Research Article

A note on the number of triangles in graphs without the suspension of a path on four vertices

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Abstract

The suspension of the path P_4 consists of a P_4 and an additional vertex adjacent to each of the four vertices, and is denoted by $\hat{P}_4$. The largest number of triangles in a $\hat{P}_4$-free n-vertex graph is denoted by $\mathrm{ex}(n, K_3, \hat{P}_4)$. Mubayi and Mukherjee showed in the preprint [arXiv:2004.11930, (2020)] that $\mathrm{ex}(n, K_3, \hat{P}_4) = n^2/8 + O(n)$. For sufficiently large n, it is shown here that $\mathrm{ex}(n, K_3, \hat{P}_4) = \lfloor n^2/8 \rfloor$.

Keywords: generalized Turán problems; Berge hypergraph; suspension of a graph.

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1. Introduction

One of the main topics in extremal graph theory was initiated by Turán [8] and deals with the quantity $\mathrm{ex}(n, F)$, which is the largest number of edges in n-vertex F-free graphs.

Generalized Turán problems deal with the quantity $\mathrm{ex}(n, H, F)$, which is the largest number of copies of a subgraph H in n-vertex F-free graphs. The systematic study of generalized Turán problems was initiated by Alon and Shikhelman [1].

The *suspension* $\hat{F}$ of a graph F is obtained by adding an additional vertex v to F and for each vertex $u \in V(F)$, adding the edge uv. Mubayi and Mukherjee [6] studied $\mathrm{ex}(n, K_3, \hat{F})$ for various different bipartite graphs F. In particular, they investigated $\mathrm{ex}(n, K_3, \hat{P}_4)$, where P_4 denotes the path on four vertices.

Proposition 1.1 (see [6]). *We have* $\lfloor n^2/8 \rfloor - O(1) \leq \mathrm{ex}(n, K_3, \hat{P}_4) < n^2/8 + 3n$.

We determine $\mathrm{ex}(n, K_3, \hat{P}_4)$ exactly if n is sufficiently large.

Theorem 1.1. *If n is sufficently large, then* $\mathrm{ex}(n, K_3, \hat{P}_4) = \lfloor n^2/8 \rfloor$.

We present the proof in Section 2. The proof uses (a special case of) a theorem of Győri [5] on Berge triangles and progressive induction. However, we can incorporate the proof of Győri's result into our proof and use a simple version of progressive induction, this way making our proof self-contained. We introduce Berge hypergraphs, state Győri's result, show the connection to our problem, and describe progressive induction in Section 3.

2. Proof of Theorem 1.1

Let us describe the lower bound first. We take an almost balanced complete bipartite graph on n vertices and add a full matching to one of the parts. Then the neighborhood of every vertex is either a star or or a matching, thus this graph if $\hat{P}_4$-free. If $n = 4k$, we take $K_{2k,2k}$ and add k independent edges to one of the parts. If $n = 4k+1$, we take $K_{2k,2k+1}$ and add k independent edges to the smaller part. If $n = 4k+2$, we take $K_{2k,2k+2}$ and add k independent edges to the smaller part. If $n = 4k+3$, we take $K_{2k+1,2k+2}$ and add $k+1$ independent edges to the larger part.

Let us continue with the proof of the upper bound. We will show that if $n \geq 525$ and $f(n) := \mathrm{ex}(n, K_3, \hat{P}_4) - \lfloor n^2/8 \rfloor > 0$, then either $f(n) < f(n-1)$ or $f(n) < f(n-4)$. Let

$$K = \max\{f(n) : n < 525\} \leq \binom{525}{3},$$

then for $n \geq 525$ we have $f(n) < K$ (note that by Proposition 1.1 we have $K \leq 1575$). Similarly for $n \geq 529$ we have $f(n) < K - 1$ and in general for $n \geq 525 + 4k$ we have $f(n) < K - k$. Therefore, for $n \geq 525 + 4K$ the statement holds.

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Let G be a $\hat{P}_4$-free n-vertex graph with $\mathrm{ex}(n, K_3, \hat{P}_4)$ triangles. Assume first that G is K_4-free. Observe that every triangle in G contains at least two edges that are not contained in any other triangle. Indeed, assume that a, b, c forms a triangle and ab and bc are contained in other triangles a, b, x and b, c, y. If $x = y$, then there is a K_4 with vertices a, b, c, x, and if $x \neq y$, then each vertex of the path $xacy$ is adjacent to b.

Let us pick for each triangle in G two edges that are not contained in any other triangle (if there are three such edges, we pick two arbitrarily). The resulting graph G' clearly has twice as many edges as the number of triangles in G. Observe that G' is triangle-free, since G' is a subgraph of G, and for every triangle of G, one of its edges was not picked. Therefore, by Mantel's theorem G' has at most $\lfloor n^2/4 \rfloor$ edges, hence G has at most $\lfloor n^2/8 \rfloor$ triangles.

Assume now that G contains a K_4 with vertices $U = \{v_1, v_2, v_3, v_4\}$. Assume first that there is a vertex v that is contained in at most $(2n - 5)/8$ triangles. Then $\mathrm{ex}(n, K_3, \hat{P}_4) \leq \mathrm{ex}(n - 1, K_3, \hat{P}_4) + (2n - 5)/8$ and thus

$$f(n) - f(n-1) = \mathrm{ex}(n, K_3, \hat{P}_4) - \lfloor n^2/8 \rfloor - \mathrm{ex}(n - 1, K_3, \hat{P}_4) + \lfloor (n-1)^2/8 \rfloor < 0.$$

As there is no P_4 in the neighborhood $N(v)$ of v in G, the connected components of $G[N(v)]$ are stars or triangles. This means that there are at most $|N(v)|$ edges inside $N(v)$, thus v is in at most $|N(v)|$ triangles. Therefore, we are done if there is a vertex with degree at most $(2n - 5)/8$. Assume now that there are at most $n - 3$ triangles containing a vertex from U. Then $\mathrm{ex}(n, K_3, \hat{P}_4) \leq \mathrm{ex}(n - 4, K_3, \hat{P}_4) + n - 3$, thus $f(n) < f(n-1)$.

Assume from now on that every vertex has degree more than $(2n - 5)/8$ (thus at least $(n - 4)/4$), and for every K_4, there are more than $n - 3$ triangles that contain at least one of its vertices. Observe that every vertex outside U is adjacent to at most one vertex of U by the $\hat{P}_4$-free property. Let $V_i = N(v_i) \setminus U$ for $i \leq 4$ and $V_5 = V(G) \setminus (U \cup V_1 \cup V_2 \cup V_3 \cup V_4)$, then the V_i's are pairwise disjoint, thus there are at most $n - 4$ triangles containing one vertex from U. There are 4 triangles inside U and no triangles containing exactly two vertices of U, thus there are n or $n - 1$ or $n - 2$ triangles containing at least one vertex from U. In particular, at least two of the v_i's have the property that v_i is in $|N(v_i)|$ triangles. This can happen only if $G[V_i]$ consists of vertex-disjoint triangles. Without loss of generality this holds for V_1 and V_2.

Observe that the above holds for every K_4, i.e. there are two vertices in every K_4 such that their neighborhood induces vertex disjoint triangles. In particular, there is a such vertex $v \in V_1$. Observe that only one of the triangles in $N(v)$ intersects V_1, and every triangle from $N(v)$ contains at most one vertex from V_2. Indeed, if a, b, c form such a triangle, with $a, b \in V_2$, then there is a triangle a, b, c' inside V_2. Then $vcac'$ is a P_4 in the neighborhood in b, a contradiction.

Recall that v has degree at least $(n - 4)/4$ and its neighborhood consists of at least $(n - 4)/12$ vertex disjoint triangles. This means that out of the at least $(n - 4)/12$ triangles in $N(v)$, one is inside $V_1 \cup \{v_1\}$, and the other at least $(n - 16)/12$ triangles each contain an edge from $V_3 \cup V_4 \cup V_5$. Observe that these are independent edges.

Let A be denote the set of vertices in G that are contained in some K_4. Then $|A| \geq |U| + |V_1| + |V_2| + (n - 16)/6 \geq (8n + 8)/12$. Since for $i \leq 4$ we have $|V_i| \geq (n - 16)/4$ but the union of these disjoint four sets have order at most $n - 4$, we have that $|V_i| \leq (n + 32)/4$. Therefore, the degree of vertices in A is at most $(n + 44)/4$. Let B be denote the set of vertices in G that are not contained in any K_4. A vertex of B is adjacent to at most one vertex in each triangle inside V_1 and V_2, and to at most one vertex of each of the $(n - 16)/12$ independent edges inside $V_3 \cup V_4 \cup V_5$ that form triangles with v. This means that the degree of a vertex of B is at most $n - 1 - (n - 3)/3 - (n - 6)/12 = (7n + 6)/12$. Therefore, the sum of the degrees in G is at most

$$\frac{8n + 8}{12}\frac{n + 44}{4} + \frac{4n - 8}{12}\frac{7n + 6}{12} = \frac{13n^2 + 262n + 252}{36}.$$

Using that the number of triangles containing a vertex is at most its degree, it gives an upper bound on three times the number of triangles in G (as we count every triangle three times this way). Therefore, the number of triangles is at most

$$\frac{13n^2 + 262n + 252}{108} < \left\lfloor \frac{n^2}{8} \right\rfloor$$

if $n \geq 525$, completing the proof.

3. Remarks

Given a graph F, a hypergraph $\mathcal{H}$ is a *Berge copy of* F (in short Berge-F) if $V(F) \subset V(\mathcal{H})$ and there is a bijection $\phi : E(F) \to E(\mathcal{H})$ such that for each edge $e \in E(F)$ we have $e \subset \phi(e)$. In other words, we can obtain $\mathcal{H}$ from F by enlarging the edges arbitrarily. Note that there may be several non-isomorphic Berge copies of F. We denote by $\mathrm{ex}_r(n, \text{Berge-}F)$ the largest number of hyperedges in an r-uniform n-vertex hypergraph that does not contain any of the Berge copies of F.

Berge hypergraphs were introduced by Gerbner and Palmer [3], extending the well-studied notion of Berge cycles and paths. The connection of Berge hypergraphs to generalized Turán problems was shown in [4], by the following simple

proposition: $\text{ex}(n, K_r, F) \le \text{ex}_r(n, \text{Berge-}F) \le \text{ex}(n, K_r, F) + \text{ex}(n, F)$. We remark that the lower bound follows by observing that the hypergraph consisting of the vertex sets of copies of K_r in an F-free graph is Berge-F-free. We will connect our problem to Berge hypergraphs differently.

Given a graph G, let us denote by $\mathcal{T}(G)$ the 3-uniform hypergraph with vertex set $V(G)$, where $\{a, b, c\}$ is a hyperedge if and only if there is a triangle with vertices a, b, c in G.

Proposition 3.1. $\mathcal{T}(G)$ *is Berge-K_3-free if and only if G is $\hat{P}_4$-free and K_4-free.*

Proof. Let us assume that there is a Berge-K_3 in $\mathcal{T}(G)$. It consists of 3 hyperedges that correspond to a graph triangle with vertices a, b, c. Then the 3 hyperedges are $\{a, b, x\}$, $\{b, c, y\}$ and $\{c, a, z\}$. At most one of these hyperedges are $\{a, b, c\}$, thus without loss of generality $x, y \notin \{a, b, c\}$. The edges $ab, ax, bx, bc, by, cy, ca, cz, az$ are all in G. If $x = y$, then there is a K_4 in G with vertices a, b, c, x. If $x \ne y$, then the vertices of the path $xacy$ are each adjacent to b, giving a $\hat{P}_4$.

If G contains a K_4 with vertices a, b, c, d, then there is a bijection from the edges of the triangle a, b, c to the hyperedges $\{a, b, d\}$, $\{b, c, d\}$, $\{c, a, d\}$ of $\mathcal{T}(G)$, giving us a Berge-K_3 in $\mathcal{T}(G)$. If G contains a path $xacy$ such that its vertices are each adjacent to a fifth vertex b, then there is a bijection from the edges of the triangle a, b, c to the hyperedges $\{a, b, x\}$, $\{b, c, y\}$, $\{c, a, b\}$ of $\mathcal{T}(G)$, giving us a Berge-K_3 in $\mathcal{T}(G)$. ∎

The above proposition immediately implies that

$$\text{ex}(n, K_3, \{\hat{P}_4, K_4\}) \le \text{ex}_3(n, \text{Berge-}K_3).$$

Győri [5] showed that $\text{ex}_3(n, \text{Berge-}K_3) = \lfloor n^2/8 \rfloor$. We remark that the lower bound is obtained by $\mathcal{T}(G)$, where G is the construction that gives the lower bound in Theorem 1.1. Using the above results, we immediately obtain that

$$\text{ex}(n, K_3, \{\hat{P}_4, K_4\}) = \text{ex}_3(n, \text{Berge-}K_3) = \left\lfloor \frac{n^2}{8} \right\rfloor.$$

To strengthen it and obtain Theorem 1.1, we need to show that if a $\hat{P}_4$-free graph contains a K_4, then it contains at most $\lfloor n^2/8 \rfloor$ triangles.

We remark that the proof of the upper bound of Proposition 1.1 in [6] also deals separately with the two cases depending on whether a $\hat{P}_4$-free graph is K_4-free or not. Our improvement in the first case is by the above observations, and Győri's proof is incorporated to our proof in Section 2. The second case is dealt by induction in [6]. However, we cannot prove our improved bound in a similar way, as our Theorem 1.1 does not hold for every n. For example, if $n = 7$, then $\lfloor n^2/8 \rfloor = 6$, but two copies of K_4 sharing a vertex is a $\hat{P}_4$-free 7-vertex graph and contains 8 triangles. Therefore, we do not have the base step of the induction, even though the induction step works almost the same as in [6].

Progressive induction, introduced by Simonovits [7], is used in such cases (it was first used in the generalized Turán setting in [2]). The basic idea is to have a stronger induction step, where we also deal with the case of equality. This means that for small values of n $\text{ex}(n, K_3, \hat{P}_4)$ can be larger than $\lfloor n^2/8 \rfloor$, but this surplus starts decreasing and eventually vanishes. The first paragraph of the proof of the upper bound of Theorem 1.1 is a simple example of this; we do not go into the details of the general version. Finally, we mention that it is likely that Theorem 1.1 holds for every $n \ge 8$. If the upper bound for $n = 8, 9, 10, 11$ is proved, either by a careful case analysis or by brute force, then ordinary induction could be used instead of progressive induction to complete the proof of Theorem 1.1.

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