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Research Article

A Wilf–Zeilberger–based solution to the Basel problem with applications

John M. Campbell*

Department of Mathematics and Statistics, York University, Toronto, Ontario, Canada

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Abstract

Wilf [Accelerated series for universal constants, by the WZ method, *Discrete Math. Theor. Comput. Sci.* **3** (1999) 189–192] applied Zeilberger's algorithm to obtain an accelerated version of the famous series $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$. However, if we write the Basel series $\sum_{k=1}^{\infty} 1/k^2$ as a ${}_3F_2(1)$-series, it is not obvious as to how to determine a Wilf–Zeilberger (WZ) pair or a WZ proof certificate that may be used to formulate a proof for evaluating this ${}_3F_2(1)$-expression. In this article, using the WZ method, we prove a remarkable identity for a ${}_3F_2(1)$-series with three free parameters that Maple 2020 is not able to evaluate directly, and we apply our WZ proof of this identity to obtain a new proof of the famous formula $\zeta(2) = \pi^2/6$. By applying partial derivative operators to our WZ-derived ${}_3F_2(1)$-identity, we obtain an identity involving binomial-harmonic sums that were recently considered by Wang and Chu [Series with harmonic-like numbers and squared binomial coefficients, *Rocky Mountain J. Math.*, In press], and we succeed in solving some open problems given by Wang and Chu on series involving harmonic-type numbers and squared binomial coefficients.

Keywords: Basel problem; WZ theory; hypergeometric series; creative telescoping; Riemann zeta function.

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1. Introduction

The Basel problem, i.e., the problem of determining a closed form for

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \tag{1}$$

is one of the most famous problems in the history of mathematics. As recorded in [3], the Basel problem had been open for 91 years, with its having first been solved in 1735 by Leonhard Euler. Letting $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$ denote the Riemann zeta function, the famous formula $\zeta(2) = \frac{\pi^2}{6}$ is due to Euler. Over the centuries, many very elegant proofs of this formula have been introduced. See [13,22,31,44,46] for some relevant literature concerning the Basel problem. Many past articles have been devoted to new and elegant proofs that $\zeta(2) = \frac{\pi^2}{6}$; see [1,3–5,10–12,15–21,23,25–30,32–34,36–40,42,43,45]. In this article, we introduce a proof of the formula $\zeta(2) = \frac{\pi^2}{6}$ using the *Wilf–Zeilberger* (WZ) method [41].

For the sake of brevity, we assume familiarity with the basics about WZ theory [41]. In 1999, Wilf [49] applied a series acceleration method based on WZ theory, and provided an infinite family of geometrically rapid series for $\zeta(2)$, via an application of Zeilberger's algorithm [41, §6]; see also [24]. This was achieved by setting $F(n,k) = (k!)^2/((n+k+1)!)^2$, so that $\sum_{k\geq 0} F(0,k)$ equals the Basel series in (1), and by using Zeilberger's algorithm to determine that

$$-(4n+2)F(n,k) + (n+1)^3 F(n+1,k) = G(n,k+1) - G(n,k), \tag{2}$$

where $G(n,k) = (3n+2k+3)F(n,k)$. However, a *WZ pair* must satisfy the difference equation

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k), \tag{3}$$

in contrast to (2). For a pair (F,G) of bivariate hypergeometric functions satisfying the required conditions for a WZ pair, the rational function $R(n,k)$ such that $G(n,k) = R(n,k)F(n,k)$ is referred to as a *WZ proof certificate*. In this article, we consider the interesting problem of determining a WZ proof for evaluating the Basel sum in (1), inspired in part by the telescoping-based approach given in [3] toward the Basel problem.

Our new WZ-based proof that $\zeta(2) = \frac{\pi^2}{6}$ has led us to discover some remarkable identities for families of hypergeometric series that Maple 2020 cannot evaluate. Writing

$${}_{p+1}F_p\left[\begin{matrix} a_0, a_1, \ldots, a_{p+1} \\ b_1, b_2, \ldots, b_p \end{matrix} \,\middle|\, z\right] = \sum_{n=0}^{\infty} \frac{(a_0)_n\,(a_1)_n \cdots (a_{p+1})_n}{n!\,(b_1)_n\,(b_2)_n \cdots (b_p)_n} z^n, \tag{4}$$

*E-mail address: jmaxwellcampbell@gmail.com

we assume some basic familiarity with *classical hypergeometric series* [2, §2.5]. From the definition in (4), it is immediate that the Basel series in (1) may be written as the following ${}_3F_2(1)$-series:

$$ {}_3F_2\left[\begin{matrix}1,1,1\\ 2,2\end{matrix}\;\middle|\;1\right]. \tag{5}$$

It is not obvious as to how classically known identities for ${}_3F_2(1)$-series with free parameters, such as Dixon's identity, Watson's hypergeometric identity, or Whipple's theorem, could be used to determine a full proof for the closed form for (5). Our WZ proof for this closed-form evaluation is very much inspired by Zeilberger's WZ proof [14] of a famous ${}_4F_3(-1)$-series for $\frac{1}{\pi}$ due to Ramanujan, and we make use of the classical result known *Carlson's theorem* [2, p. 39] in a similar way relative to the WZ proof in [14].

2. Main results

In order to construct a WZ proof for evaluating the ${}_3F_2(1)$-series in (5), our strategy is to determine a WZ proof for an equivalent form of the hypergeometric transform indicated below:

$$ {}_3F_2\left[\begin{matrix}a,b,1\\ c,2\end{matrix}\;\middle|\;1\right] = \frac{1-c}{(a-1)(b-1)} + \frac{c-1}{(a-1)(b-1)}\,{}_2F_1\left[\begin{matrix}a-1,b-1\\ c-1\end{matrix}\;\middle|\;1\right]. \tag{6}$$

Remarkably, Maple 2020 is not able to provide any evaluation for the ${}_3F_2(1)$-series on the left-hand side of the above equality, for free variables a, b, and c. This motivates the generalization of our methods in view of practical applications regarding commercially available CAS software.

To prove (6), we begin by considering the equivalent form of (6) whereby

$$\sum_{n=0}^{\infty}\left(\frac{(a)_n\,(b)_n}{(c)_n\,n!(n+1)} - \frac{c-1}{(a-1)(b-1)}\frac{(a-1)_n\,(b-1)_n}{(c-1)_n\,n!}\right) = \frac{1-c}{(a-1)(b-1)}.$$

This is equivalent to the identity

$$\sum_{k=0}^{\infty}\frac{(a)_k\,(b)_k}{(c)_k\,k!}\frac{(a+b-c-2)k+(b-1)a-b-c+2}{(k+1)(a+k-1)(b+k-1)} = \frac{1-c}{(a-1)(b-1)}. \tag{7}$$

Set $F(n,k)$ as the summand of (7) divided by the right-hand side of (7) with $a=-n$. Explicitly,

$$F(n,k) = \frac{(b-1)(-n-1)(b)_k(-n)_k(k(b-c-n-2)-(b-1)n-b-c+2)}{(1-c)(k+1)k!(b+k-1)(k-n-1)(c)_k}. \tag{8}$$

Maple is able to provide the corresponding rational certificate

$$R(n,k) = \frac{(-k-1)k(c+k-1)}{(k-n-2)(bk-bn-b-ck-c-kn-2k+n+2)},$$

and, writing $G(n,k) = F(n,k)R(n,k)$, it is easily seen that F and G satisfy the difference equation in (3) for suitably bounded real n and k. Again with $a=-n$ in (7), we apply Carlson's theorem as below.

Theorem 2.1. *The identity*

$$\sum_{k=0}^{\infty}F(n,k) = 1 \tag{9}$$

holds for suitably bounded n.

Proof. For the time being, we let $n\in\mathbb{N}_0$. For k and n such that $n-k+1$ and $n-k+2$ are nonzero, we have that the WZ difference equation in (3) is satisfied by (8) and by $G(n,k)=F(n,k)R(n,k)$. So, we find that

$$\sum_{k=0}^{n-1}(F(n+1,k)-F(n,k)) = \sum_{k=0}^{n-1}(G(n,k+1)-G(n,k)). \tag{10}$$

Writing

$$(-n)_k = (-1)^k k!\binom{n}{k}, \tag{11}$$

and using the fact that the right-hand side of (10) telescopes, we find that

$$\sum_{k=0}^{n-1}(F(n+1,k)-F(n,k)) = -\frac{(-1)^n n(n+1)\Gamma(c-1)\Gamma(b+n-1)}{2\Gamma(b-1)\Gamma(c+n-1)}.$$

This can be used to give us that

$$\sum_{k=0}^{n+1}F(n+1,k)-\sum_{k=0}^{n}F(n,k) = -\frac{(-1)^n\Gamma(c-1)(b+c+2n)\Gamma(b+n)}{\Gamma(b-1)\Gamma(c+n+1)}. \tag{12}$$

Now, expressions as in "$F(m,m+1)$" for an element $m\in\mathbb{N}_0$ are not well-defined, due to a factor of the form

$$\frac{(-n)_k}{n-k+1}$$

in the above definition for F. Rewriting this factor according to the Pochhammer/binomial identity in (11), we define

$$H(n) = \lim_{k\to n+1} F(n,k) = \frac{(-1)^{n+1}\Gamma(c-1)\Gamma(b+n)}{\Gamma(b-1)\Gamma(c+n)}$$

and

$$\mathcal{F}(m,k) = \begin{cases} F(m,k) & \text{if } k\neq m+1 \\ H(m) & \text{if } k=m+1 \end{cases}$$

from whence it may be obtained that

$$\sum_{k=0}^{\infty}\mathcal{F}(n+1,k) = \sum_{k=0}^{\infty}\mathcal{F}(n,k)$$

according to (12), noting that $\mathcal{F}(n,k)$ vanishes for integers $k>n+1\in\mathbb{N}$. So, since the expression $\sum_{k=0}^{\infty}\mathcal{F}(n,k)$ as a function of $n\in\mathbb{N}_0$ is constant, we may easily verify the following:

$$\sum_{k=0}^{\infty}\mathcal{F}(n,k) = 1.$$

Applying Carlson's theorem as in [14], we find that

$$\sum_{k=0}^{\infty}\mathcal{F}(n,k) = \sum_{k=0}^{\infty}F(n,k) = 1$$

for suitably bounded real n. □

A WZ solution to the Basel problem

We have provided, as above, a WZ proof for the hypergeometric transform in (6). This transform can be used in a direct way to show, as below, that $\zeta(2)=\frac{\pi^2}{6}$. We are to also make use of Gauss' evaluation for ${}_2F_1(1)$-series, recalling the WZ proof of this classical identity given in [41]. As below, we let

$$\psi_n(z) = \frac{d^{n+1}}{dz^{n+1}}\ln\left(\Gamma(z)\right)$$

denote the polygamma function. We recall the series expansion

$$\psi_n(z) = (-1)^{n+1}n!\sum_{k=0}^{\infty}\frac{1}{(z+k)^{n+1}}, \tag{13}$$

writing $\psi_0=\psi$ to denote the digamma function. We recall that the Euler–Mascheroni constant γ is defined so that $\gamma=\lim_{n\to\infty}\left(H_n-\ln n\right)$.

Theorem 2.2. *The Basel identity $\zeta(2)=\frac{\pi^2}{6}$ holds true.*

Proof. According to our WZ-derived transform in (6), together with Gauss' ${}_2F_1$-identity, we find that

$${}_3F_2\left[\begin{matrix}a,b,1\\c,2\end{matrix}\,\middle|\,1\right] = \frac{1-c}{(a-1)(b-1)} + \frac{(c-1)\Gamma(c-1)\Gamma(-a-b+c+1)}{(a-1)(b-1)\Gamma(c-a)\Gamma(c-b)}.$$

So, we find that

$$ {}_3F_2\left[\begin{matrix} a, b, 1 \\ 2, 2 \end{matrix} \,\middle|\, 1\right] = \frac{\frac{\Gamma(-a-b+3)}{\Gamma(2-a)\Gamma(2-b)} - 1}{(a-1)(b-1)}. $$

Making use of Euler's reflection formula

$$ \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, $$

we find that

$$ {}_3F_2\left[\begin{matrix} a, b, 1 \\ 2, 2 \end{matrix} \,\middle|\, 1\right] = \frac{\frac{(a+b-2)(a+b-1)(a+b)\sin(\pi(a-1))\sin(\pi(b-1))}{\pi(a-1)a(b-1)b\sin(\pi(a+b-2))\binom{a+b}{a}} - 1}{(a-1)(b-1)}. $$

Taking the limit as $a \to 1$, we obtain

$$ \frac{\pi\cot(\pi b) + \psi(b-1) + \gamma}{1-b}. $$

So, we have that

$$ \sum_{k=1}^{\infty} \frac{1}{k^2} = \lim_{b \to 1} \frac{\pi\cot(\pi b) + \psi(b-1) + \gamma}{1-b}. $$

Applying l'Hôpital's rule to the above limit, we find that

$$ \sum_{k=1}^{\infty} \frac{1}{k^2} = \lim_{b \to 1} \left(\pi^2 \csc^2(\pi b) - \psi_1(b-1)\right). $$

According to the usual series expansion for the polygamma function, as in (13), we have that

$$ \sum_{k=1}^{\infty} \frac{1}{k^2} = \lim_{b \to 1} \left(\pi^2 \csc^2(b\pi) - \sum_{k=0}^{\infty} \frac{1}{(b-1+k)^2}\right). $$

So, we obtain that

$$ \sum_{k=1}^{\infty} \frac{1}{k^2} = \lim_{b \to 1} \left(\pi^2 \csc^2(b\pi) - \frac{1}{(b-1)^2} - \sum_{k=1}^{\infty} \frac{1}{k^2}\right). $$

So, we have shown that

$$ 2\sum_{k=1}^{\infty} \frac{1}{k^2} = \lim_{b \to 1} \left(\pi^2 \csc^2(\pi b) - \frac{1}{(b-1)^2}\right). $$

Equivalently,

$$ 2\sum_{k=1}^{\infty} \frac{1}{k^2} = \lim_{\beta \to \infty} \pi^2 \csc^2\left(\frac{\pi}{\beta}\right) - \beta^2. $$

Using the Laurent series for the elementary function csc, and then taking the Cauchy product of this Laurent expansion times itself, we obtain the following:

$$ 2\sum_{k=1}^{\infty} \frac{1}{k^2} = \lim_{\beta \to \infty} \left(-\beta^2 + \pi^2\left(\frac{1}{3} + \frac{1}{\left(\frac{\pi}{\beta}\right)^2} + \frac{1}{15}\left(\frac{\pi}{\beta}\right)^2 + \frac{2}{189}\left(\frac{\pi}{\beta}\right)^4 + \frac{1}{675}\left(\frac{\pi}{\beta}\right)^6 + \frac{2\left(\frac{\pi}{\beta}\right)^8}{10395} + \cdots\right)\right). $$

Since $\beta \to \infty$, the right-hand side of the above equality is reducible in the following manner:

$$ 2\sum_{k=1}^{\infty} \frac{1}{k^2} = \lim_{\beta \to \infty} \left(-\beta^2 + \pi^2\left(\frac{1}{3} + \frac{1}{\left(\frac{\pi}{\beta}\right)^2}\right)\right). $$

Expanding the expression

$$ -\beta^2 + \pi^2\left(\frac{1}{3} + \frac{1}{\left(\frac{\pi}{\beta}\right)^2}\right) $$

as $\frac{\pi^2}{3}$, this completes our proof. $\square$

Relation to recent problems due to Wang and Chu

The problem of determining closed-form evaluations for series as in

$$\sum_{k=1}^{\infty} \frac{\binom{2k}{k}^2 O_k^2}{16^k (2k-1)} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k}^2 O_k^2}{16^k (2k-1)^2} \tag{14}$$

was recently given as an open problem in [48]. Interestingly, our hypergeometric identity highlighted as Theorem 2.1 may be applied in relation to this problem. Our explorations of applications of this theorem have led us to solve a number of open problems from [48].

We apply $\frac{\partial}{\partial n} \cdot \big|_{n=\frac{1}{2}}$ to both sides of the hypergeometric identity (9), and then apply $\frac{\partial}{\partial b} \cdot \big|_{b=\frac{1}{2}}$ and then set $c = 2$. After much simplification and rearrangement using closed forms from [48] and from references therein, we can prove the below linear relation involving the series in (14). We are letting $O_k = 1 + \frac{1}{3} + \cdots + \frac{1}{2k-1}$ denote the k^{th} odd harmonic number, and we let G denote Catalan's constant. Also, we let $\mathcal{G} = \Im\left(\text{Li}_3\left(\frac{1+i}{2}\right)\right)$ denote the Catalan-like constant under consideration in [8]. The aforementioned linear relation is such that

$$4\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2 O_k}{16^k (2k+1)} - \sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2 O_k^2}{16^k (2k-1)} - 2\sum_{k=1}^{\infty} \frac{\binom{2k}{k}^2 O_k^2}{16^k (2k-1)^2} \tag{15}$$

evaluates as

$$\frac{12G}{\pi} + \frac{32\mathcal{G}}{\pi} - \frac{6\ln^2(2)}{\pi} - \frac{3\pi^2}{4} - \frac{\pi}{4} - \ln^2(2).$$

Letting $H_\ell = 1 + \frac{1}{2} + \cdots + \frac{1}{\ell}$ denote the ℓ^{th} harmonic number, consider rewriting the summand of the first series in (15) according to the relation $O_k = H_{2k} - \frac{1}{2}H_k$, i.e., so that

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2 O_k}{16^k (2k+1)} = \sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2 \left(H_{2k} - \frac{1}{2}H_k\right)}{16^k (2k+1)}. \tag{16}$$

A remarkable aspect about our above identity for (15) is due to how known techniques for evaluating series containing

$$\frac{\binom{2k}{k}^2 H_{2k}}{16^k}$$

or

$$\frac{\binom{2k}{k}^2 H_k}{16^k}$$

as in [6, 7, 47, 48] cannot be applied to (16).

Quite recently, using coefficient-extraction techniques, Wang and Chu [48] introduced the remarkable evaluation

$$\sum_{k=1}^{\infty} \frac{\binom{2k}{k}^2 \left(O_k^2 - O_k^{(2)}\right)}{16^k (2k-1)} = \frac{\pi}{6} - \frac{2(\ln 2)^2}{\pi}, \tag{17}$$

inspired by the past work in [6, 47] on Ramanujan-like rational series for $\frac{1}{\pi}$ involving harmonic numbers, writing

$$O_k^{(2)} = 1 + \frac{1}{3^2} + \cdots + \frac{1}{(2k-1)^2}.$$

It is stated explicitly in [48] that the authors of [48] had failed in the problem of separately evaluating the series

$$\sum_{k=1}^{\infty} \frac{\binom{2k}{k}^2 O_k^2}{16^k (2k-1)} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k}^2 O_k^{(2)}}{16^k (2k-1)} \tag{18}$$

obtained by expanding the summand in (17) according to the summand factor $\left(O_k^2 - O_k^{(2)}\right)$. The problem of evaluating the initial series in (18) is relevant to our work concerning the WZ-derived hypergeometric identity in Theorem 2.1, in view of how we applied this Theorem to evaluate (15). As it turns out, the formula

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2 O_k^{(2)}}{16^k (2k-1)} = \frac{4G}{\pi} - \frac{\pi}{4}$$

was very recently proved by Nimbran et al. in [35], by applying integral operators to the Maclaurin series for $\arcsin^3 x$, so this provides a solution to Wang and Chu's problem on (18). We provide, as below, a completely different way of solving this problem. Our reindexing argument, as applied below, may be generalized to solve a number of other open problems from [48], as we later consider.

Theorem 2.3. *The evaluation*

$$\sum_{k=1}^{\infty} \frac{\binom{2k}{k}^2 O_k^2}{16^k(2k-1)} = \frac{4G}{\pi} - \frac{\pi}{12} - \frac{2\ln^2(2)}{\pi} \tag{19}$$

holds true.

Proof. We apply the series reindexings/rearrangements indicated below:

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{\left(\frac{1}{16}\right)^k \binom{2k}{k}^2 O_k^2}{2k-1} &= \sum_{k=0}^{\infty} \frac{\left(\frac{1}{16}\right)^{k+1} \binom{2(k+1)}{k+1}^2 O_{k+1}^2}{2(k+1)-1} \\
&= 4\sum_{k=0}^{\infty} \frac{\left(\frac{1}{16}\right)^{k+1} (2k+1)^2 \binom{2k}{k}^2 O_{k+1}^2}{(k+1)^2(2(k+1)-1)} \\
&= \frac{1}{4}\sum_{k=0}^{\infty} \frac{\left(\frac{1}{16}\right)^{k} \binom{2k}{k}^2 (2k+1) O_{k+1}^2}{(k+1)^2} \\
&= \frac{1}{4}\sum_{k=0}^{\infty} \frac{\left(\frac{1}{16}\right)^{k} \binom{2k}{k}^2 (2k+1) \left(O_k + \frac{1}{2k+1}\right)^2}{(k+1)^2} \\
&= \frac{1}{4}\sum_{k=0}^{\infty} \left(\frac{1}{16}\right)^{k} \binom{2k}{k}^2 \left(-\frac{1}{(k+1)^2} - \frac{2}{k+1} + \frac{4}{2k+1} + \frac{2O_k}{(k+1)^2} - \frac{O_k^2}{(k+1)^2} + \frac{2O_k^2}{k+1}\right).
\end{aligned}$$

Using previously known closed forms as in [6,7,48], we obtain that the desired result holds true. □

So, we immediately obtain a closed form for the latter series in (18), from the formulas in (17) and Theorem 2.3. Mimicking our proof of Theorem 2.3, we may obtain infinite families of closed forms for the series obtained by replacing the summand factors $\frac{1}{2k-1}$ in (18) with $\frac{1}{2k-(2z-1)}$ for a natural number z, solving open problems given by Wang and Chu in [48]. For example, the evaluation

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 \left(O_n^2 - O_n^{(2)}\right)}{16^n(2n-3)} = \frac{5\pi}{54} + \frac{8}{45\pi} - \frac{2(2-5\ln 2)^2}{45\pi}$$

is proved via the coefficient-extraction techniques in [48], and it is stated explicitly that the authors of [48] had failed to separately evaluate the series obtained by expanding the above summand according to the factor $\left(O_n^2 - O_n^{(2)}\right)$; mimicking our recursive argument used in Theorem 2.3, we can show that

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 O_n^2}{16^n(2n-3)} = \frac{16G}{9\pi} - \frac{10\ln^2(2)}{9\pi} + \frac{8\ln(2)}{9\pi} - \frac{2}{9\pi} - \frac{5\pi}{108},$$

solving an open problem from [48].

The problem of proving closed forms for

$$\sum_{k=1}^{\infty} \frac{\binom{2k}{k}^2 O_k^2}{16^k(2k-(2z-1))^2}$$

for $z \in \mathbb{N}$ is also left as an open problem in [48]. For the base case $z = 1$, we have shown, via our WZ-derived result in (15) together with our solution to the problem of evaluating the series shown in (19), that the open problem given in [48] of evaluating the series

$$\sum_{k=1}^{\infty} \frac{\binom{2k}{k}^2 O_k^2}{16^k(2k-1)^2}$$

is equivalent to the problem of evaluating

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2 O_k}{16^k(2k+1)}. \tag{20}$$

Series closely related to (20) are evaluated in [9], such as the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{16}\right)^n \binom{2n}{n}^2 \left(\frac{1}{2n} + \frac{1}{2n+1}\right) H_{2n},$$

and we encourage the exploration as to how the methods from [9] may be applied to evaluate (20); such an evaluation, as we have demonstrated, would solve an open problem from [48].

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