Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **10** (2022) 41–44
DOI: 10.47443/dml.2022.020

Research Article

Signed total strong Roman domination in graphs

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(Received: 19 February 2022. Received in revised form: 6 May 2022. Accepted: 7 May 2022. Published online: 12 May 2022.)

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Abstract

Let $G = (V, E)$ be a finite and simple graph of order n and maximum degree Δ. A signed total strong Roman dominating function on G is a function $f : V \to \{-1, 1, 2, \ldots, \lceil \Delta/2 \rceil + 1\}$ satisfying the conditions: (i) for every vertex v of G, $\sum_{u \in N(v)} f(u) \geq 1$, where $N(v)$ is the open neighborhood of v, and (ii) every vertex v satisfying $f(v) = -1$ is adjacent to at least one vertex u such that $f(u) \geq 1 + \lceil |N(u) \cap V_{-1}| /2 \rceil$, where $V_{-1} = \{v \in V \mid f(v) = -1\}$. The signed total strong Roman domination number of G, $\gamma^t_{ssR}(G)$, is the minimum weight of a signed total strong Roman dominating function. In this paper, some bounds for this parameter are presented.

Keywords: signed total Roman domination; signed total strong Roman domination.

2020 Mathematics Subject Classification: 05C69.

1. Introduction

Let $G = (V, E)$ be a graph without isolated vertices with vertex set V and edge set E. The *order* of G is $n = n(G) = |V|$. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = N_G(v) = \{u \in V(G) \mid uv \in E\}$. The *degree* of vertex $v \in V$ is $d(v) = d_G(v) = |N(v)|$. The *maximum* and *minimum degree* of G are denoted by $\Delta = \Delta(G)$ and $\delta = \delta(G)$, respectively. We write P_n for the *path* of order n and C_n for the *cycle* of length n.

A (total) dominating set in a graph G is a set of vertices $S \subseteq V(G)$ such that any vertex of $V - S$ (V) is adjacent to at least one vertex of S. The *(total) domination number* $\gamma(G)$ ($\gamma_t(G)$) equals the minimum cardinality of a (total) dominating set in G.

Let $f : V(G) \to \{0, 1, 2, \ldots, k\}$ be a function and let $(V_0, V_1, V_2, \ldots, V_k)$ be the ordered weak partition of $V = V(G)$ induced by f, where $V_i = \{v \in V \mid f(v) = i\}$ for $i \in \{0, 1, \ldots, k\}$. There is a 1-1 correspondence between the function $f : V \to \{0, 1, 2, \ldots, k\}$ and the ordered weak partition $(V_0, V_1, V_2, \ldots, V_k)$ of V, so we write $f = (V_0, V_1, V_2, \ldots, V_k)$.

A *signed total Roman dominating function* (STRDF) on a graph G is a function $f : V(G) \to \{-1, 1, 2\}$ such that $f(N(v)) \geq 1$ for every $v \in V(G)$, and every vertex u satisfying $f(u) = -1$ is adjacent to a vertex v for which $f(v) = 2$. The weight of a STRDF f on G is $\omega(f) = \sum_{u \in V(G)} f(u)$. The *signed total Roman domination number* $\gamma_{stR}(G)$ is the minimum weight of a STRDF on G. The concept of signed total Roman domination was introduced by Volkmann [7] and has been studied by several authors (see for instance [6, 8]). Some variants of signed Roman domination have been studied in [1–3, 9, 10]. For more details on Roman domination and its variants we refer the reader to [4, 5].

A *signed total strong Roman dominating function* (STStRD-function) on G is a function $f : V \to \{-1, 1, 2, \ldots, \lceil \Delta/2 \rceil + 1\}$ satisfying the conditions: (i) for every vertex v of G, $\sum_{u \in N(v)} f(u) \geq 1$, and (ii) every vertex v satisfying $f(v) = -1$ is adjacent to at least one vertex u such that

$$f(u) \geq 1 + \left\lceil \frac{1}{2} |N(u) \cap V_{-1}| \right\rceil,$$

where $V_{-1} = \{v \in V \mid f(v) = -1\}$. The weight of a STStRD-function f is the value $\omega(f) = f(V(G)) = \sum_{u \in V(G)} f(u)$. The *signed total strong Roman domination number* $\gamma^t_{ssR}(G)$ is the minimum weight of a STStRD-function on G. In this article, we are interested in establishing some bounds for $\gamma^t_{ssR}(G)$.

The next result immediately follows from definitions.

Observation 1.1. For any connected graph G with $\Delta(G) \leq 2$, $\gamma^t_{ssR}(G) = \gamma_{stR}(G)$.

The next corollaries are direct consequences of Observation 1.1 and Examples 5 and 6 of [7].

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Corollary 1.1. *For $n \geq 3$,*

$$\gamma_{ssR}^t(P_n) = \begin{cases} \dfrac{n}{2} & \text{if} \quad n \equiv 0 \pmod 4, \\ \left\lceil \dfrac{n+3}{2} \right\rceil & \text{if} \quad n \equiv 1,2,3 \pmod 4. \end{cases}$$

Corollary 1.2. *For $n \geq 3$,*

$$\gamma_{ssR}^t(C_n) = \begin{cases} \dfrac{n}{2} & \text{if} \quad n \equiv 0 \pmod 4, \\ \dfrac{n+3}{2} & \text{if} \quad n \equiv 1,3 \pmod 4, \\ \dfrac{n+6}{2} & \text{if} \quad n \equiv 2 \pmod 4. \end{cases}$$

2. Main results

In this section, we present some bounds on the signed total strong Roman domination number in graphs.

Theorem 2.1. *Let G be a connected graph of order n. Then the following statements hold.*

1. $\gamma_{ssR}^t(G) \leq n$.
2. *If $\delta(G) \geq 3$, then $\gamma_{ssR}^t(G) \leq n-1$.*
3. $\gamma_{ssR}^t(G) \geq 2\gamma_t(G) - n - 1$.

Proof. *1.* Clearly, assigning the weight 1 to every vertex of G produces an STStRD-function of G of weight n which implies that $\gamma_{ssR}^t(G) \leq n$.

2. Let x and y be two adjacent vertices. Define the function $f : V(G) \rightarrow \{-1, 1, \ldots, \lceil \Delta/2 \rceil + 1\}$ by $f(x) = -1$, $f(y) = 2$ and $f(z) = 1$ for $z \in V(G) - \{x, y\}$. Since $\delta(G) \geq 3$, we observed that f is an STStRD-function on G of weight $n-1$ and thus $\gamma_{ssR}^t(G) \leq n-1$.

3. Let f be a $\gamma_{ssR}^t(G)$-function. If $V_i = \emptyset$, for each $2 \leq i \leq \lceil \Delta/2 \rceil + 1$, then $V = V_1$, and we have $\gamma_{ssR}^t(G) = \omega(f) = n$. Hence we may assume that $|V_i| \geq 1$ for some $2 \leq i \leq \lceil \Delta/2 \rceil + 1$. Since

$$|V_{-1}| = n - \sum_{i=1}^{\lceil \Delta/2 \rceil + 1} |V_i|$$

and $\bigcup_{i=1}^{\lceil \Delta/2 \rceil + 1} V_i$ is a total dominating set for G, we have

$$\begin{aligned} \gamma_{ssR}^t(G) &= \sum_{i=1}^{\lceil \Delta/2 \rceil + 1} i|V_i| - |V_{-1}| \\ &= \sum_{i=1}^{\lceil \Delta/2 \rceil + 1} (i+1)|V_i| - n \\ &> 2\left(\sum_{i=1}^{\lceil \Delta/2 \rceil + 1} |V_i|\right) - n \\ &\geq 2\gamma_t(G) - n. \end{aligned}$$

Thus, $\gamma_{ssR}^t(G) \geq 2\gamma_t(G) - n - 1$. □

Theorem 2.2. *If G is a graph of order $n \geq 3$ with maximum degree Δ and minimum degree $\delta \geq 1$, then*

$$\gamma_{ssR}^t(G) \geq \max\{1 + \Delta - n, \delta + 4 - n\}.$$

Proof. Let g be a $\gamma^t_{ssR}(G)$-function. If $V_{-1} = \emptyset$, then $V = V_1$ and so $\gamma^t_{ssR}(G) = n \geq \max\{1 + \Delta - n, \delta + 4 - n\}$. Hence, we assume that $V_{-1} \neq \emptyset$. First we show that $\gamma^t_{ssR}(G) \geq 4 + \delta - n$. Let $u \in V_{-1}$. Then there exists a vertex $w \in N(u)$ with $g(w) \geq 2$, and it follows that

$$\begin{aligned}\gamma^t_{ssR}(G) &= g(w) + g(N(w)) + \sum_{x \in V(G)-N[w]} g(x)\\ &\geq 2 + 1 + \sum_{x \in V(G)-N[w]} g(x)\\ &\geq 3 - (n - d(w) - 1)\\ &\geq 4 + \delta - n.\end{aligned}$$

To prove $\gamma^t_{ssR}(G) \geq \Delta + 1 - n$, let v be a vertex of maximum degree and let $X = V(G) - N[v]$. By definition we have

$$\begin{aligned}\gamma^t_{ssR}(G) &= \sum_{x \in V(G)} g(x)\\ &= g(v) + \sum_{x \in N(v)} g(x) + \sum_{x \in X} g(x)\\ &\geq -1 + 1 + \sum_{x \in X} g(x)\\ &\geq -1 + 1 + (\Delta + 1 - n)\\ &= \Delta + 1 - n,\end{aligned}$$

as desired. □

By Corollary 1.2, we have $\gamma^t_{ssR}(C_4) = 2 = 4 + \delta(C_4) - n(C_4)$.

Theorem 2.3. *If G is a connected graph of order $n \geq 3$, then*

$$\gamma^t_{ssR}(G) \leq n - \delta + \left\lceil \frac{\delta}{4} \right\rceil + 1.$$

Proof. The result is immediate for $\delta(G) \leq 2$ by Proposition 2.1-(1). Assume that $\delta(G) \geq 3$ and $\delta(G) \equiv r \pmod 2$. Let v be a vertex with minimum degree $\delta(G)$ and let $N(v) = \{v_1, v_2, \ldots, v_{\delta(G)}\}$. If $r = 0$, then define the function

$$f : V(G) \to \{-1, 1, 2, \ldots, \lceil \Delta/2 \rceil + 1\}$$

by $f(v_1) = -1$, $f(v_2) = 2$, $f(v_i) = (-1)^i$ for $3 \leq i \leq \delta(G)$, $f(v) = \lceil \frac{\delta}{4} \rceil + 1$ and $f(x) = 1$ otherwise. If $r = 1$, then define the function $f : V(G) \to \{-1, 1, 2, \ldots, \lceil \Delta/2 \rceil + 1\}$ by $f(v_i) = (-1)^i$ for $1 \leq i \leq \delta - 1$, $f(v) = \lceil (\delta - 1)/4 \rceil + 1$ and $f(x) = 1$ otherwise. In either case, it is easy to verify that f is STStRD-function on G of weight at most $n - \delta + \lceil \delta/4 \rceil + 1$ and hence $\gamma^t_{ssR}(G) \leq n - \delta + \lceil \delta/4 \rceil + 1$, as desired. □

A set $S \subseteq V(G)$ is a 2-*packing* of the graph G if $N[u] \cap N[v] = \emptyset$ for any two distinct vertices $u, v \in S$. The maximum cardinality of a 2-packing in G is the 2-*packing number*, denoted by $\rho(G)$.

Theorem 2.4. *If G is a graph of order n, minimum degree $\delta \geq 1$ and packing number ρ, then $\gamma^t_{ssR}(G) \geq \rho(\delta + 1) - n$.*

Proof. Let $\{v_1, v_2, \ldots, v_\rho\}$ be a 2-packing of G and let f be a $\gamma^t_{ssR}(G)$-function. Take $A = \bigcup_{i=1}^{\rho} N(v_i)$. Since $\{v_1, v_2, \ldots, v_\rho\}$ is a 2-packing, we have $|A| = \sum_{i=1}^{\rho} d(v_i) \geq \rho\delta$. Thus,

$$\begin{aligned}\gamma^t_{ssR}(G) &= \sum_{x \in V(G)} f(x)\\ &= \sum_{i=1}^{\rho} f(N(v_i)) + \sum_{x \in V(G)\setminus A} f(x)\\ &\geq \rho + \sum_{x \in V(G)\setminus A} f(x)\\ &\geq \rho - n + |A|\\ &\geq \rho - n + \rho\delta\\ &= \rho(\delta + 1) - n.\end{aligned}$$

□

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