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Research Article

Gallai-Ramsey number for rainbow S_3

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Abstract

For the given graphs G and H, and for a positive integer k, the Gallai-Ramsey number is denoted by $gr_k(G : H)$ and is defined as the minimum integer n such that every coloring of the complete graph K_n using at most k colors contains either a rainbow copy of G or a monochromatic copy of H. The k-color Ramsey number for G, denoted by $R_k(G)$, is the minimum integer n such that every coloring of K_n using at most k colors contains a monochromatic copy of G in some color. Let S_n be the star graph on n edges and let P_n be the path graph on n vertices. Denote by S_n^+ the graph obtained from S_n by adding an edge between any two pendant vertices. Let T_{n+2} be the tree on $n+2$ vertices obtained from S_n by subdividing one of its edges. In this paper, we consider $gr_k(S_3 : H)$, where $H \in \{S_n, S_n^+, P_n, T_{n+2}\}$, and obtain its relation with $R_2(H)$ and $R_3(H)$. We also obtain 3-color Ramsey numbers for S_n, S_n^+, and T_{n+2}.

Keywords: Gallai-Ramsey number; coloring; rainbow copy; monochromatic copy.

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1. Introduction

In this paper, edge-colorings of finite simple graphs are considered. Throughout this paper, by coloring we mean edge-coloring. For an integer $k \geq 1$, let $\mathcal{C} : E(G) \to \{1, 2, ..., k\}$ be a k-coloring of a graph G. Thus, $\mathcal{C}$ partitions the edge set of G, $E(G)$, into k sets $C_1, C_2, \cdots, C_k$, where C_i consists of those edges of G that are colored with color i. Note that $\mathcal{C}$ need not be a proper coloring. The color i is represented at a vertex v if some edge incident with v has color i. A coloring of a graph is called monochromatic if all edges are colored the same, and a coloring is called rainbow if all edges are colored differently. Given a graph G, the k-color Ramsey number for G, denoted by $R_k(G)$, is the minimum integer n such that every coloring of the complete graph K_n using at most k colors contains a monochromatic copy of G in some color. For the given graphs G and H, and for a positive integer k, the Gallai-Ramsey number, denoted by $gr_k(G : H)$, is defined as the minimum integer n such that every coloring of K_n using at most k colors contains either a rainbow copy of G or a monochromatic copy of H. For any graph H, the inequality $gr_k(G : H) \leq R_k(H)$ holds.

In 1967, Gallai [4] investigated the structures of rainbow triangle-free (i.e., there is no rainbow K_3) colorings of complete graphs and proved the following result. In honor of Gallai's work, a coloring of a complete graph G is said to be Gallai coloring if G is rainbow triangle-free.

Theorem 1.1. [4] *In any Gallai colored complete graph G, $V(G)$ can be partitioned into non-empty sets $H_1, H_2, \cdots, H_l$, with $l \geq 2$, such that there are at most two colors between the parts, and there is only one color on the edges between every pair of parts.*

In recent years, many results on Gallai-Ramsey numbers concerning the case when G is a triangle have been reported [2, 3, 8]. However, Gallai-Ramsey numbers for other choices of G have been much less studied. In [6], the authors proved the following theorem for $G = P_4$ and posed a conjecture when $G = P_5$.

Theorem 1.2. [6] *For any graph H with no isolated vertices, $gr_k(P_4 : H) = R_2(H)$ except when $H = P_3$ and $k \geq 3$, in which case $gr_k(P_4 : P_3) = 5$.*

Conjecture 1.1. [6] *For any graph H with no isolated vertices, $gr_k(P_5 : H) = R_3(H)$.*

Gyárfás et al. [5] proved the next result concerning 3-color Ramsey numbers of paths, which was conjectured by Faudree and Schelp in [1].

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Theorem 1.3. [5] *For sufficiently large* n, $R_3(P_n) = \begin{cases} 2n-1 & \text{if } n \text{ is odd,} \\ 2n-2 & \text{if } n \text{ is even.} \end{cases}$

In this paper, we consider $gr_k(G : H)$ for rainbow S_3 and monochromatic stars, paths and some extensions of stars. Few results are known for the case when $G = S_3$ and finding this number for a path is a fundamental work. Let S_n be the star on $n+1$ vertices and n edges. Denote by S_n^+ the graph obtained from S_n by adding an edge between any two pendant vertices. Let P_n be the path on n vertices and T_{n+2} be the tree on $n+2$ vertices obtained from the star S_n with one edge subdivided. Let $V = \{v_1, v_2, \cdots, v_n\}$ be the vertex set of the complete graph K_n. For any non-empty subset V' of V, the subgraph of K_n whose vertex set is V' and edge set is the set of those edges of K_n that have both ends in V' is called the subgraph of K_n induced by V', denoted by $K_n[V']$.

2. Main results

In this section, 3-color Ramsey numbers for S_n, S_n^+, and T_{n+2} are obtained. Also, in this section, it is shown that for all $k \geq 3$, the inequality $R_2(H) \leq gr_k(S_3 : H) \leq R_3(H)$ holds when $H \in \{S_n, S_n^+, P_n, T_{n+2}\}$. It is clear that $gr_2(S_3 : H) = R_2(H)$.

Theorem 2.1. $R_3(S_n) = 3n - 1$.

Proof. To prove $R_3(S_n) \geq 3n-1$, it is enough to show that there exist a 3-coloring of K_{3n-2} that does not contain a monochromatic copy of S_n. Let us take $G_1 = K_{3n-2}[\{v_1, v_2, \cdots, v_{n-1}\}]$, $G_2 = K_{3n-2}[\{v_n, v_{n+1}, \cdots, v_{2n-2}\}]$ and $G_3 = K_{3n-2}[\{v_{2n-1}, v_{2n}, \cdots, v_{3n-3}\}]$. Color the edges of G_i with color i where $i = 1, 2, 3$. The edge $e = uv$ is colored with color 1 if $u \in G_2, v \in G_3$, with color 2 if $u \in G_1, v \in G_3$ and with color 3 if $u \in G_1, v \in G_2$. Now, the edge $e = uv_{3n-2}$ is assigned color 1 if $u \in G_1$, color 2 if $u \in G_2$ and color 3 if $u \in G_3$. Under this coloring each vertex in K_{3n-2} is represented by color i where $i = 1, 2, 3$, at most $n-1$ times. Thus, K_{3n-2} does not contain a monochromatic copy of S_n. Hence, $R_3(S_n) \geq 3n-1$.

Now, consider any 3-coloring of K_{3n-1} and let v be any vertex in K_{3n-1}. Since $deg(v) = 3n-2$, at least n edges incident with v must be of same color giving a monochromatic copy of S_n. Thus, $R_3(S_n) \leq 3n-1$ and hence $R_3(S_n) = 3n-1$. □

Theorem 2.2. $R_3(T_{n+2}) = 3n$.

Proof. The lower bound can be proved by showing that there exist a 3-coloring of K_{3n-1} that does not contain a monochromatic copy of T_{n+2}. Let $G_1 = K_{3n-1}[\{v_1, v_2, \cdots, v_{n-1}\}]$, $G_2 = K_{3n-1}[\{v_n, v_{n+1}, \cdots, v_{2n-2}\}]$ and $G_3 = K_{3n-1}[\{v_{2n-1}, v_{2n}, \cdots, v_{3n-3}\}]$. Color the edges of G_i and the edges $w_i v_{3n-2}, w_i v_{3n-1}$, $w_i \in V(G_i)$ with color i where $i = 1, 2, 3$. The edge $e = uv$ is colored with color 1 if $u \in G_2, v \in G_3$, with color 2 if $u \in G_1, v \in G_3$ and with color 3 if $u \in G_1, v \in G_2$. Assign color 1 for the edge $v_{3n-2}v_{3n-1}$. Under this coloring K_{3n-1} does not contain a monochromatic copy of T_{n+2}. So, $R_3(T_{n+2}) \geq 3n$.

To prove the upper bound consider a 3-coloring $\mathcal{C} = \{C_1, C_2, C_3\}$ of K_{3n}. Since $deg(v_{3n}) = 3n-1$, at least n edges incident with v_{3n} must be of same color. Let $\{v_{3n}v_1, v_{3n}v_2, \cdots, v_{3n}v_n\} \subseteq C_1$. If there is an edge $v_iv_j \in C_1$, $1 \leq i \leq n, n+1 \leq j \leq 3n-1$, then K_{3n} contains a monochromatic copy of T_{n+2}.

Now, suppose that each edge v_iv_j, $1 \leq i \leq n, n+1 \leq j \leq 3n-1$ belongs to C_2 or C_3. Then a monochromatic copy of T_{n+2} in K_{3n} can be obtained as follows. For $i = 1, 2, 3$, let $E_i = \{v_iv_j, n+1 \leq j \leq 3n-1\}$. Then $|E_i| = 2n-1$ and the edges of E_i are colored with color 2 or color 3. So, in each E_i, n edges are of same color. Let $E'_i \subset E_i$ be such that $|E'_i| = n$ and all edges of E'_i are of same color. Among E'_1, E'_2, E'_3, two of the sets must have edges in same color. Suppose C_2 contains E'_1 and E'_2. Then for some $r, n+1 \leq r \leq 3n-1$ there exists a vertex v_r such that the edges $v_1v_r \in E'_1$ and $v_2v_r \in E'_2$. If such a vertex v_r does not exist, then the set of n end vertices of edges in E'_1 and the set of n end vertices of edges in E'_2 are disjoint. This implies that there exist $2n$ vertices in the set $\{v_j, n+1 \leq j \leq 3n-1\}$, which is not possible. Then $E'_1 \cup \{v_rv_2\}$ will give a monochromatic copy of T_{n+2} in K_{3n} in color 2. Thus, $R_3(T_{n+2}) \leq 3n$. Hence, $R_3(T_{n+2}) = 3n$. □

Lemma 2.1. *Any* 2*-coloring of* K_{2k+1} *contains a monochromatic copy of* S_k^+.

Proof. Consider a 2-coloring $\mathcal{C} = \{C_1, C_2\}$ of K_{2k+1}. Suppose there is a vertex v in K_{2k+1} such that $k+1$ edges incident with v have same color. Let $\{v_{2k+1}v_1, v_{2k+1}v_2, \cdots, v_{2k+1}v_{k+1}\} \subseteq C_1$. If there exist some edge v_iv_j, $1 \leq i < j \leq k+1$, in C_1, K_{2k+1} contains a monochromatic copy of S_k^+ in color 1. Suppose such an edge does not exist. This will imply that every edge of the induced subgraph $G' = K_{2k+1}[\{v_1, v_2, \cdots, v_{k+1}\}]$ is in C_2. Thus, G' and hence K_{2k+1} contains a monochromatic copy of S_k^+ in color 2.

Now, suppose there is no vertex in K_{2k+1} incident with $k+1$ edges in same color. Then every vertex is incident with exactly k edges in C_1 and k edges in C_2. Let $\{v_{2k+1}v_1, v_{2k+1}v_2, \cdots, v_{2k+1}v_k\} \subseteq C_1$. As in the case above if there exist some edge v_iv_j, $1 \leq i < j \leq k$, in C_1, K_{2k+1} contains a monochromatic copy of S_k^+ in color 1. If not, then every edge of $K_{2k+1}[\{v_1, v_2, \cdots, v_k\}]$ is colored with color 2. Since v_k is incident to k edges that are colored with color 2, there exist an edge v_kv_t in C_2, where $k+1 \le t \le 2k$. Thus, $\{v_kv_i,\ 1 \le i \le k-1\} \cup \{v_kv_t\} \cup \{v_1v_2\}$ is a monochromatic copy of S_k^+ in color 2 contained in K_{2k+1}. □

Theorem 2.3. $R_3(S_n^+) = 5n+1$.

Proof. To prove the lower bound consider K_{5n}. Let $G_1 = K_{5n}[\{v_1, v_2, \cdots, v_n\}]$, $G_2 = K_{5n}[\{v_{n+1}, v_{n+2}, \cdots, v_{2n}\}]$, $G_3 = K_{5n}[\{v_{2n+1}, v_{2n+2}, \cdots, v_{3n}\}]$, $G_4 = K_{5n}[\{v_{3n+1}, v_{3n+2}, \cdots, v_{4n}\}]$ and $G_5 = K_{5n}[\{v_{4n+1}, v_{4n+2}, \cdots, v_{5n}\}]$. Assign color 1 to the edges in G_i for $1 \le i \le 5$. All edges in K_{5n} between G_1 and G_2, G_1 and G_3, G_2 and G_4, G_3 and G_5, G_4 and G_5 are colored with color 2. Remaining edges in K_{5n} are colored with color 3. This gives a 3-coloring of K_{5n} which contains a monochromatic copy of S_n but does not contain a monochromatic copy of S_n^+. So, $R_3(S_n^+) \ge 5n+1$.

Consider a 3-coloring $\mathcal{C} = \{C_1, C_2, C_3\}$ of K_{5n+1}. Since $deg(v_{5n+1}) = 5n$ and for $n \ge 3$, $3(n+2) \le 5n$, at least $n+2$ edges incident with v_{5n+1} must have same color. Now, either $n+2$ or $n+1$ must be an odd number and let that odd number be $2k+1$ for some integer k. Let $\{v_{5n+1}v_1, v_{5n+1}v_2, \cdots, v_{5n+1}v_{n+2}\} \subseteq C_1$. If there is an edge $v_iv_j \in C_1$, $1 \le i < j \le n+2$, then K_{5n+1} contains a monochromatic copy of S_n^+.

If there is no such edge, $G_1 = K_{5n+1}[\{v_1, v_2, \cdots, v_{2k+1}\}]$ must be 2-colored. Also G_1 is isomorphic to the complete graph K_{2k+1}. Then by Lemma 2.1, G_1 contains a monochromatic copy of S_k^+ in color 2 and let $\{v_1, v_2, \cdots, v_k, v_{k+1}\}$ be the vertices of $S_k^+ \subseteq G_1$, where v_{k+1} is the hub vertex. If there are $n-k$ edges in $K_{5n+1} \setminus S_k^+$ in color 2 incident with v_{k+1}, then K_{5n+1} contains a monochromatic copy of S_n^+.

Otherwise at most $n-k-1$ edges in color 2 are incident with v_{k+1}. So, at least $4n+1$ edges incident with v_{k+1} are in C_1 or C_3. Among these, $2n+1$ edges must be in C_t where $t = 1$ or 3. Let $\{v_{k+1}v_{5n}, v_{k+1}v_{5n-1}, \cdots, v_{k+1}v_{3n}\} \subseteq C_t$ and let $G_2 = K_{5n+1}[\{v_{3n}, v_{3n+1}, \cdots, v_{5n}\}]$. If there is an edge v_rv_s, $3n \le r < s \le 5n$ in color t, then K_{5n+1} contains a monochromatic copy of S_n^+.

If there is no such edge, then G_2 is 2-colored. Then by Lemma 2.1, there is a monochromatic copy of S_n^+ in G_2 and hence in K_{5n+1}. So, $R_3(S_n^+) \le 5n+1$. Hence, $R_3(S_n^+) = 5n+1$. □

Lemma 2.2. *$gr_k(S_3 : H) \ge R_2(H)$, where $H \in \{S_n, T_{n+2}, P_n, S_n^+\}$.*

Proof. By the definition of $R_2(H)$, there is a 2-coloring of K_m where $m = R_2(H)-1$ which has no monochromatic copy of H. Since only two colors are used, K_m cannot have a rainbow copy of S_3. So, $gr_k(S_3 : H) \ge R_2(H)$. □

Theorem 2.4. $gr_k(S_3 : S_n) = 2n$.

Proof. Consider K_{2n-1}. Color the edges of the induced subgraphs $G_1 = K_{2n-1}[\{v_1, v_2, \cdots, v_{n-1}\}]$ and $G_2 = K_{2n-1}[\{v_n, v_{n+1}, \cdots, v_{2n-2}\}]$ with color 1 and color 2 respectively. Use color 3 for the edges between G_1 and G_2. The edges between the vertices of G_1 and v_{2n-1} are colored with color 1 and those between G_2 and v_{2n-1} are colored with color 2. Now, every vertex of K_{2n-1} are two colored and hence there does not exist a rainbow S_3 in K_{2n-1}. Only a monochromatic S_{n-1} could be obtained with the above coloring. Hence, $gr_k(S_3 : S_n) \ge 2n$.

Let $\mathcal{C}$ be a k-coloring of K_{2n}. If there is a vertex in K_{2n} represented by at least 3 colors, a rainbow copy of S_3 is obtained. If not, $\mathcal{C}$ is such that every vertex of K_{2n} is at most 2-colored. Let v be a vertex of K_{2n}. Since degree of v is $2n-1$, n edges incident with v must be of same color. These n edges gives a monochromatic copy of S_n in K_{2n}. Hence, $gr_k(S_3 : S_n) \le 2n$. Thus, $gr_k(S_3 : S_n) = 2n$. □

Theorem 2.5. $R_2(S_n) \le gr_k(S_3 : S_n) \le R_3(S_n)$.

Proof. From Lemma 2.2, Theorem 2.1, and Theorem 2.4, the result follows. □

Theorem 2.6. $gr_k(S_3 : T_{n+2}) = 2n+1$.

Proof. Consider the complete graph K_{2n}. Color the edges of the induced subgraph $G_1 = K_{2n}[\{v_1, v_2, \cdots, v_{n+1}\}]$ with color 1. Now, color all the edges except the edge v_1v_{n+1} of the induced subgraph $G_2 = K_{2n}[\{v_{n+1}, v_{n+2}, \cdots, v_{2n}, v_1\}]$ with color 2. Use color 3 for the edges connecting the vertices of $G_1 \setminus \{v_1, v_{n+1}\}$ and $G_2 \setminus \{v_1, v_{n+1}\}$. Only a monochromatic S_n is obtained with the above coloring in color 1 and color 2. In color 3 a monochromatic S_{n-1} is obtained. So, $gr_k(S_3 : T_{n+2}) \ge 2n+1$.

Let $\mathcal{C} = \{C_1, C_2, \cdots, C_k\}$ be a k-coloring of K_{2n+1}. If there is a vertex in K_{2n+1} represented by at least 3 colors, a rainbow copy of S_3 is obtained. If not, $\mathcal{C}$ is such that every vertex of K_{2n+1} is at most 2-colored. Since degree of v_{2n+1} is $2n$, at least n edges incident with v_{2n+1} must be of same color. Without loss of generality, let the edges $v_{2n+1}v_i, 1 \le i \le n$ be in C_1. Let $W_1 = \{v_1, v_2, \cdots, v_n\}$ and $W_2 = \{v_{n+1}, v_{n+2}, \cdots, v_{2n}\}$. If there is an edge in C_1 with one end in W_1 and other end in W_2, a monochromatic copy of T_{n+2} in color 1 exist. If not, each $v_1w, w \in W_2$ must be in C_2. Now, if each $v_2w, w \in W_2$ is in C_2, $\{v_1w : w \in W_2\} \cup \{v_2v_{2n}\}$ gives a monochromatic copy of T_{n+2} in color 2. If each $v_2w, w \in W_2$ is in

C_3, v_3v_{2n} must be in C_2 or C_3. If $v_3v_{2n} \in C_2$, $\{v_1w : w \in W_2\} \cup \{v_3v_{2n}\}$ gives a monochromatic copy of T_{n+2} in color 2. Otherwise $\{v_2w : w \in W_2\} \cup \{v_3v_{2n}\}$ gives a monochromatic copy of T_{n+2} in color 3. Hence, $gr_k(S_3 : T_{n+2}) \leq 2n+1$. Thus, $gr_k(S_3 : T_{n+2}) = 2n+1$. □

Theorem 2.7. $R_2(T_{n+2}) \leq gr_k(S_3 : T_{n+2}) \leq R_3(T_{n+2})$.

Proof. From Lemma 2.2, Theorem 2.2, and Theorem 2.6, the result follows. □

Theorem 2.8. $gr_k(S_3 : S_n^+) = 2n+1$, *where* S_n^+ *is obtained from* S_n *by adding an edge between any two pendant vertices.*

Proof. Consider the complete graph K_{2n}. Color the edges of the induced subgraphs $G_1 = K_{2n}[\{v_1, v_2, \cdots, v_n\}]$ and $G_2 = K_{2n}[\{v_{n+1}, v_{n+2}, \cdots, v_{2n}\}]$ with color 1 and color 2 respectively. Use color 3 for the edges between G_1 and G_2. Now, every vertex of K_{2n} are two colored and hence there does not exist a rainbow S_3 in K_{2n}. Only a monochromatic S_n could be obtained with the above coloring. Hence, $gr_k(S_3 : S_n^+) \geq 2n+1$.

Let $\mathcal{C} = \{C_1, C_2, \cdots, C_k\}$ be a k-coloring of K_{2n+1}. If there is a vertex in K_{2n+1} represented by at least 3 colors, a rainbow copy of S_3 is obtained. If not, $\mathcal{C}$ is such that every vertex of K_{2n+1} is at most 2-colored.

Assume that there is a vertex in K_{2n+1} incident with $n+1$ edges and all these edges have the same color. Let $\{v_1v_{2n+1}, v_2v_{2n+1}, \cdots, v_{n+1}v_{2n+1}\} \subseteq C_1$ and let $G_1 = K_{2n+1}[\{v_1, v_2, \cdots, v_{n+1}\}]$. If there is an edge in C_1 which belongs to G_1, we get a monochromatic copy of S_n^+ in color 1. If not, every edge of G_1 must be in C_2. Then G_1 contains a monochromatic copy of S_n^+ in color 2.

Now, assume that there does not exist such a vertex. Then each vertex must have n edges in one color and n edges in another color. Let these edges be $v_1v_{2n+1}, v_2v_{2n+1}, \cdots, v_nv_{2n+1}$ in C_1 and let $G_2 = K_{2n+1}[\{v_1, v_2, \cdots, v_n\}]$. If there is an edge in C_1 which belongs to G_2, a monochromatic copy of S_n^+ is obtained in color 1. If not, every edge of G_2 is in C_2. Now, v_n is incident with $n-1$ edges in C_2. Since v_n must have n edges in color 2, there must exist an edge v_rv_n in C_2 for some r, $n+1 \leq r \leq 2n$. Then $v_1v_n, v_2v_n, \cdots, v_{n-1}v_n, v_rv_n$ and v_1v_2 gives a monochromatic copy of S_n^+ in color 2. Hence, $gr_k(S_3 : S_n^+) \leq 2n+1$. So, $gr_k(S_3 : S_n^+) = 2n+1$. □

Theorem 2.9. $R_2(S_n^+) \leq gr_k(S_3 : S_n^+) \leq R_3(S_n^+)$.

Proof. From Lemma 2.2, Theorem 2.3, and Theorem 2.8, the result follows. □

Theorem 2.10. *For* $n \geq 3$, $R_2(P_n) \leq gr_k(S_3 : P_n) \leq R_3(P_n)$.

Proof. The lower bound is clear from Lemma 2.2. When at most three colors are used, from the definition of $R_3(P_n)$ it is clear that $gr_k(S_3 : P_n) \leq R_3(P_n)$. Suppose at least four colors are used. The upper bound is established by applying induction on n. $R_3(P_3) = 5$ (from [7]) and in any k-coloring of K_5 without a rainbow S_3, each vertex of K_5 must be incident with at most 2 colors. Since $deg(v) = 4 \ \forall \ v \in K_5$, at least two edges incident to v must be of same color, which is a monochromatic copy of P_3. Thus, $gr_k(S_3 : P_3) \leq R_3(P_3)$.

Suppose that $gr_k(S_3 : P_{n-1}) \leq R_3(P_{n-1})$. The inequality $gr_k(S_3 : P_n) \leq R_3(P_n)$ is to be proved. Let $m = R_3(P_n)$. It is enough to show that any k-coloring of K_m contains a rainbow copy of S_3 or a monochromatic copy of P_n. Let $\mathcal{C} = \{C_1, C_2, \cdots, C_k\}$ be a k-coloring of K_m. Suppose that K_m does not contain a rainbow copy of S_3. Then at most two colors are represented at each vertex of K_m. Here it will be proved that K_m contains a monochromatic copy of P_n. Observe that $R_3(P_{n-1}) \leq R_3(P_n)$. Then from the induction hypothesis we get $gr_k(S_3 : P_{n-1}) \leq R_3(P_n) = m$. Since K_m does not contain a rainbow copy of S_3, it must contain a monochromatic copy of P_{n-1}. Without loss of generality, let $v_1v_2 \cdots v_{n-1}$ be a monochromatic copy of P_{n-1} in color 1. Let $G_1 = K_m[\{v_2, v_3, \cdots, v_{n-2}\}]$ and $G_2 = K_m[\{v_n, v_{n+1}, \cdots, v_m\}]$. If there is an edge v_1w or $v_{n-1}w$ for some $w \in G_2$ in color 1, then K_m contains a monochromatic copy of P_n. If not, for all $w \in G_2$ the edges $v_1w \notin C_1$ and $v_{n-1}w \notin C_1$. Since $v_1v_2 \in C_1$, all the edges v_1w, $w \in G_2$ must belong to C_i for some fixed $i, i \geq 2$ (otherwise a rainbow copy of S_3 is obtained at v_1). Same argument holds for $v_{n-1}w$, $w \in G_2$. Consider the following cases.

Case 1. For all $w \in G_2$, $v_1w \in C_2$ and $v_{n-1}w \in C_3$.
The colors, color 2 and color 3 are represented at each vertex of G_2, color 1 and color 2 at v_1, color 1 and color 3 at v_{n-1} (see Figure 1). The edges $v_nu, u \in G_1$ must be in C_2 or C_3 and hence two colors are represented at each vertex of G_1. Thus, two colors are represented at each vertex of K_m using color 1, color 2 or color 3. So, in this case $k \geq 4$ is not possible (If $k \geq 4$, then K_m contains a rainbow copy of S_3). When $k = 3$ the existence of a monochromatic copy of P_n in K_m is assured by the definition of $R_3(P_n)$, since $m = R_3(P_n)$ is the smallest integer such that every coloring of K_m with at most 3 colors will contain a monochromatic copy of P_n.

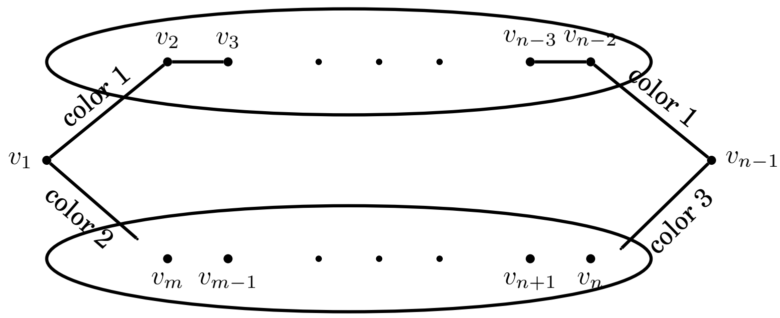

Figure 1: Case 1 of the proof of Theorem 2.10.

Case 2. For all $w \in G_2$, both v_1w and $v_{n-1}w$ are in C_2.

Subcase 2.1. For some $i \geq 3$, K_m has an edge in C_i with one end in G_1 and the other in G_2.
Without loss of generality suppose that K_m has an edge in C_3 with one end in G_1 and the other in G_2. Let v_rv_s belong to C_3 where $v_r \in G_1, v_s \in G_2$. Then color 1 and color 3 are represented at v_r, color 2 and color 3 are represented at v_s (see Figure 2). So, each edge $v_su, u \in G_1$ must be in C_2 or C_3 (otherwise a rainbow copy of S_3 is obtained at v_s) and the edges $v_rw, w \in G_2$ must be in C_1 or C_3 (otherwise a rainbow copy of S_3 is obtained at v_r). Then two colors are represented at each vertex of K_m. So, as in case 1, $k \geq 4$ is not possible and when $k = 3$, by definition of $R_3(P_n)$ there exist a monochromatic copy of P_n in K_m.

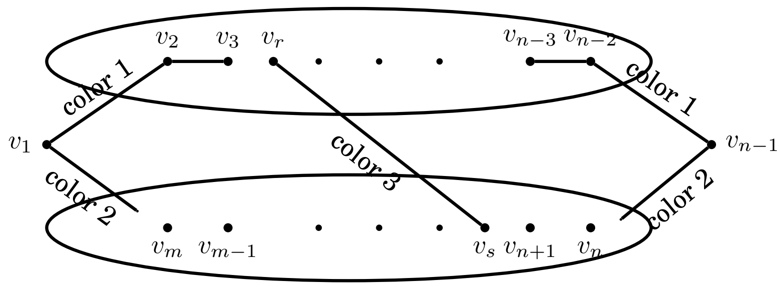

Figure 2: Subcase 2.1 of the proof of Theorem 2.10.

Subcase 2.2. For any $i, i \geq 3$, K_m has no edge in C_i with one end in G_1 and the other in G_2.
Since at least four colors are used to color the edges of K_m, C_3 is non empty. From the supposition of this subcase, the edges having color 3 must belong to G_1 or G_2 (or both). Then two cases are to be considered.

Subcase 2.2.1. Suppose G_2 contains an edge that belongs to C_3.
Let v_rv_s be the edge of G_2 that belongs to C_3 (see Figure 3).

Claim 1. Two colors, color 1 and color 2 are represented at every vertex of $V(G_1) \cup \{v_1, v_{n-1}\}$.
From the supposition of case 2, $v_1v_r \in C_2$, so color 2 is represented at v_r. Thus, two colors, color 2 and color 3 are represented at v_r. Consider the edges $v_ru, u \in G_1$. Then v_ru must have color 2 or color 3 (otherwise a rainbow copy of S_3 is obtained at v_r). From the supposition of subcase 2.2, $v_ru \notin C_3$ and hence $v_ru \in C_2$ for all $u \in G_1$. Since $u \in G_1$, color 1 is represented at u. Thus, two colors, color 1 and color 2, are represented at each vertex of G_1. So, any edge from G_1 to G_2 must be in C_1 or C_2 (otherwise a rainbow copy of S_3 is obtained). Also color 1 and color 2 are represented at the vertices v_1, v_{n-1} (from the supposition of case 2). Thus, two colors, color 1 and color 2 are represented at the vertices of $V(G_1) \cup \{v_1, v_{n-1}\}$.

Let $W = \{w \in G_2 : uw \in C_2 \; \forall \; u \in G_1\}$. Since $v_ru \in C_2$ for all $u \in G_1$, $v_r \in W$ and hence $W \neq \emptyset$. Consider the set $K_m \backslash W$.

Claim 2. Two colors, color 1 and color 2 are represented at every vertex of $K_m \backslash W$.
$V(K_m \backslash W) = V(G_1) \cup \{v_1, v_{n-1}\} \cup V(G_2 \backslash W)$. If $G_2 \backslash W = \emptyset$, then $V(K_m \backslash W) = V(G_1) \cup \{v_1, v_{n-1}\}$. Hence, from claim 1, color 1 and color 2 are represented at every vertex of $V(K_m \backslash W)$. Suppose $G_2 \backslash W \neq \emptyset$. Let x be a vertex of $G_2 \backslash W$. Since $x \in G_2$, color 2 is represented at x and since $x \notin W$, there exist some $u \in G_1$ such that $ux \notin C_2$. So, $ux \in C_1$, since any edge from G_1 to G_2 must be in C_1 or C_2. Thus, two colors, color 1 and color 2, are represented at each vertex of $G_2 \backslash W$. Also from claim 1, color 1 and color 2 are represented at each vertex of G_1 and at the vertices v_1, v_{n-1}. Hence, color 1 and color 2 are

represented at every vertex of $K_m \backslash W$. Thus, claim 2 is proved.

So, every edge that is not colored using color 1 or color 2 must be in $K_m[W]$ (otherwise a rainbow copy of S_3 is obtained at a vertex of $K_m \backslash W$).

i) Let $|W| \geq \lfloor \frac{n}{2} \rfloor$. Then $v_1 w_1 v_2 w_2 \dots v_{\frac{n}{2}} w_{\frac{n}{2}}$ is a monochromatic copy of P_n in color 2 when n is even and $v_1 w_1 v_2 w_2 \dots v_{\lfloor \frac{n}{2} \rfloor} w_{\lfloor \frac{n}{2} \rfloor} v_{\lfloor \frac{n}{2} \rfloor + 1}$ is a monochromatic copy of P_n in color 2 when n is odd, where $w_i \in W$ for $i \geq 1$.

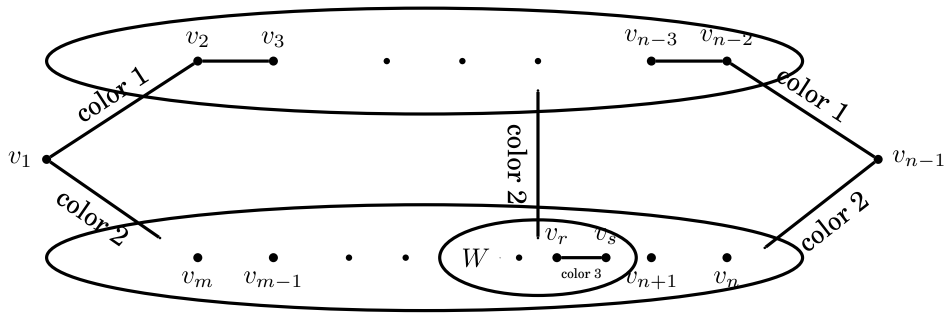

Figure 3: Subcase 2.2.1 of the proof of Theorem 2.10.

ii) Let $|W| < \lfloor \frac{n}{2} \rfloor$. It will be proved that K_m contains a monochromatic copy of P_n in color 1 or color 2. For that construct a 3-coloring of K_m from $\mathcal{C}$ using color 1, color 2 and color 3. Under $\mathcal{C}$ every edge of $E(K_m) \backslash E(W)$ is in color 1 or color 2 (from claim 2). Recolor the edges of $K_m[W]$ alone using color 3. This recoloring gives a new 3-coloring, $\mathcal{C}'$, of K_m. Then, from the definition of $R_3(P_n)$, K_m contains a monochromatic copy of P_n under $\mathcal{C}'$. All the edges of K_m having color 3 under $\mathcal{C}'$ belongs to $K_m[W]$ and hence if the monochromatic copy of P_n under $\mathcal{C}'$ is in color 3, then it must be contained in $K_m[W]$. But $|W| < \lfloor \frac{n}{2} \rfloor$. So, the monochromatic copy of P_n under $\mathcal{C}'$ is not in $K_m[W]$. This implies that the monochromatic copy of P_n in K_m under $\mathcal{C}'$ is not in color 3 and hence it is either in color 1 or in color 2. Without loss of generality suppose that the monochromatic copy of P_n under $\mathcal{C}'$ is in color 1 and let $e_1 e_2 \dots e_{n-1}$ be the edges in P_n. It is to be noted that every edge of K_m having color 1 or color 2 under $\mathcal{C}'$ had the same color under $\mathcal{C}$. Then these e_i's will have color 1 in K_m under $\mathcal{C}$ and hence a monochromatic copy of P_n in color 1 is obtained under $\mathcal{C}$.

Subcase 2.2.2. Suppose that G_2 does not contain an edge that belongs to C_3.
From the supposition in subcase 2.2, every edge in C_3 must be in G_1. Let $v_r v_s$ be an edge in G_1 that belong to C_3. Then color 1 and color 3 is represented at v_r. So, the edges $v_r w, w \in G_2$ must be in C_1 or C_3 (otherwise a rainbow copy of S_3 is obtained at v_r). From the supposition of subcase 2.2 $v_r w$ cannot have color 3. So, for all $w \in G_2$, $v_r w$ is in color 1. Thus, two colors, color 1 and color 2, are represented at each vertex in G_2 and at the vertices v_1, v_{n-1}. Recolor G_1 with color 3 to obtain a 3-coloring $\mathcal{C}'$ of K_m. Then from the definition of $R_3(P_n)$, K_m contains a monochromatic copy of P_n under $\mathcal{C}'$. Since $|G_1| < n$, this monochromatic copy of P_n is not in color 3 and hence it is either in color 1 or in color 2. Then the same monochromatic copy of P_n in K_m under $\mathcal{C}'$ can be obtained under $\mathcal{C}$. Thus, in all cases $gr_k(S_3 : P_n) \leq R_3(P_n)$. □

Remark 2.1. *Let us consider an example for which strict inequality holds in Theorem 2.10. We have* $R_3(P_3) = 5$. *But,* $gr_k(S_3 : P_3) = 4$. *Consider a* k*-coloring of* K_4 *that does not contain a rainbow* S_3. *Then at most two colors are represented at each vertex of* K_4. *Since the degree of each vertex of* K_4 *is three, there exist at least two edges in the same color incident with each vertex of* K_4*, giving a monochromatic copy of* P_3. *So,* $gr_k(S_3 : P_3) \leq 4$. *Now, the complete graph on three vertices,* C_3 *does not contain a rainbow copy of* S_3 *or a monochromatic copy of* P_3 *in any* 3*-coloring. Hence,* $gr_k(S_3 : P_3) = 4$.

References

[1] R. J. Faudree, R. H. Schelp, Path Ramsey numbers in multicolorings, *J. Combin. Theory Ser. B* **19** (1975) 150–160.
[2] S. Fujita, C. Magnant, Gallai-Ramsey numbers for cycles, *Discrete Math.* **311** (2011) 1247–1254.
[3] S. Fujita, C. Magnant, K. Ozeki, Rainbow generalizations of Ramsey theory – a dynamic survey, *Theory Appl. Graphs* **0** (2018) #1.
[4] T. Gallai, Transitiv orientierbare graphen, *Acta Math. Hungar.* **18** (1967) 25–66.
[5] A. Gyárfás, M. Ruszinko, G. Sarkozy, E. Szemeredi, Three-color Ramsey numbers for paths, *Combinatorica* **27** (2007) 35–69.
[6] X. Li, P. Besse, C. Magnant, L. Wang, N. Watts, Gallai-Ramsey numbers for rainbow paths, *Graphs Combin.* **36** (2020) 1163–1175.
[7] S. P. Radziszowski, Small Ramsey numbers, *Electron. J. Combin.* **DS1** (2021) #DS1.16.
[8] Z. Wang, Y. Mao, C. Magnant, J. Zou, Ramsey and Gallai-Ramsey numbers for two classes of unicyclic graphs, *Graphs Combin.* **37** (2021) 337–354.