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Research Article

A new proof of Boole's additive combinatorics formula

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Abstract

The Boole's additive combinatorics formula is given by

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^m = \begin{cases} 0 & \text{if} \quad m<n, \\ n! & \text{if} \quad m=n. \end{cases}$$

A new proof of this formula is presented in this paper.

Keywords: combinatorial identities; Boole's formula; factorial function.

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1. Introduction

Binomial coefficients are related to statistical and probabilistic aspects of mathematics very intently and are considered very significant. The binomial coefficient $\binom{n}{k}$ is often described as picking k event without order out of n possibilities.The binomial coefficient $\binom{n}{k}$ is defined as

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!} & \text{if} \quad n \geq k, \\ 0 & \text{if} \quad n<k. \end{cases}$$

The formula

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^m = \begin{cases} 0 & \text{if} \quad m<n, \\ n! & \text{if} \quad m=n, \end{cases} \tag{1}$$

is known as Boole's formula becasue it shows up in Boole's book [5]. On the other hand, the binomial studies have always been important in mathematics since Euler's era. As being one of the earliest sources, Gould debates the binomial identity (1) in [8] and named this identity as *Euler's formula*. Binomial coefficients still gather interest and new proofs of this identity have been studies in the recent decades; for example, Anglani and Barile [2] gave two new different proofs of the identity in 2015. Phoata [10] used Lagrange's interpolation polynomials to obtain an extension of (1) as given below

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}P(a+kb) = a_0 b^n \cdot n! \quad (n \in \mathbb{N}), \tag{2}$$

where $a, b \in \mathbb{R}$, b is nonzero, and P is an nth-degree polynomial with a_0 being the leading coefficient. Katsuro [9] proved this extension formula for real and complex numbers as given below

$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}(xk+y)^m = \begin{cases} 0 & \text{if} \quad 0 \leq m<n, \\ (-1)^n x^n n! & \text{if} \quad m=n, \end{cases}$$

where $m, n \in \mathbb{Z}^+$ and x and y can be real and complex numbers. Note that Katsuuro presented a spacial case of Phoata's formula. Although Phoata's and Katsuuro's results seem to be new but Gould presented a more general form of these formulas in his book [7]. Gould demonstrated that for any mth-degree polynomial $f(t) = c_0 + c_1 t + \dots + c_m t^m$, the identity

$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}f(k) = \begin{cases} 0 & \text{if} \quad m<n, \\ (-1)^n n! c_n & \text{if} \quad m=n, \end{cases}$$

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holds. All these results are actually natural consequences of identity (1). Also, for $g(x) = c_0 + c_1x + c_2x^2 + ... + c_nx^n$ we have

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\binom{n}{k}g(k) &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{j=0}^{n}b_jk^j = \sum_{j=0}^{n}b_j\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^j \\ &= b_n\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^n + \sum_{j=0}^{n-1}b_j\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^j \\ &= (-1)^n n!b_n + 0 = (-1)^n n!b_n.\end{aligned}$$

Alzer and Chapman [1] gave a new proof of (1). Recently, Batir [3] provided another proof of (1). Readers may consult [4,6] for different arguments. The identity (1) is closely associated with Stirling's partition numbers denoted by $S(m,n)$ which satisfies

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^m = n!S(m,n).$$

Our aim in this paper is to provide another new proof of (1). In our proof, we use only differentiation, integration, and the following identities involving binomial coefficients:

$$\binom{n+1}{k} = \frac{n+1}{k}\binom{n}{k-1}, \tag{3}$$

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}. \tag{4}$$

2. Main result

The main result of this paper is the following theorem.

Theorem 2.1. *If $m,n \in \mathbb{N}$, then*

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^m = \begin{cases} 0 & \text{if} \quad m < n, \\ n! & \text{if} \quad m = n. \end{cases}$$

Proof. For $x \in \mathbb{R}$, define

$$f_n(x) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}(x-k)^n.$$

Differentiation gives

$$f_n'(x) = n\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x-k)^{n-1}.$$

We want to show that $f_n'(x) = 0$ for all $x \in \mathbb{R}$. Clearly, $f_1'(x) = 0$. We assume that $f_n'(x) = 0$ for all $x \in \mathbb{R}$. Then, we find by utilizing (3) and (4) that

$$\begin{aligned}\frac{f_{n+1}'}{n+1} &= \sum_{k=1}^{n+1}(-1)^k\binom{n+1}{k}(x-k)^n + x^n = \sum_{k=1}^{n+1}(-1)^k\binom{n+1}{k}(x-k)(x-k)^{n-1} + x^n \\ &= x\sum_{k=1}^{n+1}(-1)^k\binom{n+1}{k}(x-k)^{n-1} - \sum_{k=1}^{n+1}(-1)^k\binom{n+1}{k}k(x-k)^{n-1} + x^n + x^n \\ &= x\sum_{k=1}^{n+1}(-1)^k\left[\binom{n}{k} + \binom{n}{k-1}\right](x-k)^{n-1} - (n+1)\sum_{k=1}^{n+1}(-1)^k\binom{n}{k-1}(x-k)^{n-1} + x^n.\end{aligned}$$

Making appropriate changes in dummy indices k, this can be rewritten in the following form

$$\begin{aligned}\frac{f_{n+1}'}{n+1} &= x\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x-k)^{n-1} + x\sum_{k=0}^{n}(-1)^{k-1}\binom{n}{k}(x-1-k)^{n-1} - (n+1)\sum_{k=0}^{n}(-1)^{k-1}\binom{n}{k}(x-1-k)^{n-1} \\ &= \frac{x}{n}f_n'(x) - \frac{x}{n}f_{n+1}'(x-1) + \frac{n+1}{n}f_n'(x-1) = 0,\end{aligned}$$

that is $f'_{n+1}(x) = 0$ for all $x \in \mathbb{R}$. This proves that $f'_n(x) = 0$ for all $x \in \mathbb{R}$. Therefore, f_n is a constant function. Let's define $a_n = f_n(x)$. Then, we have

$$(n+1)a_n = (n+1)\int_n^{n+1} f_n(x)dx = (n+1)\int_n^{n+1} \sum_{k=0}^{n}(-1)^k \binom{n}{k}(x-k)^n dx.$$

Inverting the order of integration and summation, and then utilizing (3) and (4), we get

$$\begin{aligned}(n+1)a_n &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k+1)^{n+1} - \sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)^{n+1}\\ &= \sum_{k=0}^{n}(-1)^k\left[\binom{n+1}{k} - \binom{n}{k-1}\right](n-k+1)^{n+1} - \sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)^{n+1}\\ &= \sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}(n-k+1)^{n+1} - \sum_{k=0}^{n}(-1)^k\binom{n}{k-1}(n-k+1)^{n+1} - \sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)^{n+1}\\ &= a_{n+1} + \sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)^{n+1} - \sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)^{n+1}\\ &= a_{n+1}.\end{aligned}$$

Since $a_1 = 1$, this implies that $a_n = n!$. Furthermore, since $f_n^{(k)}(x) = 0$ for all $m = 0, 1, 2, \cdots, n-1$, we have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x-k)^m = 0$$

for $m < n$. □

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