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Research Article

A binomial formula for evaluating integrals

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Abstract

In this paper, a special formula for transforming integrals to series is presented. The resulting series involves binomial transforms with the Taylor coefficients of the integrand. Five applications are provided for evaluating challenging integrals.

Keywords: binomial transform; binomial identities; integral evaluation; harmonic numbers; dilogarithm; trilogarithm.

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1. Main theorem

The following result gives a rule for evaluating integrals in terms of series with binomial expressions.

Theorem 1.1. *Let $f(x)$ be a function defined and integrable on the interval $(-r, \lambda]$ for some $r > 0,\ \lambda > 0$. Also, let $f(x)$ be analytic in a neighborhood of the origin with Taylor series*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then, we have

$$\int_0^{\lambda} f(x)dx = \sum_{n=0}^{\infty} \left(\frac{\lambda}{\lambda+1}\right)^{n+1} \frac{1}{(n+1)} \sum_{m=0}^{n} b_m = \sum_{n=0}^{\infty} \left(\frac{\lambda}{\lambda+1}\right)^{n+1} \sum_{k=0}^{n} \binom{n}{k} \frac{a_k}{k+1}$$

where the sequence $\{b_n\}$ is the binomial transform of the sequence $\{a_n\}$:

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k.$$

In particular, for $\lambda = 1$, we have

$$\int_0^{1} f(x)dx = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}(n+1)} \sum_{m=0}^{n} b_m = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{a_k}{k+1}$$

and for $\lambda \to \infty$, we have

$$\int_0^{\infty} f(x)dx = \sum_{n=0}^{\infty} \frac{1}{(n+1)} \sum_{m=0}^{n} b_m = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{a_k}{k+1}.$$

Proof. With the substitution $x = \frac{t}{1-t},\ t = \frac{x}{x+1}$, we get

$$\int_0^{\lambda} f(x)dx = \int_0^{\lambda/(\lambda+1)} \frac{1}{(1-t)^2} f\left(\frac{t}{1-t}\right) dt = \int_0^{\lambda/(\lambda+1)} \frac{1}{1-t} \left\{ \frac{1}{1-t} f\left(\frac{t}{1-t}\right) \right\} dt$$

$$= \int_0^{\lambda/(\lambda+1)} \frac{1}{1-t} \left\{ \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \binom{n}{k} a_k \right\} dt = \int_0^{\lambda/(\lambda+1)} \frac{1}{1-t} \left\{ \sum_{n=0}^{\infty} b_n t^n \right\} dt$$

by using Euler's series transformation formula

$$\frac{1}{1-t} f\left(\frac{t}{1-t}\right) = \sum_{n=0}^{\infty} t^n \left\{ \sum_{k=0}^{n} \binom{n}{k} a_k \right\} = \sum_{n=0}^{\infty} b_n t^n$$

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where the sequence $\{b_n\}$ is the binomial transform of the sequence $\{a_n\}$as described above.
Expanding $(1-t)^{-1}$ as geometric series and using Cauchy's rule for multiplication of two power series, we write

$$\int_0^{\lambda/(\lambda+1)} \frac{1}{1-t}\left\{\sum_{n=0}^{\infty} b_n t^n\right\} dt = \int_0^{\lambda/(\lambda+1)} \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n} b_k\right\} t^n dt = \sum_{n=0}^{\infty}\left(\frac{\lambda}{\lambda+1}\right)^{n+1}\frac{1}{n+1}\sum_{k=0}^{n} b_k$$

by the property

$$\sum_{k=0}^{n}\binom{n}{k}\frac{a_k}{k+1} = \frac{1}{n+1}\sum_{k=0}^{n} b_k$$

(see p. 61 in [1]). The interchange of integration and summation is justified as we work with power series. This way the theorem is proved. □

Differentiating in Theorem 1.1 with respect to λ, we come to the following result.

Corollary 1.1. *Under the conditions of Theorem 1.1, we have the representation*

$$f(\lambda) = \frac{1}{(\lambda+1)^2}\sum_{n=0}^{\infty}\left(\frac{\lambda}{\lambda+1}\right)^n \sum_{m=0}^{n} b_m.$$

2. Applications

In this section, we give some applications of Theorem 1.1 in the form of examples.

Example 2.1. In this example, we evaluate the integral

$$\int_0^{\infty} \frac{\log(1+t)}{t(1+t)} dt.$$

We start from the well-known series

$$\sum_{n=1}^{\infty} H_n t^n = \frac{-\log(1-t)}{1-t},$$

(here $H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ and $H_0 = 0$ are the harmonic numbers). Replacing t by $-t$ and dividing both sides by t, we get

$$\frac{\log(1+t)}{t(1+t)} = \sum_{k=0}^{\infty}(-1)^k H_{k+1} t^k$$

and we take $a_k = (-1)^k H_{k+1}$. Equation (9.32) in [1] says that

$$\sum_{k=0}^{n}\binom{n}{k}\frac{a_k}{k+1} = \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k H_{k+1}}{k+1} = \frac{1}{(n+1)^2}.$$

This way

$$\int_0^{\infty}\frac{\log(1+t)}{t(1+t)}dt = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{a_k}{k+1} = \sum_{n=0}^{\infty}\frac{1}{(n+1)^2} = \frac{\pi^2}{6},$$

(see Entry 4.291.12 in [3]). □

The integral given in the next example can be reduced to the one given in Example 2.1 using integration by parts, but we evaluate it independently for illustrating the method.

Example 2.2. Here, we evaluate the difficult integral

$$\int_0^{\infty}\left(\frac{\log(1+t)}{t}\right)^2 dt.$$

We start from the well-known power series (see (5.5.28) in [4])

$$\frac{\log^2(1-t)}{2t} = \sum_{n=1}^{\infty}\frac{H_n t^n}{n+1}$$

where we replace t by $-t$ and then divide both sides by t to write

$$\frac{\log^2(1+t)}{2t^2} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}H_n t^{n-1}}{n+1} = \sum_{k=0}^{\infty}\frac{(-1)^k H_{k+1} t^k}{k+2}.$$

So, we take

$$a_k = \frac{(-1)^k H_{k+1}}{k+2}, \quad \sum_{k=0}^{n} \binom{n}{k} \frac{a_k}{k+1} = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k H_{k+1}}{(k+1)(k+2)}.$$

Now,

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k H_{k+1}}{(k+1)(k+2)} = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k H_{k+1}}{k+1} - \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k H_{k+1}}{k+2}$$

$$= \frac{1}{(n+1)^2} - \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k+2} \left(H_k + \frac{1}{k+1} \right)$$

(using again (9.32) of [1]). Next, applying property (5.5) from [1], we get

$$-\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k+2} \left(H_k + \frac{1}{k+1} \right) = -\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k H_k}{k+2} - \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(k+1)(k+2)}$$

$$= \frac{n+H_n}{(n+1)(n+2)} - \frac{1}{n+2} = \frac{H_n - 1}{(n+1)(n+2)}.$$

It is easy to see that

$$\sum_{n=0}^{\infty} \frac{H_n - 1}{(n+1)(n+2)} = \sum_{n=0}^{\infty} \frac{H_n}{(n+1)(n+2)} - \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = 1 - 1 = 0$$

and we compute

$$\int_0^{\infty} \frac{\log^2(1+t)}{2t^2}\, dt = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{a_k}{k+1} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{6}.$$

Finally,

$$\int_0^{\infty} \left(\frac{\log(1+t)}{t} \right)^2 dt = \frac{\pi^2}{3}.$$

□

Example 2.3. Using some well-known generating functions, we evaluate the integral

$$\int_0^1 Li_2\left(\frac{t}{1+t} \right) dt.$$

Here $Li_2(x)$ is the dilogarithm [5]:

$$Li_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} \quad (|x| < 1).$$

We have

$$Li_2\left(\frac{t}{1+t} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} H_n t^n}{n} \quad (|t| < 1)$$

so that we take

$$a_n = \frac{(-1)^{n-1} H_n}{n}, \quad \frac{a_k}{k+1} = \frac{(-1)^{k-1} H_k}{k(k+1)} = \frac{(-1)^{k-1} H_k}{k} - \frac{(-1)^{k-1} H_k}{k+1},$$

and using two binomial transform formulas (9.4a) and (9.32) from [1], we have

$$\sum_{k=0}^{n} \binom{n}{k} \frac{a_k}{k+1} = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{k-1} H_k}{k} - \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{k-1} H_k}{k+1} = H_n^{(2)} - \frac{H_n}{n+1}.$$

Here,

$$H_n^{(2)} = 1 + \frac{1}{2^2} + \ldots + \frac{1}{n^2}, H_0^{(2)} = 0.$$

This way

$$\int_0^1 Li_2\left(\frac{t}{1+t} \right) dt = \sum_{n=0}^{\infty} \frac{H_n^{(2)}}{2^{n+1}} - \sum_{n=0}^{\infty} \frac{H_n}{2^{n+1}(n+1)}.$$

These two series are easy to evaluate. We have (see p. 292 in [2])

$$\sum_{n=0}^{\infty} H_n^{(2)} x^n = \frac{Li_2(x)}{1-x}, \quad \sum_{n=0}^{\infty} \frac{H_n\, x^n}{n+1} = \frac{\log^2(1-x)}{2x} \quad (|x| < 1)$$

and we compute with $x = \frac{1}{2}$,

$$\int_0^1 Li_2\left(\frac{t}{1+t}\right) dt = Li_2\left(\frac{1}{2}\right) - \frac{\log^2(2)}{2} = \frac{\pi^2}{12} - \log^2(2),$$

here we used the well-known formula:

$$Li_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{\log^2 2}{2}.$$

□

Example 2.4. Let q be a positive integer. In this example, we evaluate the integral

$$\int_0^1 \frac{x^q}{(1+x)^{q+1}}\, dx.$$

Let $a_k = \left(\begin{array}{c} k \\ q \end{array}\right)(-1)^k$. Then

$$f(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \left(\begin{array}{c} k \\ q \end{array}\right)(-1)^k x^k = \sum_{k=q}^{\infty} \left(\begin{array}{c} k \\ q \end{array}\right)(-x)^k = \frac{(-x)^q}{(1+x)^{q+1}} = \frac{(-1)^q x^q}{(1+x)^{q+1}}.$$

$$\int_0^1 f(x)dx = (-1)^q \int_0^1 \frac{x^q}{(1+x)^{q+1}}\, dx = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) \frac{a_k}{k+1}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) \left(\begin{array}{c} k \\ q \end{array}\right) \frac{(-1)^k}{k+1} = (-1)^q \sum_{n=q}^{\infty} \frac{1}{2^{n+1}(n+1)}$$

because (see Equation (10.28) in [1])

$$\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) \left(\begin{array}{c} k \\ q \end{array}\right) \frac{(-1)^k}{k+1} = \frac{(-1)^q}{n+1}.$$

This way

$$\int_0^1 \frac{x^q}{(1+x)^{q+1}}\, dx = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}(n+1)} - \sum_{n=0}^{q-1} \frac{1}{2^{n+1}(n+1)} = \log 2 - \sum_{n=1}^{q} \frac{1}{2^n n},$$

(see Entry 3.194.8 in [3]). □

Example 2.5. In this example, we evaluate the challenging integral

$$\int_0^1 \frac{\log^2(1+x)}{2x}\, dx.$$

We have the expansion (see (5.5.2) in [4])

$$\frac{\log^2(1+t)}{2t} = \sum_{k=0}^{\infty} \frac{(-1)^{k-1} H_k t^k}{k+1} \qquad (|t| < 1)$$

and here

$$a_k = \frac{(-1)^{k-1} H_k}{k+1}, \qquad \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) \frac{(-1)^{k-1} H_k}{k+1} = \frac{H_n}{n+1}$$

(see (9.32) in [1]). Also, (see (5.7) in [1])

$$\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) \frac{a_k}{k+1} = \frac{1}{n+1} \sum_{k=0}^{n} \frac{H_k}{k+1}.$$

Therefore,

$$\int_0^1 \frac{\log^2(1+x)}{2x}\, dx = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}(n+1)} \sum_{k=0}^{n} \frac{H_k}{k+1}.$$

A simple computation shows that

$$\sum_{k=0}^{n} \frac{H_k}{k+1} = \frac{1}{2}\left(H_n^2 - H_n^{(2)}\right) + \frac{H_n}{n+1}$$

(an identity interesting by itself). From this, one gets

$$\int_0^1 \frac{\log^2(1+x)}{2x}\,dx = \frac{1}{2}\sum_{n=0}^{\infty}\frac{H_n^2 - H_n^{(2)}}{2^{n+1}(n+1)} + \sum_{n=0}^{\infty}\frac{H_n}{2^{n+1}(n+1)^2}.$$

These two sums can be evaluated easily by using the generating functions

$$\sum_{n=0}^{\infty}\frac{(H_n^2 - H_n^{(2)})\,t^{n+1}}{n+1} = -\frac{1}{3}\log^3(1-t),$$

$$\sum_{n=0}^{\infty}\frac{H_n\,t^{n+1}}{(n+1)^2} = \frac{1}{2}\log(t)\log^2(1-t) + \log(1-t)Li_2(1-t) - Li_3(1-t) + \zeta(3)$$

(see p. 303 in [5]). Here,

$$Li_3(x) = \sum_{n=1}^{\infty}\frac{x^n}{n^3} \quad (|x| < 1)$$

is the trilogarithm [5]. Setting here $t = \frac{1}{2}$ and using the values

$$Li_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{\log^2 2}{2}, Li_3\left(\frac{1}{2}\right) = \frac{7}{8}\zeta(3) - \frac{\pi^2}{12}\log(2) + \frac{1}{6}\log^2(2),$$

we come to the evaluation

$$\int_0^1 \frac{\log^2(1+x)}{2x}\,dx = \frac{1}{8}\zeta(3).$$

□

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