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Research Article

Distance signless Laplacian eigenvalues, diameter, and clique number

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Abstract

Let G be a connected graph of order n. Let $\mathcal{D}iag(Tr)$ be the diagonal matrix of vertex transmissions and let $\mathcal{D}(G)$ be the distance matrix of G. The distance signless Laplacian matrix of G is defined as $\mathcal{D}^{\mathcal{Q}}(G) = \mathcal{D}iag(Tr) + \mathcal{D}(G)$ and the eigenvalues of $\mathcal{D}^{\mathcal{Q}}(G)$ are called the distance signless Laplacian eigenvalues of G. Let $\partial_1^{\mathcal{Q}}(G) \geq \partial_2^{\mathcal{Q}}(G) \geq \cdots \geq \partial_n^{\mathcal{Q}}(G)$ be the distance signless Laplacian eigenvalues of G. The largest eigenvalue $\partial_1^{\mathcal{Q}}(G)$ is called the distance signless Laplacian spectral radius. We obtain a lower bound for $\partial_1^{\mathcal{Q}}(G)$ in terms of the diameter and order of G. With a given interval I, denote by $m_{\mathcal{D}^{\mathcal{Q}}(G)}I$ the number of distance signless Laplacian eigenvalues of G which lie in I. For a given interval I, we also obtain several bounds on $m_{\mathcal{D}^{\mathcal{Q}}(G)}I$ in terms of various structural parameters of the graph G, including diameter and clique number.

Keywords: distance matrix; distance signless Laplacian matrix; spectral radius; diameter; clique number.

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1. Introduction

Let $G = (V(G), E(G))$ be a simple connected graph with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The *order* and *size* of G are $|V(G)| = n$ and $|E(G)| = m$, respectively. The *degree* of a vertex v, denoted by $d_G(v)$ is the number of edges incident to the vertex v. In G, $N_G(v)$ is the set of all vertices which are adjacent to v. Further, K_n denotes the complete graph on n vertices. In a graph G, the subset $M \subseteq V(G)$ is called an *independent set* if no two vertices of M are adjacent. A *clique* is a complete subgraph of a given graph G. The cardinality of the maximum clique is called the *clique number* of G and is denoted by ω. A vertex $u \in V(G)$ is called a pendant vertex if $d_G(u) = 1$. For other standard definitions, we refer the reader to [6, 11].

For $v_i, v_j \in V(G)$, the *distance* between v_i and v_j, denoted by d_{ij} or $d_G(v_i, v_j)$, is the length of a shortest path between v_i and v_j. The *diameter* d (or $d(G)$) of a graph G is the maximum distance between any two vertices of G. The *distance matrix* of G, denoted by $\mathcal{D}(G)$, is defined as $\mathcal{D}(G) = (d_{ij})_{v_i, v_j \in V(G)}$. The *transmission* $Tr_G(v_i)$ (we will write $Tr(v_i)$ if the graph G is understood) of a vertex v_i is defined as the sum of the distances from v_i to all other vertices in G, that is,

$$Tr_G(v_i) = \sum_{v_j \in V(G)} d_G(v_i, v_j).$$

Let $Tr(G) = diag(Tr(v_1), Tr(v_2), \ldots, Tr(v_n))$ be the diagonal matrix of vertex transmissions of G. For a connected graph G, Aouchiche and Hansen [4] defined the *distance Laplacian matrix* of G as $D^L(G) = Diag(Tr) - D(G)$ (or simply D^L) and the *distance signless Laplacian matrix* as $\mathcal{D}^{\mathcal{Q}}(G) = Tr(G) + \mathcal{D}(G)$ (or simply $\mathcal{D}^{\mathcal{Q}}$). The eigenvalues of $\mathcal{D}^{\mathcal{Q}}(G)$ are called the distance signless Laplacian eigenvalues of G. Clearly, $\mathcal{D}^{\mathcal{Q}}(G)$ is a real symmetric matrix. We denote its eigenvalues by $\partial_i^{\mathcal{Q}}(G)$'s and order them as $\partial_1^{\mathcal{Q}}(G) \geq \partial_2^{\mathcal{Q}}(G) \geq \cdots \geq \partial_{n-1}^{\mathcal{Q}}(G) \geq \partial_n^{\mathcal{Q}}(G)$. The largest eigenvalue $\partial_1^{\mathcal{Q}}(G)$ is called the distance signless Laplacian spectral radius. Recent work on distance Laplacian matrix can be seen in [13, 14]. For more work done on distance signless Laplacian matrix of a graph G, we refer the reader to [1–3, 7–9, 12, 15–19]. If the graph G is understood, we may write $\partial_i^{\mathcal{Q}}$ in place of $\partial_i^{\mathcal{Q}}(G)$ and refer the distance signless Laplacian eigenvalues as $\mathcal{D}^{\mathcal{Q}} - eigenvalues$. Let $m_{\mathcal{D}^{\mathcal{Q}}(G)}I$ be the number of distance signless Laplacian eigenvalues of G that lie in the interval I. Also, let $m_{\mathcal{D}^{\mathcal{Q}}(G)}(\partial_i^{\mathcal{Q}}(G))$ be the multiplicity of the distance signless Laplacian eigenvalue $\partial_i^{\mathcal{Q}}(G)$.

In this paper, we obtain a lower bound for the distance signless Laplacian spectral radius of the graph G in terms of diameter d and order n. We show that the number of distance signless Laplacian eigenvalues in the interval $[n-2, dn]$ is at least $d+1$, where d is the diameter of the graph G. We also obtain a lower bound for the number of distance signless Laplacian eigenvalues which fall in the interval $(n-2, 2n-2)$, in terms of the order n and the number of vertices having

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degree $n-1$. Moreover, we show that the number of distance signless Laplacian eigenvalues in the interval $[n-2, 2n-\omega-2)$ is at most $n-\omega+2$, where n is the order and ω is the clique number of the graph G.

2. Distribution of distance signless Laplacian eigenvalues

We require the following lemmas to prove our main results.

Lemma 2.1. [5] *Let G be a connected graph on $n \geq 3$ vertices. Then, $\partial_1^Q(G) \geq \partial_1^Q(K_n) = 2n-2$ and $\partial_i^Q(G) \geq \partial_i^Q(K_n) = n-2$ for all $2 \leq i \leq n$.*

A particular case of the well known *min − max theorem* is the following result.

Lemma 2.2. [20] *If N is a symmetric $n \times n$ matrix with eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, then for any $x \in R^n$ ($x \neq 0$), we have*

$$\mu_1 \geq \frac{x^T N x}{x^T x},$$

where the equality holds if and only if x is an eigenvector of N corresponding to the largest eigenvalue μ_1.

Lemma 2.3. [10] *Let $M = (m_{ij})$ be a $n \times n$ complex matrix having $l_1, l_2, \ldots, l_p$ as its distinct eigenvalues. Then,*

$$\{l_1, l_2, \ldots, l_p\} \subset \bigcup_{i=1}^{n} \Big\{z : |z - m_{ii}| \leq \sum_{j \neq i} |m_{ij}|\Big\}.$$

If we apply Lemma 2.3 for the distance signless Laplacian matrix of a graph G with n vertices, we get

$$\partial_1^L(G) \leq 2Tr_{max} \tag{1}$$

Theorem 2.1 (Cauchy Interlacing Theorem). *Let M be a real symmetric matrix of order n, and let A be a principal submatrix of M with order $s \leq n$. Then*

$$\lambda_i(M) \geq \lambda_i(A) \geq \lambda_{i+n-s}(M) \qquad (1 \leq i \leq s).$$

In the following theorem, we give the lower bound for the distance signless Laplacian spectral radius of the graph G in terms of diameter d and order n.

Theorem 2.2. *Let G be a connected graph on n vertices having diameter d. Then*

$$\partial_1^Q(G) \geq \frac{2n + d(d+1) - 2}{2}.$$

Proof. Let $P_{d+1} : v_1 v_2 \ldots v_{d+1}$ be a diametral path in G such that $d_G(v_1, v_{d+1}) = d$. Consider the n-vector

$$y = (y_1, y_2, \ldots, y_{d-1}, y_d, y_{d+1}, \ldots, y_n)^T$$

defined by

$$y_i = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } i = 1,\ d+1 \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 2.2, we have

$$\partial_1^Q(G) \geq \frac{y^T \mathcal{D}^Q y}{y^T y} = \frac{Tr(v_1) + Tr(v_{d+1})}{2} + d_G(v_1, v_{d+1}). \tag{2}$$

Now, we have

$$Tr(v_1) + Tr(v_{d+1}) \geq 2(1 + 2 + \cdots + d) + 2(n-d-1) = d(d+1) + 2(n-d-1)$$

On substituting the above inequality in Inequality (2), we get

$$\partial_1^Q(G) \geq \frac{d(d+1) + 2(n-d-1)}{2} + d = \frac{2n + d(d+1) - 2}{2}.$$

□

The next result shows that the number of distance signless Laplacian eigenvalues in the interval $[n-2, dn]$ is at least $d+1$, where d is the diameter of the graph G.

Theorem 2.3. *Let G be a connected graph on $n \geq 3$ vertices having diameter d, then*

$$m_{\mathcal{D}^{\mathcal{Q}}(G)}[n-2, dn] \geq d+1.$$

Proof. We consider the principal submatrix, say M, corresponding to the vertices $v_1, v_2, \ldots, v_{d+1}$ which belong to the induced path P_{d+1} in the distance signless Laplacian matrix of G. Clearly,

$$Tr(v_i) \leq 1+2+\cdots+d+d(n-d-1)$$
$$= \frac{d(2n-d-1)}{2},$$

for all $i = 1, 2, \ldots, d+1$. Also, the sum of the off diagonal elements of any row of M is less than or equal to $d(d+1)/2$. Using Lemma 2.3, we conclude that the maximum eigenvalue of M is at most dn. Using Lemma 2.1 and Theorem 2.1, we see there are at least $d+1$ distance signless Laplacian eigenvalues of G which are greater than or equal to $n-2$ and less than or equal to dn, that is

$$m_{\mathcal{D}^{\mathcal{Q}}(G)}[n-2, dn] \geq d+1.$$

□

An immediate consequence of Theorem 2.3 is the following result.

Corollary 2.1. *Let G be a connected graph on $n \geq 3$ vertices having diameter d. If $dn < 2Tr_{max}$, then*

$$m_{\mathcal{D}^{\mathcal{Q}}(G)}(dn, 2Tr_{max}] \leq n-d-1.$$

Proof. Since $dn < 2Tr_{max}$, by Lemma 2.1 and Inequality (1), we have

$$m_{\mathcal{D}^{\mathcal{Q}}(G)}[n-2, dn] + m_{\mathcal{D}^{\mathcal{Q}}(G)}(dn, 2Tr_{max}] = n.$$

Thus, using Theorem 2.3, we get

$$m_{\mathcal{D}^{\mathcal{Q}}(G)}(dn, 2Tr_{max}] \leq n-d-1.$$

□

For proving the next result, we need the following lemma which can be found in [5].

Lemma 2.4. *Let G be a connected graph with n vertices. If $K = \{v_1, v_2, \ldots, v_p\}$ is a clique of G such that $N_G(v_i) - K = N_G(v_j) - K$ for all $i, j \in \{1, 2, \ldots, p\}$, then $\partial = Tr(v_i) = Tr(v_j)$ for all $i, j \in \{1, 2, \ldots, p\}$ and $\partial - 1$ is an eigenvalue of $\mathcal{D}^{\mathcal{Q}}(G)$ with multiplicity at least $p-1$.*

Now, we obtain a lower bound for the number of distance signless Laplacian eigenvalues which fall in the interval $(n-2, 2n-2)$, in terms of the order n and the number of vertices having degree $n-1$.

Theorem 2.4. *Let G be a connected graph on n vertices. If $m_d = |\{u \in V(G) : d_G(u) = n-1\}|$, where $1 \leq m_d \leq n$, then*

$$m_{\mathcal{D}^{\mathcal{Q}}(G)}(n-2, 2n-2) \leq n - m_d.$$

Equality holds when $m_d = n$, that is, $G \cong K_n$.

Proof. We consider the following two cases.

Case 1. Let $m_d = n$, that is, $G \cong K_n$. By Lemma 2.1, we see that the equality holds.

Case 2. Let $1 \leq m_d \leq n-1$. Since G contains m_d vertices of degree $n-1$, therefore, G contains a clique, say S, of size m_d. Let $S = \{v_1, v_2, \ldots, v_{m_d}\}$. Clearly,

$$n-1 = Tr(v_1) = Tr(v_2) = \cdots = Tr(v_{m_d}).$$

By Lemma 2.4, we observe that $n-2$ is a distance signless Laplacian eigenvalue of G with multiplicity at least $m_d - 1$. Also, we know that the distance signless Laplacian matrix corresponding to any connected graph H is symmetric, positive and irreducible. Therefore, by the Perron-Frobenius Theorem, $\partial_1^{\mathcal{Q}}(H - uv) > \partial_1^{\mathcal{Q}}(H)$ whenever $uv \in E(H)$ and $H - uv$ is connected. As $m_d \leq n-1$, therefore, $G \ncong K_n$. Thus, from the above information $\partial_1^{\mathcal{Q}}(G) > \partial_1^{\mathcal{Q}}(K_n) = 2n-2$. Hence,

$$m_{\mathcal{D}^{\mathcal{Q}}(G)}(n-2, 2n-2) \leq n - (m_d - 1) - 1 = n - m_d.$$

□

The following lemma is used in proving Theorem 2.5.

Lemma 2.5. [5] *Let G be a graph with n vertices. If $K = \{v_1, v_2, \ldots, v_p\}$ is an independent set of G such that $N_G(v_i) = N_G(v_j)$ for all $i, j \in \{1, 2, \ldots, p\}$, then $\partial = Tr(v_i) = Tr(v_j)$ for all $i, j \in \{1, 2, \ldots, p\}$ and $\partial - 2$ is an eigenvalue of $\mathcal{D}^{\mathcal{Q}}(G)$ with multiplicity at least $p - 1$.*

The next result shows that the number of distance signless Laplacian eigenvalues in the interval $[n-2, 2n-4)$ is at most $n - p + 1$, where $n \geq 3$ is the order of G and p is the number of pendant vertices adjacent to common neighbour.

Theorem 2.5. *Let G be a connected graph of order $n \geq 3$. If $S = \{v_1, v_2, \ldots, v_p\} \subseteq V(G)$, where $|S| = p \leq n - 1$, is the set of pendant vertices such that every vertex in S has the same neighbourhood in $V(G) \setminus S$, then*

$$m_{\mathcal{D}^{\mathcal{Q}}(G)}[n-2, 2n-4) \leq n - p + 1.$$

Proof. Clearly all the vertices in S form an independent set. Since all the vertices in S are adjacent to same vertex, therefore, all the vertices of S have the same transmission . Now, for any v_i $(i = 1, 2, \ldots, p)$ of S, we have

$$T = Tr(v_i) \geq 2(p-1) + 1 + 2(n-p-1) = 2n - 3.$$

From Lemma 2.5, there are at least $p - 1$ distance signless Laplacian eigenvalues of G which are equal to $T - 1$. From above we have $T - 1 \geq 2n - 3 - 1 = 2n - 4$. Thus, there are at least $p - 1$ distance signless Laplacian eigenvalues of G which are greater than or equal to $2n - 4$. Using Lemma 2.1, we get $m_{\mathcal{D}^{\mathcal{Q}}(G)}[n-2, 2n-4) \leq n - p + 1$. □

Next, we show that the number of distance signless Laplacian eigenvalues in the interval $[n-2, 2n\ \omega - 2)$ is at most $n - \omega + 2$, where n is the order and ω is the clique number of the graph G.

Theorem 2.6. *Let G be a connected graph of order n having clique number $\omega \leq n - 1$. If only one vertex of the corresponding maximum clique is adjacent to the vertices outside of the clique, then*

$$m_{\mathcal{D}^{\mathcal{Q}}(G)}[n-2, 2n-\omega-2) \leq n - \omega + 2.$$

Proof. Let $S = \{v_1, v_2, \ldots, v_\omega\}$ be the set of vertices of the maximum clique such that v_ω is the only vertex having neighbours outside of S. Clearly, the set of vertices $N = \{v_1, v_2, \ldots, v_{\omega-1}\}$ also form a clique such that every vertex of N is adjacent to v_ω only outside of N. It is easy to see that all the vertices belonging to N have the same transmission. For any $v_i \in N$, $i = 1, 2, \ldots, \omega - 1$, we have

$$T = Tr(v_i) \geq \omega - 1 + 2(n - \omega) = 2n - \omega - 1. \tag{3}$$

Using Lemma 2.4, we observe that $T - 1$ is a distance signless Laplacian eigenvalue of G of multiplicity at least $\omega - 2$. From Inequality (3), we have $T - 1 \geq 2n - 1 - \omega - 1 = 2n - \omega - 2$. So there are at least $\omega - 2$ distance signless Laplacian eigenvalues of G which are greater than or equal to $2n - \omega - 2$. From Inequality (1), we get $m_{\mathcal{D}^{\mathcal{Q}}(G)}[2n - \omega - 2, 2Tr_{max}] \geq \omega - 2$. Thus, by the above observation and Lemma 2.1, we have $m_{\mathcal{D}^{\mathcal{Q}}(G)}[n-2, 2n-\omega-2) \leq n - \omega + 2$, which completes the proof. □

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