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Research Article

On the permanental polynomial and permanental sum of signed graphs

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Abstract

Let $\dot{G} = (G, \sigma)$ be a signed graph, where G is its underlying graph and σ is its sign function (defined on the edge set $E(G)$ of G). Let $A(\dot{G})$ be the adjacency matrix of $\dot{G}$. The polynomial $\pi(\dot{G}, x) = \text{per}(xI - A(\dot{G}))$ is called the permanental polynomial of $\dot{G}$, where I is the identity matrix and per denotes the permanent of a matrix. In this paper, we obtain the coefficients of the permanental polynomial of a signed graph in terms of its structure. We also establish the recursion formulas for the permanental polynomial of a signed graph. Moreover, we investigate the permanental sum $PS(\dot{G})$ of a signed graph $\dot{G}$, give the recursion formulas for the permanental sum $PS(\dot{G})$, and show that the equation $PS(\dot{G}) = PS(G)$ holds for trees and unicyclic graphs, where $PS(G)$ is the permanental sum of the underlying graph G of $\dot{G}$.

Keywords: adjacency matrix; permanental polynomial; coefficient; recursion formulas; permanental sum.

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1. Introduction

The permanent of an $n \times n$ real matrix $A = (a_{ij})$, with $i, j \in \{1, 2, \ldots, n\}$, is defined as

$$\text{per}(A) = \sum_{\tau} \Pi_{i=1}^{n} a_{i\tau(i)}$$

where the sum is taken over all permutations τ of $\{1, 2, \ldots, n\}$. Let $A(G)$ be the adjacency matrix of a graph $G = (V, E)$ with n vertices. The polynomial

$$\Phi(G, x) = \det(xI - A(G)) = \sum_{k=0}^{n} a_k x^{n-k}, \tag{1}$$

is called the characteristic polynomial of G, where I is the $n \times n$ identity matrix. The polynomial

$$\pi(G, x) = \text{per}(xI - A(G)) = \sum_{k=0}^{n} b_k x^{n-k}, \tag{2}$$

is called the permanental polynomial of G [7]. The characteristic polynomial and the permanental polynomial are important among the well-studied graph polynomials. Valiant [9] has shown that computing the permanent of matrices is ♯P-complete even when restricted to $(0, 1)$-matrices. It is difficult to compute the permanental polynomials of graphs. Numerous works were done on the adjacency permanental polynomials of graphs, including the relations between the adjacency permanental and characteristic polynomials of graphs, the coefficients and roots of the adjacency permanental polynomial of a graph [1–3, 16–18]. It was shown that the coefficients of the characteristic and permanental polynomials of graphs are related to graphs' structures [3, 4, 7].

A linear graph (or a Sachs graph) is a graph in which each component is a single edge or a cycle. A linear subgraph of a graph G is termed as a subgraph whose components are cycles or single edges. A linear subgraph with k vertices is denoted by U_k. Then

$$a_k = \sum_{U_k \subseteq G} (-1)^{p(U_k)} 2^{c(U_k)} (1 \le k \le n), \tag{3}$$

and

$$b_k = (-1)^k \sum_{U_k \subseteq G} 2^{c(U_k)} (1 \le k \le n), \tag{4}$$

where the summations range over all linear subgraphs U_k of G, $p(U_k)$ is the number of components of U_k and $c(U_k)$ is the

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number of cycles of U_k. For a bipartite graph G, b_n is equal to the square of the number $m(G)$ of perfect matchings of G [6], i.e.

$$b_n = m^2(G).$$

Let $\sigma : E(G) \to \{+, -\}$ be a mapping defined on the edge set of G and $\dot{G} = (G, \sigma)$ a signed graph, where G is its underlying graph and σ is its sign function (or signature). Hence, signed graphs are sometimes treated as weighted graphs, whose (edge) weights belong to $\{1, -1\}$. An edge e is positive (negative) if $\sigma(e) = +$ (resp. $\sigma(e) = -$). If all edges in $\dot{G}$ are positive (negative), then $\dot{G}$ is denoted by $(G, +)$ (resp. $(G, -)$). A cycle of $\dot{G}$ is said to be balanced (or positive) if it contains an even number of negative edges, otherwise it is unbalanced (or, negative). A signed graph is said to be balanced if all its cycles are balanced; otherwise, it is unbalanced.

Let $\dot{G}$ be a signed graph on n vertices. The adjacency matrix of $\dot{G}$ is $A(\dot{G}) = (a^{\sigma}_{ij})$ with $a^{\sigma}_{ij} = \sigma(v_iv_j)a_{ij}$, where (a_{ij}) is the adjacency matrix of the underlying graph $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$. Then

$$\Phi(\dot{G}, x) = \det(xI - A(\dot{G})) = \sum_{k=0}^{n} c_k x^{n-k}, \tag{5}$$

is the characteristic polynomial [5] of $\dot{G}$, $c_0 = 1$ and

$$c_k = \sum_{U_k \subseteq \dot{G}} (-1)^{p(U_k)+s(U_k)} 2^{c(U_k)} = \sum_{U_k \subseteq \dot{G}} (-1)^{p(U_k)+c^-(U_k)} 2^{c(U_k)} (1 \le k \le n), \tag{6}$$

for any $k(1 \le k \le n)$, where $s(U_k)$ is the number of negative edges in cycles of U_k, $p(U_k)$ is the number of components of U_k, $c(U_k)$ is the number of cycles of U_k and $c^-(U_k)$ is the number of negative cycles of U_k.

Similarly, we introduce the permanental polynomial of signed graph $\dot{G}$ defined as

$$\pi(\dot{G}, x) = \mathrm{per}(xI - A(\dot{G})) = \sum_{k=0}^{n} s_k x^{n-k} \tag{7}$$

and its permanental sum $PS(\dot{G})$ as

$$PS(\dot{G}) = \sum_{k=0}^{n} |s_k|.$$

In this paper, we investigate the permanental polynomial of a signed graph and obtain its coefficients in terms of the graph structure, and establish the recursion formulas for the permanental polynomial and the permanental sum $PS(\dot{G})$ of signed graph $\dot{G}$, and show that $PS(\dot{G}) = PS(G)$ for trees and unicyclic graphs, where $PS(G)$ is the permanental sum of its underlying graph G introduced in [12].

2. The coefficients of the permanental polynomial of a signed graph

In this section, we give a graphical interpretation of the coefficients of the permanental polynomial of a signed graph using its linear subgraphs.

Let A be an $m \times n$ matrix. If $S \subseteq \{1, \ldots, m\}$ and $T \subseteq \{1, \ldots, n\}$, then $A[S|T]$ denotes the submatrix of A determined by the rows corresponding to S and the columns corresponding to T.

Lemma 2.1 (see [8]). *If A is an $n \times n$ matrix and $\mathrm{per}(xI - A) = \sum_{k=0}^{n} b_k x^{n-k}$, then $b_0 = 1$ and $b_k = (-1)^k \sum_{|T|=k} \mathrm{per}(A[T|T])$ for $1 \le k \le n$.*

Theorem 2.1. *Let $\dot{G}$ be a signed graph with n vertices and $\pi(\dot{G}, x) = \sum_{k=1}^{n} s_k x^{n-k}$ be its permanental polynomial. Then $s_0 = 1$ and*

$$s_k = \sum_{U_k \subseteq \dot{G}} (-1)^{k+c^-(U_k)} 2^{c(U_k)} \tag{8}$$

for $1 \le k \le n$, where $c^-(U_k)$ is the number of negative cycles of U_k and $c(U_k)$ is the number of cycles of U_k.

Proof. Let $\dot{A} = A(\dot{G})$ be the adjacency matrix of $\dot{G}$. Clearly, $s_0 = 1$. By Lemma 2.1, we have $s_k = (-1)^k \sum_{|T|=k} \mathrm{per}(\dot{A}[T|T])$ for $1 \le k \le n$. We first prove that

$$s_n = (-1)^n \mathrm{per}(\dot{A}) = \sum_{U_n \subseteq \dot{G}} (-1)^{n+c^-(U_n)} 2^{c(U_n)}.$$

Let $\pi = \pi_1\pi_2 \ldots \pi_n$ be a permutation of $\{1, 2, \ldots, n\}$. Then $\mathrm{per}(\dot{A}) = \sum_{\pi} \Pi_{i=1}^{n} a_{i,\pi_i}$, where the sum is taken over all permutations π of $\{1, 2, \ldots, n\}$. The term vanishes if $a_{i,\pi_i} = 0$, i.e., $\{v_i, v_{\pi_i}\}$ is not an edge of $\dot{G}$ for some $i \in \{1, 2, \ldots, n\}$.

Thus, if the term corresponding to a permutation π is non-zero, then π can be expressed uniquely as the composition of disjoint cycles or some K_2. Each K_2 corresponds to the factors $a_{i,j}a_{j,i}$, which in turn signifies a single edge v_iv_j in $\dot{G}$. Each cycle $(pqr\dots t)$ of length greater than two corresponds to the factors $a_{pq}a_{qr}\dots a_{tp}$, and signifies a simple circuit $v_p, v_q, \dots, v_t$ in $\dot{G}$. Consequently, each non-vanishing term in the permanent expansion gives rise to a linear subgraph U_n of G with $V(S) = V(G)$. The number of such π's – arising from a given U_n is $2^{c(U_n)}$, since there are two ways of choosing the corresponding cycle in π for each circuit in U_n. Thus, each U_n contributes $(-1)^{c^-(U_n)}2^{c(U_n)}$ to the permanent of $\dot{A}$, and we have that

$$s_n = (-1)^n\mathrm{per}(\dot{A}) = \sum_{U_n\subseteq \dot{G}} (-1)^{n+c^-(U_n)}2^{c(U_n)}.$$

Similarly, we can deduce

$$s_k = \sum_{U_k\subseteq \dot{G}} (-1)^{k+c^-(U_k)}2^{c(U_k)}$$

for $1 \le k \le n-1$. □

Theorem 2.2. *For a graph G on n vertices, there is a signed graph $\dot{G}$ of G with $|s_k| = |b_k|$ if and only if the number of negative cycles in any linear subgraph with k vertices has the same parity.*

Proof. First, $b_k = 0$ if and only if no linear subgraph U_k with k vertices exists in G, which is the case if and only if no linear subgraph U_k with k vertices exists in $\dot{G}$. Thus, it is trivial for the case $b_k = 0$. Next, we consider the case $b_k > 0$.

If the number of negative cycles in any linear subgraph U_k with k vertices has the same parity, then $(-1)^{c^-(U_k)} = 1$ (resp. -1) for all U_k. So, we have

$$|s_k| = |\sum_{U_k\subseteq \dot{G}} (-1)^{k+c^-(U_k)}2^{c(U_k)}| = \sum_{U_k\subseteq \dot{G}} 2^{c(U_k)} = \sum_{U_k\subseteq G} 2^{c(U_k)}$$

and

$$|b_k| = |(-1)^k \sum_{U_k\subseteq G} 2^{c(U_k)}| = \sum_{U_k\subseteq G} 2^{c(U_k)}.$$

So we have $|s_k| = |b_k|$.

On the other hand, if $|s_k| = |b_k|$, then all the linear subgraphs U_k satisfy

$$\sum_{U_k\subseteq G} 2^{c(U_k)} = \sum_{U_k\subseteq \dot{G}} 2^{c(U_k)} = \sum_{U_k\subseteq \dot{G}} (-1)^{c^-(U_k)}2^{c(U_k)}$$

or

$$\sum_{U_k\subseteq G} 2^{c(U_k)} = \sum_{U_k\subseteq \dot{G}} 2^{c(U_k)} = -\sum_{U_k\subseteq \dot{G}} (-1)^{c^-(U_k)}2^{c(U_k)}.$$

This means that the number of negative cycles in any linear subgraph U_k with k vertices has the same parity. □

By the definition of balanced signed graph and Theorem 2.2, we can get the following results.

Corollary 2.1. *Let a graph G on n vertices and its signed graph $\dot{G}$ is a balanced signed graph. Then $s_k = b_k$, $1 \le k \le n$.*

Corollary 2.2. *Let $\dot{T}$ be any signed graph of a tree T. Then $\mathrm{per}(xI - A(\dot{T})) = \mathrm{per}(xI - A(T))$.*

Let G be a bipartite graph with n vertices. We know $b_n = m^2(G)$ in [6]. From Theorem 2.2, we can obtain

Corollary 2.3. *For a bipartite graph G on n vertices, there is a signed graph $\dot{G}$ of G with $|s_n| = m^2(G)$ if and only if the number of negative cycles in any linear subgraph with n vertices has same parity.*

Corollary 2.4. *There is a signed graph $\dot{G}$ of the bipartite perfect matching n order graph G with $s_n = m^2(G)$ if and only if the number of negative cycles in any the linear subgraph with n vertices is even.*

Proof. Let $n = 2p$. Since G has a perfect matching, pK_2 is a linear subgraph U_n of $\dot{G}$ and the number of negative cycles of pK_2 is zero. By Corollary 2.3, the result holds. □

We know that the perfect matching of a tree is unique if it exists.

Corollary 2.5. *Any signed tree $\dot{T}$ of the perfect matching n order tree T with $s_n = \mathrm{per}(A(\dot{T})) = \mathrm{per}(A(T)) = 1$.*

3. Recursion formulas for the permanental polynomial of a signed graph

The relationships between the permanental polynomial of a signed graph and the ones of its subgraphs are discussed in this section. Let G be a graph with a vertex subset $S \subseteq V(G)$. Denote by $G - S$ the graph obtained by deleting all the vertices in S from G together with all edges incident with S. In particular, if $S = \{v\}$ with $v \in V(G)$, we write $G - \{v\}$ simply by $G - v$.

Theorem 3.1. *Let $e = (u, v)$ be an edge of a signed graph $\dot{G}$ and $\mathcal{C}_e(\dot{G})$ the set of cycles in $\dot{G}$ containing e. Then*

$$\pi(\dot{G}, x) = \pi(\dot{G} - e, x) + \pi(\dot{G} - u - v, x) + 2\Sigma_{C \in \mathcal{C}_e(\dot{G})}(-1)^{|V(C)|+r(C)}\pi(\dot{G} - V(C), x), \tag{9}$$

where $r(C) = 1$ when C is negative in $\dot{G}$, and $r(C) = 0$ when C is positive.

Proof. Let $\pi(\dot{G}, x) = \sum_{k=0}^{n} s_k x^{n-k}$. By Theorem 2.1, the coefficient s_k can be expressed in terms of the linear subgraphs of $\dot{G}$. We show that if a linear subgraph U_k contributes to s_k on the left side of (9), then there is a linear subgraph that contributes a corresponding amount to one of the terms on the right. Suppose that U_k contributes m to the coefficient s_k of x^{n-k} on the left. We consider the following cases of U_k.

Case 1: $e \notin U_k$. Let $W_k = U_k$, then W_k is a linear subgraph of $\dot{G} - e$ and it contributes m to the coefficient of x^{n-k} in $\pi(\dot{G} - e, x)$.

Case 2: $e = uv$ is a component of U_k. Let $W_{k-2} = U_k - u - v$, then W_{k-2} is a linear subgraph of $\dot{G} - u - v$. By (8), W_{k-2} contributes

$$(-1)^{k-2+c^-(W_{k-2})}2^{c(W_{k-2})} = (-1)^{k+c^-(U_k)}2^{c(U_k)} = m$$

to the coefficient of x^{n-k} in $\pi(\dot{G} - u - v, x)$.

Case 3: $e = uv$ belongs to a cycle C of U_k. Then $W^e = U_k - V(C)$ is a linear subgraph of $\dot{G} - V(C)$. In this case, W^e contributes

$$(-1)^{k-|V(C)|+c^-(W^e)}2^{c(W^e)} = (-1)^{k-|V(C)|+c^-(W^e)}2^{c(U_k)-1}$$

to the coefficient of x^{n-k} in $\pi(\dot{G} - V(C), x)$. If C is negative in $\dot{G}$, then

$$(-1)^{k-|V(C)|+c^-(W^e)}2^{c(W^e)} = (-1)^{k-|V(C)|+c^-(U_k)-1}2^{c(U_k)-1} = \frac{1}{2}(-1)^{|V(C)|+1}m.$$

If C is positive in $\dot{G}$, then

$$(-1)^{k-|V(C)|+c^-(W^e)}2^{c(W^e)} = (-1)^{k-|V(C)|+c^-(U_k)}2^{c(U_k)-1} = \frac{1}{2}(-1)^{|V(C)|}m.$$

Thus, W^e contributes m to the coefficient of x^{n-k} in $2(-1)^{|V(C)|+r(C)}\pi(\dot{G} - V(C), x)$, where $r(C) = 1$ when C is negative in $\dot{G}$, and $r(C) = 0$ when C is positive.

By all the above cases, the result holds. □

Similarly, we can get the following result.

Theorem 3.2. *Let v be a vertex of a signed graph $\dot{G}$ and $\mathcal{C}_v(\dot{G})$ the set of cycles in $\dot{G}$ containing v. Then*

$$\pi(\dot{G}, x) = x\pi(\dot{G} - v, x) + \Sigma_{u \sim v}\pi(\dot{G} - u - v, x) + 2\Sigma_{C \in \mathcal{C}_v(\dot{G})}(-1)^{|V(C)|+r(C)}\pi(\dot{G} - V(C), x), \tag{10}$$

where $r(C) = 1$ when C is negative in $\dot{G}$, and $r(C) = 0$ when C is positive.

4. Permanental sum $PS(\dot{G})$ of a signed graph

Some works were done on the permanental sum of graphs [10–15]. The permanental sum $PS(\dot{G})$ of signed graphs is discussed in this section. By definition of the permanental sum $PS(\dot{G})$ of a signed graph $\dot{G}$ and Theorem 2.1, we have

$$PS(\dot{G}) = \sum_{k=0}^{n} |s_k| = \sum_{k=0}^{n} |\sum_{U_k \subseteq \dot{G}} (-1)^{k+c^-(U_k)}2^{c(U_k)}| = \sum_{k=0}^{n} |\sum_{U_k \subseteq \dot{G}} (-1)^{c^-(U_k)}2^{c(U_k)}|,$$

and

$$PS(\dot{G}) = \mathrm{per}(I + A(\dot{G})).$$

By Theorem 2.2 and definition of the permanental sum $PS(\dot{G})$ of a signed graph $\dot{G}$, we have the next result.

Theorem 4.1. *Let $\dot{G}$ be a signed graph. If the number of negative cycles in any linear subgraph U_k with k vertices has same parity for $1 \le k \le n$, then $PS(\dot{G}) = PS(G)$.*

In [12], Wu and Lai showed that the permanental sum of a graph satisfies the following identities.

Lemma 4.1 (see [12]). *(i) Let G and H be two vertex disjoint connected graphs. Then*

$$PS(G \cup H) = PS(G)PS(H).$$

(ii) Let $e = uv$ be an edge of graph G. Then

$$PS(G) = PS(G-e) + PS(G-v-u) + 2\sum_{C \in \mathcal{C}_e(G)} PS(G - V(C)).$$

(iii) Let v be a vertex of graph G. Then

$$PS(G) = PS(G-v) + \sum_{u \in N_G(v)} PS(G-v-u) + 2\sum_{C \in \mathcal{C}_v(G)} PS(G - V(C)).$$

Next, we will consider that the recursion formulas for the permanental sum $PS(\dot{G})$ of signed graph $\dot{G}$.

Theorem 4.2. *(i) Let $\dot{G}$ and $\dot{H}$ be two vertex disjoint connected balanced signed graphs. Then*

$$PS(\dot{G} \cup \dot{H}) = PS(\dot{G})PS(\dot{H}).$$

(ii) Let $e = uv$ be an edge of signed graph $\dot{G}$. Then

$$PS(\dot{G}) = PS(\dot{G}-e) + PS(\dot{G}-v-u) + 2\sum_{C \in \mathcal{C}_e(\dot{G}))} (-1)^{r(C)} PS(\dot{G} - V(C)).$$

(iii) Let v be a vertex of signed graph $\dot{G}$. Then

$$PS(\dot{G}) = PS(\dot{G}-v) + \sum_{u \in N_{\dot{G}}(v)} PS(\dot{G}-v-u) + 2\sum_{C \in \mathcal{C}_v(\dot{G})} (-1)^{r(C)} PS(\dot{G} - V(C)).$$

where $r(C) = 1$ when C is negative in $\dot{G}$, and $r(C) = 0$ when C is positive.

Proof. (i) Each linear subgraph with k vertices in $\dot{G} \cup \dot{H}$ consists of a linear subgraph with t in $\dot{G}$ together with a linear subgraph with $k-t$ vertices in $\dot{H}$, where $0 \le t \le k$. Since $\dot{G}$ and $\dot{H}$ are connected balanced signed graphs, by Theorem 2.1, we have

$$PS(\dot{G} \cup \dot{H}) = \sum_{k=0}^{n} |s_k(\dot{G} \cup \dot{H})| = \sum_{k=0}^{n}\sum_{t=0}^{k} |s_t(\dot{G}) s_{k-t}(\dot{H})| = PS(\dot{G})PS(\dot{H}).$$

Now, let L_i denote the collection of all linear subgraphs of $\dot{G}$ with i vertices.

(ii) Let $e = uv \in E(G)$ be a given edge, and $L_i'(G,e) = \{H \in L_i : e \in E(H)\}$ and $L_i''(G,e) = \{H \in L_i : e \notin E(H)\}$. Then, $|L_i''(G,e)| = |s_i(G-uv)|$. For each $H \in L_i'(G,e)$, either $e = uv$ itself is a component of H, or e lies in a cycle of H. It follows that $|\{H \in L_i'(G,e) : e \text{ is a component of } H\}| = |s_{i-2}(G-u-v)|$, and

$$|\{H \in L_i'(G,e) : e \text{ lies in a cycle of } H\}| = 2\sum_{k=0}^{i} \left| \sum_{C_k \in \mathcal{C}_e(\dot{G}))} (-1)^{r(C_k)} s_{i-k}(G - V(C_k)) \right|.$$

Thus,

$$|s_i(\dot{G})| = |s_i(\dot{G}-uv)| + |s_{i-2}(\dot{G}-u-v)| + 2\sum_{k=0}^{i} \left| \sum_{C_k \in \mathcal{C}_e(\dot{G}))} (-1)^{r(C_k)} s_{i-k}(\dot{G} - V(C_k)) \right|.$$

for all positive integers i. It follows that

$$\begin{aligned} PS(\dot{G}) &= \sum_{k=0}^{n} |s_i(\dot{G})| \\ &= \sum_{k=0}^{n} (|s_i(\dot{G}-uv)| + |s_{i-2}(\dot{G}-u-v)| + 2\sum_{k=0}^{i} |\sum_{C_k \in \mathcal{C}_e(\dot{G}))} (-1)^{r(C_k)} s_{i-k}(\dot{G} - V(C_k))|) \end{aligned}$$

$$= PS(\dot{G}-e) + PS(\dot{G}-v-u) + 2\sum_{C\in\mathcal{C}_e(\dot{G}))}(-1)^{r(C)}PS(\dot{G}-V(C)).$$

(iii) Let $v \in V(G)$ be a given vertex, and $L'_i(G,v) = \{H \in L_i : v \in E(H)\}$ and $L''_i(G,v) = \{H \in L_i : v \notin V(H)\}$. Then, $|L''_i(G,v)| = |s_i(G-v)|$. For each $H \in L_i(G,v)$, either v is an endpoint of some single, or v lies in a cycle of H. It follows that $|\{H \in L'_i(G,v) : v \text{ is an endpoint of some single } e = uv \text{ of } H\}| = |s_{i-2}(G-u-v)|$, and

$$|\{H \in L'_i(G,v) : v \text{ lies in a cycle of } H\}| = 2\sum_{k=0}^{i}\left|\sum_{C_k\in\mathcal{C}_v(\dot{G}))}(-1)^{r(C_k)}s_{i-k}(G-V(C_k))\right|.$$

Thus,

$$|s_i(\dot{G})| = |s_i(\dot{G}-v)| + |s_{i-2}(\dot{G}-u-v)| + 2\sum_{k=0}^{i}\left|\sum_{C_k\in\mathcal{C}_v(\dot{G}))}(-1)^{r(C_k)}s_{i-k}(\dot{G}-V(C_k))\right|.$$

Substituting this into the definition of $PS(\dot{G})$ yields

$$PS(\dot{G}) = PS(\dot{G}-v) + \sum_{u\in N_{\dot{G}}(v)} PS(\dot{G}-v-u) + 2\sum_{C\in\mathcal{C}_v(\dot{G})}(-1)^{r(C)}PS(\dot{G}-V(C)).$$

□

From Theorem 4.2 the next two corollaries immediately follows.

Corollary 4.1. *(i) Let $e = uv$ be an edge of signed graph $\dot{G}$. If all cycles containing e are positive (or negative), then*

$$PS(\dot{G}) = PS(\dot{G}-e) + PS(\dot{G}-v-u) + 2\sum_{C\in\mathcal{C}_e(\dot{G}))} PS(\dot{G}-V(C)).$$

(ii) Let v be a vertex of signed graph $\dot{G}$. If all cycles containing v are positive (or negative), then

$$PS(\dot{G}) = PS(\dot{G}-v) + \sum_{u\in N_{\dot{G}(v)}} PS(\dot{G}-v-u) + 2\sum_{C\in\mathcal{C}_v(\dot{G})} PS(\dot{G}-V(C)).$$

Corollary 4.2. *(i) Let $\dot{G}$ be a signed graph and e an edge of $\dot{G}$. Then $PS(\dot{G}-e) < PS(\dot{G})$.*

(ii) Among all signed graphs with n vertices, the graph nK_1 and the balance complete signed graph K_n have, respectively, minimum and maximum permanental sum.

We know $PS(K_2) = PS(\dot{K}_2) = 2$. From Theorem 4.2 the next result immediately follows.

Corollary 4.3. *If G is a tree or a unicyclic graph, then $PS(G) = PS(\dot{G})$ for any signed graph $\dot{G}$ of G.*

Wu and Lai [12] determine the largest and smallest permanental sums among unicyclic graphs (trees). Denote by $F(n)$ the n^{th} Fibonacci number.

Lemma 4.2 (see [12]). *(i) Let G be a tree with n vertex, then*

$$n \le PS(G) \le F(n+1),$$

where the left equality holds if and only if $T = S_n$, and the right equality holds if and only if $T = P_n$.

(ii) Let G be a unicyclic graph with n vertex, then

$$2n \le PS(G) \le 6F(n-2) + 2F(n-3),$$

where the left equality holds if and only if $T = S_n^+$ (the graph obtained by adding a new edge to the star S_n), and the right equality holds if and only if $T = D_{3,n-3}$ (the graph obtained from the disjoint union of a cycle C_3 and a path P_{n-3} by identifying one end of P_{n-3} with one of the vertices of C_3).

By Corollary 4.3 and Lemma 4.2, we have

Corollary 4.4. *(i) Let G be a tree with n vertex and $\dot{G}$ a signed graph of G, then*

$$n \le PS(\dot{G}) \le F(n+1),$$

where the left equality holds if and only if $G = S_n$, and the right equality holds if and only if $G = P_n$.

(ii) Let G be a unicyclic graph with n vertex and $\dot{G}$ a signed graph of G, then

$$2n \le PS(\dot{G}) \le 6F(n-2) + 2F(n-3),$$

where the left equality holds if and only if $G = S_n^+$, and the right equality holds if and only if $G = D_{3,n-3}$.

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