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Research Article

Product of conjugacy classes of complete cycles in the alternating group

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Abstract

The product of conjugacy classes of a finite group can be written as a linear combination of conjugacy classes with integer coefficients. For the symmetric group, some explicit expressions for these coefficients are known only in particular cases. The aim of this paper is to give explicit expressions for the product of the conjugacy classes in the alternating group $\mathcal{A}_n$ corresponding to cycles of length n.

Keywords: structure constants; representation theory of finite groups; conjugacy classes; alternating group.

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1. Introduction

Let G be a finite group, $G_\star$ a set of representatives of the conjugacy classes of G and $\mathbb{C}[G]$ the algebra of G over the field of complex numbers $\mathbb{C}$. For $x \in G_\star$, denote by $C_x(G)$ the conjugacy class of x in G. The center of the group algebra $\mathbb{C}[G]$, usually denoted $Z(\mathbb{C}[G])$, is linearly generated by the formal sums

$$\mathbf{C_x(G)} = \sum_{g \in C_x(G)} g$$

where x runs through the elements of $G_\star$. The structure constants $c_{xy}^z(G)$ of the center of the group algebra of G are the non-negative integers defined by the following product

$$\mathbf{C_x(G)C_y(G)} = \sum_{z \in G_\star} c_{xy}^z(G)\mathbf{C_z(G)}$$

where $x, y \in G_\star$. By definition, $c_{xy}^z(G)$ counts the number of ways z can be obtained as a product of two elements fh where $(f, h) \in C_x(G) \times C_y(G)$. However, as the product of G may be complicated, counting the structure constants directly by this way may be a very difficult problem, see [5] and [7] for example where this method is used to compute particular structure constants of the center of the symmetric group algebra.

If $G^\star$ denotes the set of all non-isomorphic irreducible characters of G, then the representation theory of G offers the following formula, due to Frobenius, that expresses the structure constants $c_{xy}^z(G)$ in terms of elements of $G^\star$

$$c_{xy}^z(G) = \frac{|C_G(x)||C_G(y)|}{|G|} \sum_{\chi \in G^\star} \frac{\chi(x)\chi(y)\overline{\chi(z)}}{\chi(1)}. \tag{1}$$

We refer to the appendix [8] of Zagier where this formula is presented in a more general form. Evaluating irreducible characters is also known to be a difficult problem. However the Frobenius formula is very efficient in some particular cases where the direct computation is out of reach, see [3] and [2] for computing structure constants involving cycles of length n in the symmetric group $\mathcal{S}_n$.

The representation theory of a subgroup H of index 2 of G is strongly related to that of G, see [1] or [4]. First, each conjugacy class of G (that contains elements in H) gives either a conjugacy class of H or splits and break up to yield two conjugacy classes of H of the same size. Second, $H^\star$ can be completely determined by restricting the irreducible characters of G to H. The restriction of any element $\chi \in G^\star$ to H is either an element of $H^\star$ or splits and produce two different elements of $H^\star$ of the same degree. This suggests that the structure constants of the center of the subgroup H are closely related to the structure constants of G. We show this relation and applied it in the case of the alternating subgroup $\mathcal{A}_n$ of $\mathcal{S}_n$. This allows us to obtain, in Theorem 3.2, explicit expressions for the product of the conjugacy classes corresponding to cycles of length n for the alternating group $\mathcal{A}_n$.

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2. Representation theory of subgroups of index 2

In this section we recall some results of the representation theory of subgroups of index 2. We refer the reader to one of the following two books [1] and [4] for more details and complete proofs. Throughout this section G will be a finite group and H a subgroup of G of index 2. Thus H is a normal subgroup of G and $G = H \sqcup gH$ for some element $g \in G$ verifying $(gH)^2 = H$.

Conjugacy classes

The conjugacy classes of H can be obtained from the conjugacy classes of G using one of the following two ways:

1. Suppose that $x \in H$ commutes with an element $t \in gH$. In this case, if $y = \alpha^{-1}x\alpha$ for some $\alpha \in G$ then either $\alpha \in H$ and $y \in C_H(x)$ or $\alpha \in gH$ and $y = \alpha^{-1}t^{-1}xt\alpha = (t\alpha)^{-1}xt\alpha$ with $t\alpha \in gHgH = H$. Thus $y \in C_H(x)$ and as $C_H(x) \subset C_G(x)$, we deduce that $C_G(x) = C_H(x)$.

2. If x does not commute with any $t \in gH$ then the centralizer of x in G is the same as the centralizer of x in H and $\frac{|G|}{|C_G(x)|} = \frac{|H|}{|C_H(x)|}$. This implies that $|C_G(x)| = 2|C_H(x)|$. In addition,
$$C_G(x) = C_H(x) \sqcup \{h^{-1}g^{-1}xgh; h \in H\} = C_H(x) \sqcup C_H(g^{-1}xg).$$
This means that $C_G(x)$ splits into two disjoint conjugacy classes in H which are $C_H(x)$ and $C_H(g^{-1}xg)$.

Let $G_\star$ denote a set of representatives, chosen to be in H if possible, of the conjugacy classes of G. If we decompose $G_\star$ into two disjoint sets L and L' where L contains representatives which commute with some element $t \in gH$ and L' contains representatives which do not commute with any $t \in gH$, then from the above discussion, we have
$$H_\star = H \cap (L \sqcup L' \sqcup L'_g) \text{ and } |C_H(x^g)| = \left(\frac{1}{2}\right)^{1-\delta_{xL}} |C_G(x)|$$
where
$$x^g = \begin{cases} g^{-1}xg & \text{if } x^g \in L'_g \\ x & \text{if not.} \end{cases} \quad \text{and} \quad \delta_{xL} = \begin{cases} 1 & \text{if } x \in L \\ 0 & \text{if not.} \end{cases}$$

Irreducible characters

Let $G^\star$ denote the set of all non isomorphic irreducible representations of G. Decompose $G^\star$ into two disjoint sets S and S' where S contains irreducible characters χ satisfying $\chi(gh) \neq 0$ for some $h \in H$ and S' contains irreducible characters satisfying $\chi(gh) = 0$ for any $h \in H$. The restriction $\mathrm{Res}_H^G \chi$ of any irreducible character $\chi \in S$ is an irreducible character of H. Note that if $\chi \in S$ then the character χ' of G defined by
$$\chi'(x) = \begin{cases} \chi(x) & \text{if } x \in H \\ -\chi(x) & \text{if } x \notin H \end{cases}$$
is an irreducible character of G in S different than χ but has the same degree. In addition, it is easy to see that $\mathrm{Res}_H^G \chi' = \mathrm{Res}_H^G \chi$.

Now, if $\chi \in S'$ then $\mathrm{Res}_H^G \chi = \chi_1 + \chi_2$, where χ_1 and χ_2 are non-isomorphic irreducible characters of H of the same degree. This can be obtained from the general fact that if K is a subgroup of G and $\psi_1, \cdots, \psi_r$ is a complete list of irreducible characters of K then for any irreducible character χ of G we have
$$\mathrm{Res}_K^G \chi = \sum_{i=1}^r d_i\psi_i$$
where the non-negative integers d_i satisfy
$$\sum_{i=1}^r (d_i)^2 \leq \frac{|G|}{|H|}.$$
Thus the complete set $H^\star$ of irreducible characters of H is
$$H^\star = \{\mathrm{Res}_H^G \chi; \chi \in S\} \sqcup \{\chi_1, \chi_2; \chi \in S'\}.$$

The next lemma shows a relation between the structure constants of H and those of G.

Lemma 2.1. *Let H be a subgroup of index 2 of a finite group $G = H \sqcup gH$. If $x^g, y^g, z^g \in H_\star$ then we have the following relation between the structure constants in $Z(\mathbb{C}[G])$ and $Z(\mathbb{C}[H])$*

$$c^{z^g}_{x^g y^g}(H) = \Big(\frac{1}{2}\Big)^{2-\delta_{xL}-\delta_{yL}} c^z_{xy}(G) + \frac{|C_H(x^g)||C_H(y^g)|}{|H|}\mathcal{P}^z_{xy}(S')$$

where

$$\mathcal{P}^{z^g}_{x^g y^g}(S') = \sum_{\chi\in S'} \frac{1}{\chi(1)}\Big(2\chi_1(x^g)\chi_1(y^g)\overline{\chi_1(z^g)} + 2\chi_2(x^g)\chi_2(y^g)\overline{\chi_2(z^g)} - \frac{\chi(x)\chi(y)\overline{\chi(z)}}{2}\Big).$$

Proof. By Equation (1) of Frobenius, we have:

$$c^{z^g}_{x^g y^g}(H) = \frac{|C_H(x^g)||C_H(y^g)|}{|H|}\sum_{\chi\in H^\star}\frac{\chi(x^g)\chi(y^g)\overline{\chi(z^g)}}{\chi(1)}.$$

Using the fact that $H^\star = \{\mathrm{Res}^G_H\,\chi; \chi \in S\} \sqcup \{\chi_1, \chi_2; \chi \in S'\}$ and that both $\chi \in S$ and $\chi' \in S$ restrict to the same irreducible character of H, the sum over $H^\star$ in the above equation can be written:

$$\frac{1}{2}\sum_{\chi\in S}\frac{\chi(x)\chi(y)\overline{\chi(z)}}{\chi(1)} + \sum_{\chi\in S'}\frac{\chi_1(x^g)\chi_1(y^g)\overline{\chi_1(z^g)} + \chi_2(x^g)\chi_2(y^g)\overline{\chi_2(z^g)}}{\chi_1(1)}.$$

Notice that we have used the fact that $\chi_1(1) = \chi_2(1)$. But from the decomposition $G^\star = S \sqcup S'$, we have:

$$\sum_{\chi\in S}\frac{\chi(x)\chi(y)\overline{\chi(z)}}{\chi(1)} = \sum_{\chi\in G^\star}\frac{\chi(x)\chi(y)\overline{\chi(z)}}{\chi(1)} - \sum_{\chi\in S'}\frac{\chi(x)\chi(y)\overline{\chi(z)}}{\chi(1)}.$$

Thus,

$$\sum_{\chi\in H^\star}\frac{\chi(x^g)\chi(y^g)\overline{\chi(z^g)}}{\chi(1)} = \frac{1}{2}\Big(\sum_{\chi\in G^\star}\frac{\chi(x)\chi(y)\overline{\chi(z)}}{\chi(1)}\Big) + \sum_{\chi\in S'}\frac{\chi_1(x^g)\chi_1(y^g)\overline{\chi_1(z^g)} + \chi_2(x^g)\chi_2(y^g)\overline{\chi_2(z^g)}}{\chi_1(1)} - \frac{\chi(x)\chi(y)\overline{\chi(z)}}{2\chi(1)}.$$

The result follows using the relations $|G| = 2|H|$, $|C_H(x^g)| = \left(\frac{1}{2}\right)^{1-\delta_{xL}}|C_G(x)|$ and $\chi(1) = 2\chi_1(1)$ for any $\chi \in S'$. □

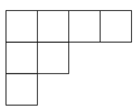
3. Application to the alternating group $\mathcal{A}_n$

The alternating group $\mathcal{A}_n$ is the subgroup of index 2 of the symmetric group $\mathcal{S}_n$ containing even permutations. We start by reviewing some useful results from the rich representation theory of the symmetric group $\mathcal{S}_n$.

Partitions

A *partition* λ is a weakly decreasing list of positive integers $(\lambda_1, \ldots, \lambda_l)$ where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l \geq 1$. The λ_i are called the *parts* of λ; the *size* of λ, denoted by $|\lambda|$, is the sum of all of its parts. If $|\lambda| = n$, we say that λ is a partition of n. We will also use the exponential notation $\lambda = (1^{m_1(\lambda)}, 2^{m_2(\lambda)}, 3^{m_3(\lambda)}, \ldots)$, where $m_i(\lambda)$ is the number of parts equal to i in the partition λ. If $\lambda = (1^{m_1(\lambda)}, 2^{m_2(\lambda)}, 3^{m_3(\lambda)}, \ldots, n^{m_n(\lambda)})$ is a partition of n then $\sum_{i=1}^n i m_i(\lambda) = n$. We will dismiss $i^{m_i(\lambda)}$ from λ when $m_i(\lambda) = 0$. The partial order $\lhd$ on the set of partitions of all numbers is defined by saying that for $\mu = (1^{m_1(\mu)}, 2^{m_2(\mu)}, 3^{m_3(\mu)}, \ldots, r^{m_r(\mu)})$, $\lambda \rhd \mu$ if and only if for all $i = 1, 2, \ldots, m_i(\lambda) \geq m_i(\mu)$.

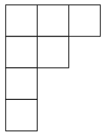
Any partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ of n can be represented by a Young diagram. This is an array of n squares having l left-justified rows with row i containing λ_i squares for $1 \leq i \leq l$. For example, the following is the Young diagram of the partition $\lambda = (4, 2, 1)$ of 7



If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is a partition of n then we define its conjugate $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{\lambda_1})$, where

$$\lambda'_i := Card\{j : \lambda_j \geq i\} \text{ for any } 1 \leq i \leq \lambda_1.$$

The conjugate of a partition is obtained by reflecting its Young diagram about its main diagonal. As an example, the conjugate of the partition $\lambda = (4, 2, 1)$ is $\lambda' = (3, 2, 1, 1)$ which is represented below



Irreducible representations of $\mathcal{A}_n$

It is well known that partitions of n are used to index both the conjugacy classes and the irreducible representations of $\mathcal{S}_n$. We recall that the *cycle-type* $\mathrm{ct}(\omega)$ of a permutation of $\omega \in \mathcal{S}_n$ is the partition of n obtained from the lengths of the cycles that appear in its decomposition into product of disjoint cycles. For example, the permutation $(4, 1, 6)(3, 5)(2)$ of $\mathcal{S}_6$ has cycle-type $(3, 2, 1)$. If λ is a partition of n, the conjugacy class C_λ of $\mathcal{S}_n$ associated to λ contains all the permutations of $\mathcal{S}_n$ of cycle-type λ,

$$C_\lambda := \{\sigma \in \mathcal{S}_n \mid \mathrm{ct}(\sigma) = \lambda\}.$$

The reader should remark that we omit the group $\mathcal{S}_n$ from our notation for conjugacy classes here. We decided to write so for sake of simplification. In case any confusion may arise, we will use the notation $C_\lambda(\mathcal{S}_n)$.

The cardinal of C_λ is given by:

$$|C_\lambda| = \frac{n!}{z_\lambda}, \tag{2}$$

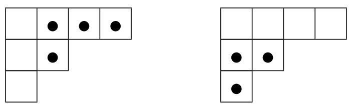
where

$$z_\lambda = 1^{m_1(\lambda)} m_1(\lambda)! 2^{m_2(\lambda)} m_2(\lambda)! \cdots n^{m_n(\lambda)} m_n(\lambda)!.$$

Let $\mathcal{E}(n)$ denote the set of partitions with even number of even parts. It would be easy to verify that a permutation of n is even if and only if its cycle-type correspond to a partition in $\mathcal{E}(n)$. According to the general discussion in Section 2, if $\lambda \in \mathcal{E}(n)$, the conjugacy class C_λ in $\mathcal{S}_n$ splits into two disjoint conjugacy classes in $\mathcal{A}_n$ if and only if there is no odd permutation commuting with a permutation of cycle-type λ. This happens if and only if all the parts of λ are odd integers and no two parts are equal. If we denote by $\mathcal{OD}(n)$ the set of all such partitions of n then the conjugacy classes of $\mathcal{A}_n$ are

$$\{C_\lambda; \lambda \in \mathcal{E}(n) \setminus \mathcal{OD}(n)\} \sqcup \{C^1_\lambda, C^2_\lambda; \lambda \in \mathcal{OD}(n)\}.$$

The irreducible representation of $\mathcal{S}_n$ that corresponds to the partition λ of n is called the *Specht module* and usually denoted by S^λ, see [6, Theorem 2.4.6]. The hooklength $h_{i,j}(\lambda)$ is the number of squares on the right and below the square in the i^{th} row and j^{th} column in the diagram of λ. For example, $h_{1,2}(4, 2, 1) = 4$ and $h_{2,1}(4, 2, 1) = 3$ as can be seen respectively from the following diagrams



The dimension f^λ of the Specht module S^λ, which is equal to the value of the character χ^λ evaluated on the identity element, is given by the following usually called hook length formula

$$f^\lambda = \chi^\lambda(1) = \frac{n!}{\prod_{(i,j)\in\lambda} h_{i,j}(\lambda)}. \tag{3}$$

As an example, we have

$$f^{(4,2,1)} = \frac{7!}{6 \times 4 \times 2 \times 3 \times 1 \times 1} = 35.$$

Since the hooklength $h_{i,j}(\mu) = h_{j,i}(\mu')$, we can easily deduce from the hook length formula that

$$f^\mu = f^{\mu'} \text{ for any partition } \mu \text{ of } n.$$

To find all the irreducible characters of $\mathcal{A}_n$, let us decompose the set $\mathcal{E}(n)$ of partitions of n into the following two disjoint sets

$$\mathcal{E}(n) = \{\lambda; \lambda \neq \lambda'\} \sqcup \{\lambda; \lambda' = \lambda\}$$

If $\lambda \neq \lambda'$, then $\mathrm{Res}_{\mathcal{A}_n}^{\mathcal{S}_n} S^{\lambda} = \mathrm{Res}_{\mathcal{A}_n}^{\mathcal{S}_n} S^{\lambda'}$ is an irreducible representation of $\mathcal{A}_n$. In addition, if λ is a self-conjugate partition, i.e. $\lambda' = \lambda$, then $\mathrm{Res}_{\mathcal{A}_n}^{\mathcal{S}_n} S^{\lambda} = S_1^{\lambda} + S_2^{\lambda}$ where S_1^{λ} and S_2^{λ} are two irreducible representations of $\mathcal{A}_n$ of the same degree. Thus,

$$\{\mathrm{Res}_{\mathcal{A}_n}^{\mathcal{S}_n} S^{\lambda}; \lambda \neq \lambda'\} \sqcup \{S_1^{\lambda}, S_2^{\lambda}; \lambda' = \lambda\}$$

is a complete set of non-isomorphic irreducible representations of $\mathcal{A}_n$. In fact there is a bijection h from $\{\lambda; \lambda' = \lambda\}$ to $\mathcal{OD}(n)$ which sends a self-conjugate partition λ to $h(\lambda) = (h_{11}(\lambda), h_{22}(\lambda), \cdots)$ where $h_{ii}(\lambda) = 2(\lambda_i - i + 1) - 1$ is the hook length of the diagonal box with coordinates (i, i) in the Young diagram corresponding to λ.

The character table of $\mathcal{A}_n$ can be obtained from that of $\mathcal{S}_n$ using the following results

Theorem 3.1. *1. If $\lambda \neq \lambda'$ then $\mathrm{Res}_{\mathcal{A}_n}^{\mathcal{S}_n} \chi^{\lambda}(C_\mu) = \chi^{\lambda}(C_\mu)$ if $\mu \in \mathcal{P}(n) \setminus \mathcal{OD}(n)$ and $\mathrm{Res}_{\mathcal{A}_n}^{\mathcal{S}_n} \chi^{\lambda}(C_\mu) = \chi^{\lambda}(C_\mu^1) = \chi^{\lambda}(C_\mu^2)$ if $\mu \in \mathcal{OD}(n)$.*

2. If λ is self-conjugate then $\chi_1^{\lambda}(C_\mu) = \chi_2^{\lambda}(C_\mu) = \frac{1}{2}\chi^{\lambda}(C_\mu)$ if $\mu \in \mathcal{P}(n) \setminus \mathcal{OD}(n)$ and if $\mu \in \mathcal{OD}(n)$ we distinguish the following two cases:

(a) if $\mu \neq h(\lambda)$ then $\chi_1^{\lambda}(C_\mu^1) = \chi_2^{\lambda}(C_\mu^1) = \chi_1^{\lambda}(C_\mu^2) = \chi_2^{\lambda}(C_\mu^2) = \frac{1}{2}\chi^{\lambda}(C_\mu)$.

(b) if $\mu = h(\lambda)$ then $\chi_1^{\lambda}(C_\mu^1) = \chi_2^{\lambda}(C_\mu^2) = x$ and $\chi_1^{\lambda}(C_\mu^2) = \chi_2^{\lambda}(C_\mu^1) = y$ where

$$\{x, y\} = \left\{\frac{1}{2}\Big((-1)^m \pm \sqrt{(-1)^m \mu_1 \times \cdots \times \mu_{l(\mu)}}\Big)\right\} \text{ and } m = \frac{1}{2}(n - l(\mu)).$$

Proof. We refer to [1, Proposition 5.3.] for a complete proof of this result. □

Corollary 3.1. *If n is an odd integer then the partition $(n)^c := (1^{\frac{n-1}{2}}, \frac{n+1}{2})$ is self conjugate and*

1. $\chi_1^{(n)^c}(C_\mu) = \chi_2^{(n)^c}(C_\mu) = \frac{1}{2}\chi^{(n)^c}(C_\mu)$ *if* $\mu \in \mathcal{P}(n) \setminus \mathcal{OD}(n)$,
2. $\chi_1^{(n)^c}(C_\mu^1) = \chi_2^{(n)^c}(C_\mu^1) = \chi_1^{(n)^c}(C_\mu^2) = \chi_2^{(n)^c}(C_\mu^2) = \frac{1}{2}\chi^{(n)^c}(C_\mu)$ *if* $\mu \in \mathcal{OD}(n) \setminus (n)$,
3. $\chi_1^{(n)^c}(C_{(n)}^1) = \chi_2^{(n)^c}(C_{(n)}^2) = x$ *and* $\chi_1^{(n)^c}(C_{(n)}^2) = \chi_2^{(n)^c}(C_{(n)}^1) = y$ *where*

$$\{x, y\} = \left\{\frac{1}{2}\Big((-1)^m \pm \sqrt{(-1)^m n}\Big)\right\} \text{ with } m = \frac{n-1}{2}.$$

In addition, $x + y = (-1)^m$, $xy = \frac{1}{4}(1 - (-1)^m n)$, $x^2 + y^2 = \frac{1}{2}(1 + (-1)^m n)$, $x^2\overline{x} + y^2\overline{y} = \frac{1}{4}\big((-1)^m + (1 + 2(-1)^m)n\big)$, $x^2\overline{y} + y^2\overline{x} = \frac{1}{4}\big((-1)^m + (1 - 2(-1)^m)n\big)$ *and* $xy\overline{x} + yx\overline{y} = \frac{1}{4}((-1)^m - n)$.

Explicit formulas for the product of conjugacy classes of complete cycles in $\mathcal{A}_n$

In [2], Goupil shows that for any partition ρ of n

$$c_{(n)(n)}^{\rho}(\mathcal{S}_n) = \frac{(n-1)!}{(n+1)|C_\rho|} \sum_{\mu \lhd \rho} \mathrm{sgn}(\mu)|C_\mu||C_{\rho-\mu}|, \tag{4}$$

where $\mathrm{sgn}(\mu)$ is the signature of any permutation of C_μ. Using Equation (4) along with Lemma 2.1, we obtain the following explicit expressions for the structure coefficients of products of cycles of length n in the alternating group $\mathcal{A}_n$.

Theorem 3.2. *If n is an odd integer and $m = \frac{n-1}{2}$ then for any partition $\rho \in \mathcal{E}(n) \setminus (n)$, we have*

$$c_{(n)^1(n)^1}^{\rho}(\mathcal{A}_n) = c_{(n)^2(n)^2}^{\rho}(\mathcal{A}_n) = \frac{1}{4}\Big(c_{(n)(n)}^{\rho}(\mathcal{S}_n) + (-1)^m (m!)^2 \chi^{(1^m, m+1)}(\rho)\Big)$$

and

$$c_{(n)^1(n)^2}^{\rho}(\mathcal{A}_n) = c_{(n)^2(n)^1}^{\rho}(\mathcal{A}_n) = \frac{1}{4}\Big(c_{(n)(n)}^{\rho}(\mathcal{S}_n) - (-1)^m (m!)^2 \chi^{(1^m, m+1)}(\rho)\Big).$$

In addition,

$$c_{(n)^1(n)^1}^{(n)^1}(\mathcal{A}_n) = c_{(n)^2(n)^2}^{(n)^2}(\mathcal{A}_n) = \frac{1}{4}\left(c_{(n)(n)}^{(n)}(\mathcal{S}_n) + (m!)^2(1 + 2(-1)^m)\right),$$

$$c_{(n)^1(n)^1}^{(n)^2}(\mathcal{A}_n) = c_{(n)^2(n)^2}^{(n)^1}(\mathcal{A}_n) = \frac{1}{4}\left(c_{(n)(n)}^{(n)}(\mathcal{S}_n) + (m!)^2(1 - 2(-1)^m)\right)$$

and

$$c_{(n)^1(n)^2}^{(n)^1}(\mathcal{A}_n) = c_{(n)^2(n)^1}^{(n)^2}(\mathcal{A}_n) = \frac{1}{4}\left(c_{(n)(n)}^{(n)}(\mathcal{S}_n) - (m!)^2\right).$$

Proof. By Lemma 2.1, if n is odd then $(n) \in \mathcal{OD}(n)$ and if $\rho \notin \mathcal{OD}(n)$ then for $i = 1, 2$

$$c^{\rho}_{(n)^i(n)^i}(\mathcal{A}_n) = \frac{1}{4}c^{\rho}_{(n)(n)}(\mathcal{S}_n) + \frac{(n-1)!}{2n}\frac{\chi^{(n)^c}(\rho)(x^2+y^2-\frac{1}{2})}{f^{(n)^c}}$$

and

$$c^{\rho}_{(n)^1(n)^2}(\mathcal{A}_n) = c^{\rho}_{(n)^2(n)^1}(\mathcal{A}_n) = \frac{1}{4}c^{\rho}_{(n)(n)}(\mathcal{S}_n) + \frac{(n-1)!}{2n}\frac{\chi^{(n)^c}(\rho)(2xy-\frac{1}{2})}{f^{(n)^c}}.$$

Using the identities $x^2+y^2 = \frac{1}{2}(1+(-1)^m n)$ and $xy = \frac{1}{4}(1-(-1)^m n)$ of Corollary 3.1 and the fact that

$$f^{(n)^c} = \frac{(n-1)!}{(m!)^2}$$

by Equation (3), we obtain the first two formulas of our theorem. In the same way, one obtains

$$c^{(n)^1}_{(n)^1(n)^1}(\mathcal{A}_n) = c^{(n)^2}_{(n)^2(n)^2}(\mathcal{A}_n) = \frac{1}{4}c^{\rho}_{(n)(n)}(\mathcal{S}_n) + \frac{(m!)^2}{2n}\left(2x^2\overline{x} + 2y^2\overline{y} - \frac{(-1)^m}{2}\right),$$

$$c^{(n)^2}_{(n)^1(n)^1}(\mathcal{A}_n) = c^{(n)^1}_{(n)^2(n)^2}(\mathcal{A}_n) = \frac{1}{4}c^{\rho}_{(n)(n)}(\mathcal{S}_n) + \frac{(m!)^2}{2n}\left(2x^2\overline{y} + 2y^2\overline{x} - \frac{(-1)^m}{2}\right)$$

and

$$c^{(n)^1}_{(n)^1(n)^2}(\mathcal{A}_n) = c^{(n)^2}_{(n)^2(n)^1}(\mathcal{A}_n) = \frac{1}{4}c^{\rho}_{(n)(n)}(\mathcal{S}_n) + \frac{(m!)^2}{2n}\left(2xy\overline{x} + 2yx\overline{y} - \frac{(-1)^m}{2}\right).$$

The proof is completed using the identities $x^2\overline{x} + y^2\overline{y} = \frac{1}{4}\big((-1)^m + (1+2(-1)^m)n\big)$, $x^2\overline{y} + y^2\overline{x} = \frac{1}{4}\big((-1)^m + (1-2(-1)^m)n\big)$ and $xy\overline{x} + yx\overline{y} = \frac{1}{4}((-1)^m - n)$ of Corollary 3.1. □

In the above theorem, characters that correspond to partitions $(1^r, n-r)$ appear in the given formulas. The following explicit result to compute the value of $\chi^{(1^r,n-r)}$ on elements of cycle type $\mu = (1^{\mu_1}, \cdots, n^{\mu_n})$, where μ is a partition of n, appears in [2]

$$\chi^{(1^r,n-r)}_{\mu} = (-1)^r \sum_{\rho=(1^{\rho_1},\cdots,r^{\rho_r})\vdash r} (-1)^{l(\rho)}\binom{\mu_1-1}{\rho_1}\binom{\mu_2}{\rho_2}\cdots\binom{\mu_r}{\rho_r}.$$

For example, none of the five partitions $(1^4), (1^2,2), (1,3), (2^2)$ and (4) of 4 contributes to $\chi^{(1^4,5)}_{(1^3,3^2)}$ except of $(1,3)$. Therefore,

$$\chi^{(1^4,5)}_{(1^3,3^2)} = (-1)^4\left((-)^2\binom{2}{1}\binom{2}{1}\right) = 4.$$

Example 3.1. *The symmetric group $\mathcal{S}_5$ has 7 conjugacy classes indexed by partitions of 5 which are:*

$$(5), (1,4), (1^2,3), (1,2^2), (1^3,2), (1^5).$$

The alternating group $\mathcal{A}_5$ has 5 conjugacy classes indexed by the partitions: $(5)^1, (5)^2, (1^2,3), (1,2^2)$ and (1^5). Here $C_{(5)^1}(\mathcal{A}_5)$ (respectively $C_{(5)^2}(\mathcal{A}_5)$) will be considered to be the the subset of $C_{(5)}(\mathcal{S}_5)$ containing the conjugates of $(1,2,3,4,5)$ (respectively $(1,2)(1,2,3,4,5)(1,2) = (1,3,4,5,2)$) in $\mathcal{A}_n$. By Equation (4) of Goupil, we have the following explicit expression

$$\mathbf{C}_{(5)}(\mathcal{S}_5)\mathbf{C}_{(5)}(\mathcal{S}_5) = 24\mathbf{C}_{(1^5)}(\mathcal{S}_5) + 8\mathbf{C}_{(1,2^2)}(\mathcal{S}_5) + 12\mathbf{C}_{(1^2,3)}(\mathcal{S}_5) + 8\mathbf{C}_{(5)}(\mathcal{S}_5).$$

Using it along with our result in Theorem 3.2, we get the following explicit products in $Z(\mathbb{C}[\mathcal{A}_5])$:

$$\mathbf{C}_{(5)^1}(\mathcal{A}_5)\mathbf{C}_{(5)^1}(\mathcal{A}_5) = 12\mathbf{C}_{(1^5)}(\mathcal{A}_5) + 3\mathbf{C}_{(1^2,3)}(\mathcal{A}_5) + 5\mathbf{C}_{(5)^1}(\mathcal{A}_5) + \mathbf{C}_{(5)^2}(\mathcal{A}_5),$$

$$\mathbf{C}_{(5)^2}(\mathcal{A}_5)\mathbf{C}_{(5)^2}(\mathcal{A}_5) = 12\mathbf{C}_{(1^5)}(\mathcal{A}_5) + 3\mathbf{C}_{(1^2,3)}(\mathcal{A}_5) + \mathbf{C}_{(5)^1}(\mathcal{A}_5) + 5\mathbf{C}_{(5)^2}(\mathcal{A}_5),$$

$$\mathbf{C}_{(5)^1}(\mathcal{A}_5)\mathbf{C}_{(5)^2}(\mathcal{A}_5) = 4\mathbf{C}_{(1,2^2)}(\mathcal{A}_5) + 3\mathbf{C}_{(1^2,3)}(\mathcal{A}_5) + \mathbf{C}_{(5)^1}(\mathcal{A}_5) + \mathbf{C}_{(5)^2}(\mathcal{A}_5),$$

and

$$\mathbf{C}_{(5)^2}(\mathcal{A}_5)\mathbf{C}_{(5)^1}(\mathcal{A}_5) = 4\mathbf{C}_{(1,2^2)}(\mathcal{A}_5) + 3\mathbf{C}_{(1^2,3)}(\mathcal{A}_5) + \mathbf{C}_{(5)^1}(\mathcal{A}_5) + \mathbf{C}_{(5)^2}(\mathcal{A}_5).$$

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