap S|} = 2(3k - 1) = |A|.$$

Thus, $A = SA_0$. Since the index of S in A is 2, S is a normal subgroup of A. Now, one has $A \cong S \rtimes A_0 \cong \mathbb{Z}_{3k-1} \rtimes \mathbb{Z}_2 \cong \mathbb{D}_{2n}$, where $n = 3k - 1$. $\square$

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