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Research Article

Odd facial total-coloring of unicyclic plane graphs

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Abstract

A facial total-coloring of a plane graph G is a coloring of the vertices and edges such that no facially adjacent edges (edges that are consecutive on the boundary walk of a face of G), no adjacent vertices, and no edge and its endvertices are assigned the same color. A facial total-coloring of G is odd if for every face f and every color c, either no element or an odd number of elements incident with f is colored by c. In this paper, it is proved that every unicyclic plane graph admits an odd facial total-coloring with at most 10 colors. It is also shown that this bound is tight.

Keywords: facial coloring; odd facial coloring; plane graph.

2020 Mathematics Subject Classification: 05C10, 05C15.

1. Introduction

All graphs considered in this paper are simple connected plane graphs. A *plane graph* is a particular drawing of a planar graph in the Euclidean plane such that no edges intersect except at their endvertices. Let G be a connected plane graph with vertex set $V(G)$, edge set $E(G)$, and face set $F(G)$. The *boundary* of a face f is the boundary in the usual topological sense. It is the collection of all edges and vertices contained in the closure of f that can be organized into a closed walk in G traversing along a simple closed curve lying just inside the face f. This closed walk is unique up to the choice of the initial vertex and the direction, and is called the *boundary walk* of the face f (see [5], p. 101).

Two vertices (two edges) are *adjacent* if they are connected by an edge (have a common endvertex). A vertex and an edge are *incident* if the vertex is an endvertex of the edge. A vertex (or an edge) and a face are incident if the vertex (or the edge) lies on the boundary of the face. Two edges of a plane graph G are *facially adjacent* if they are consecutive on the boundary walk of a face of G.

Colorings of plane graphs under constrains given by faces have recently attracted much attention, see e.g. [1, 2]. The concept of *facial total-coloring* of plane graphs was introduced by Fabrici, Jendroľ, and Vrbjarová [4]. A facial total-coloring of a plane graph G is a coloring of the vertices and edges such that no facially adjacent edges, no adjacent vertices, and no edge and its endvertices are assigned the same color.

An *odd facial total-coloring* of a plane graph is a facial total-coloring such that for every face f and every color c, either no element or an odd number of elements incident with f is colored by c. Odd facial total-coloring of acyclic plane graphs was studied in [3], the authors proved that every acyclic plane graph has an odd facial total-coloring with at most 5 colors. Moreover, this bound is tight.

In this paper, we investigate odd facial total-coloring of unicyclic plane graphs. We prove that 10 colors are sufficient in this case. Moreover, we show that infinitely many unicyclic plane graphs need 10 colors for such a coloring.

2. Main results

Let $\chi''_o(G)$ denote the minimum number of colors required in an odd facial total-coloring of a plane graph G.

Theorem 2.1. *If C_n is the cycle on $n \geq 3$ vertices, then*

$$\chi''_o(C_n) = \begin{cases} 8 & \textit{if} \quad n = 4, \\ 6 & \textit{if} \quad n \in \{3, 7\}, \\ 4 & \textit{otherwise}. \end{cases}$$

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Proof. Let $C_n = v_1e_1v_2e_2\dots v_ne_nv_1$, where $v_1, v_2, \dots, v_n$ are the vertices, $e_1, e_2, \dots, e_n$ are the edges, $e_i = v_iv_{i+1}$ for $i = 1, 2, \dots, n-1$ and $e_n = v_nv_1$. Since C_n has an even number of elements (vertices and edges), every its odd facial total-coloring uses an even number of colors. The elements v_1, e_1, v_2 receive distinct colors in each odd facial total-coloring of C_n, therefore $\chi''_o(C_n) \geq 4$ for every $n \geq 3$. In the following, we define an odd facial total-coloring of C_n, we color the elements in order $v_1, e_1, v_2, e_2, \dots, v_n, e_n$.

If $n = 3k$, $k \geq 2$, then we use three times the pattern $1, 2, 3, 4$ and then $2k-4$ times the pattern $1, 2, 3$.

If $n = 3k+1$, $k \geq 3$, then we use five times the pattern $1, 2, 3, 4$ and then $2k-6$ times the pattern $1, 2, 3$.

If $n = 3k+2$, $k \geq 1$, then we use once the pattern $1, 2, 3, 4$ and then $2k$ times the pattern $1, 2, 3$.

If $n = 7$, then we use three times the pattern $1, 2, 3, 4$ and then the pattern $5, 6$.

If $n = 3$ or $n = 4$, the we use the pattern $1, 2, \dots, 2n$.

It is easy to see that in the cases $n = 3$ and $n = 4$ no color can appear more than once on C_n, and no color can appear more than three times on C_7 in any odd facial total-coloring. □

In the proof of Theorem 2.2 we will use the following properties of the above defined odd facial total-coloring of C_n:

P1: There is no vertex of color $4, 6, 8$.

P2: There is no edge of color $5, 7$.

P3: If a vertex is incident with edges of colors 4 and 6, then its color is 5.

P4: If a vertex has color 5 or 7, then it is not incident with edges of color $1, 2, 3$.

We define an (a, b, c, d)-coloring of a rooted tree T in the following way. Let v be the root of T. We color the edges and vertices of T starting from the root to the leaves. The root does not receive any color.

In the first step we color the edges incident with v with colors a and b such that facially adjacent edges receive different colors. Next, if an edge received color a (respectively, b), then we color the uncolored endvertex (different from v) of this edge with b (respectively, a).

In the second step we use colors c and d. We color all edges with one uncolored endvertex (different from v) with colors c and d such that facially adjacent edges receive different colors. Then, if an edge received color c (respectively, d), then we color the uncolored endvertex of this edge with d (respectively, c).

In the next step a and b play the role of c and d from the previous step, and so on, see Figure 1 for illustration.

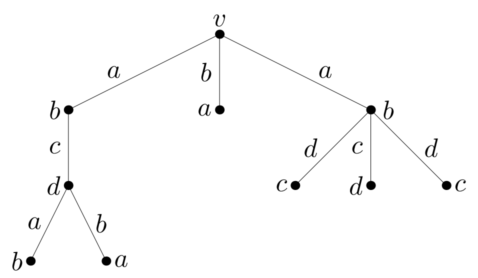

Figure 1: An (a, b, c, d)-coloring of a rooted tree with root v.

Theorem 2.2. *If G is a connected unicyclic plane graph, then $\chi''_o(G) \leq 10$.*

Proof. Let C be the cycle in G. First we color the vertices and edges of C using the coloring defined in the proof of Theorem 2.1. The other uncolored elements are either in the interior or in the exterior of C. First we remove all uncolored elements from the exterior of C and also remove the edges of C. In such a way we obtain some trees. Each such tree has exactly one colored vertex (a vertex which lies on C). Let the colored vertex be the root of the tree. By property P1 each root has color $1, 2, 3, 5$ or 7.

If the root of the tree has color $1, 2$ or 3, then we find its $(5, 7, 2, 3)$-coloring. If the root of the tree has color 5 or 7, then we find its $(2, 3, 5, 7)$-coloring.

Now consider the face f which is incident with vertices and edges lying in the interior of C. Let X_i be the set of elements of color i incident with f. Let c be a color which appears on an element incident with f. We say that c is bad if it appears on an even number of elements incident with f, otherwise it is good. Observe that if c is bad, then $c \in \{2, 3, 5, 7\}$. In the following we recolor some vertices and edges incident with f, so that there will be no bad color on f.

Case 1. If four colors appear on C, then for each bad color c we choose one element of f of color c which does not lie on C and recolor it with a color from the set $\{6, 8, 9, 10\}$. We use different colors for distinct bad colors.

Case 2. Now assume that six colors appear on C. In this case at least one of the colors $5, 7$ is good, because 5 appears on C but 7 does not appear on C. Similarly as above, for each bad color c we choose one element of f of color c which does not lie on C and recolor it with a color from the set $\{8, 9, 10\}$.

Case 3. Finally, assume that eight colors appear on C. This implies $|X_2| = |X_3|$ and $|X_5| = |X_7|$. We distinguish three cases.

Case 3.1. $|X_2|$ and $|X_5|$ are odd.
In this case there is no bad color.

Case 3.2. Either $|X_2|$ or $|X_5|$ is odd.
Without loss of generality, assume that $|X_2|$ is odd and $|X_5|$ is even. Consequently, only the colors $5, 7$ are bad. For each bad color c we choose one element of f of color c which does not lie on C and recolor it with a color from the set $\{9, 10\}$.

Case 3.3. $|X_2|$ and $|X_5|$ are even.
In this case there are four bad colors, namely $2, 3, 5, 7$. This implies that there are edges $e_1 = u_1v_1$ and $e_2 = u_2v_2$ in the interior of C such that:

- the vertices v_1 and v_2 do not lie on C,
- e_1 has color 2 and v_1 has color 3,
- e_2 has color 5 and v_2 has color 7.

Case 3.3.1. First assume that no two such edges are adjacent in the interior of C.
In this case we recolor e_1 with 9, e_2 with 10, and both of the vertices v_1, v_2 with color 4 (by property P1, the vertices u_1 and u_2 have color different from 4), see Figure 2 for illustration.

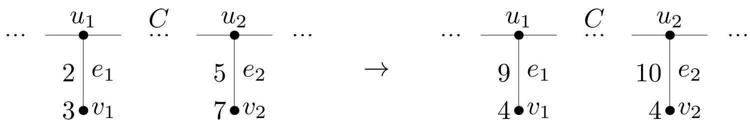

Figure 2: Recoloring of edges e_1, e_2 and vertices v_1, v_2.

Case 3.3.2. Now assume that e_1 and e_2 are adjacent.
Observe that we can choose e_1 and e_2 in such a way that one of them is incident with C, i.e. either u_1 or u_2 lies on C.

Case 3.3.2.1. First assume that u_1 is incident with C.
In this case u_1 is a root of a tree which is hanging on u_1 in the interior of C and this tree was colored by a $(2, 3, 5, 7)$-coloring (since e_1 has color 2). Consequently, u_1 has color 5 or 7. By property P4, no edge incident with u_1 has color 1. Therefore, it is sufficient to recolor e_1 with 1, v_1 with 9, e_2 with 10, and v_2 with 1, see Figure 3 for illustration.

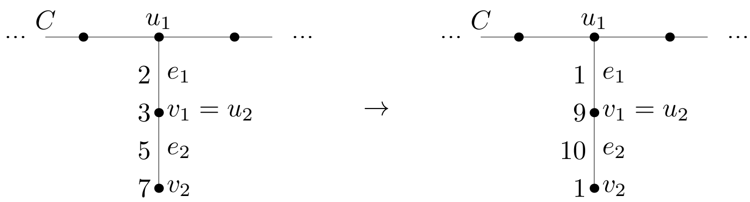

Figure 3: Recoloring of edges e_1, e_2 and vertices v_1, v_2.

Case 3.3.2.2. Finally, assume that u_2 is incident with C.
Then the tree in the interior of C hanging on u_2 was colored by a $(5, 7, 2, 3)$-coloring (since e_2 has color 5). This means that u_2 has color $1, 2$ or 3. By property P3, at least one of the colors $4, 6$ does not appear on the edges of C incident with u_2. Without loss of generality, assume that 4 does not appear on the edges incident with u_2. Now we recolor e_2 with 4, v_2 with 9, e_1 with 10, and v_1 with 4, see Figure 4 for illustration.

In each case we obtain a coloring without bad colors. Since we do not recolor any element of C, we can use the same recoloring technique in the exterior of C. □

Theorem 2.3. *For every positive integer n, there is a connected unicyclic plane graph G with at least n vertices, such that $\chi''_o(G) = 10$.*

Proof. Consider the graph G depicted in Figure 5. Let φ be an odd facial total-coloring of G. By the definition of the facial total-coloring we have $\varphi(y) \notin \{\varphi(x), \varphi(e_i), \varphi(v_i), i = 1, 2, \ldots, n\}$. The vertices and edges of the cycle $u_1u_2u_3u_4u_1$ are colored

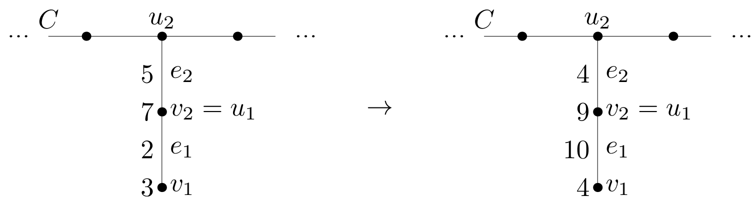

Figure 4: Recoloring of edges e_1, e_2 and vertices v_1, v_2.

with eight different colors, consequently, the color $\varphi(y)$ is unique. Therefore, φ uses at least nine colors. On the other hand, φ uses an even number of colors, since $|V(G)|+|E(G)|$ is even and the outer face of G is incident with all vertices and edges of G. Consequently, $\chi''_o(G) \geq 10$. □

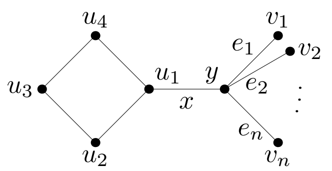

Figure 5: A unicyclic plane graph G with $\chi''_o(G) = 10$.

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