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Research Article

New recurrence relation for partitions into distinct parts

Teerapat Srichan*

Department of Mathematics, Faculty of Science, Kasetsart University, Bangkok, Thailand

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Abstract

Denote by Q_n the set of partitions of a positive integer n into distinct parts. For $k \in \mathbb{N}$, denote by $Q_{n,k}$ the set of partitions of n into distinct parts whose least part is $k+1$ and not equal to n. Let $q(n)$ and $q(n,k)$ be the number of elements in Q_n and $Q_{n,k}$, respectively. In this paper, several new recurrence relations for partitions into distinct parts are derived from the partition function $q(n,k)$.

Keywords: partition of integers; recursion formula; restricted partition function.

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1. Introduction and statements of the main results

A partition of a positive integer n is a finite sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_m$ (see [2, Chapter 14]) such that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_m = n.$$

The λ_i's are called the parts of the partition. The number of parts is unrestricted, repetition is allowed and the order of the parts is not taken into account. The corresponding partition function $p(n)$ is defined as the number of unordered partitions of n. The paper [4] gives a new recursion formula connecting the unrestricted partition function $p(n)$ with a restricted partition function $P(n,k)$, which defined by

$$P(n,k) = \begin{cases} \text{the number of partitions } (\lambda_1, \lambda_2, \ldots) \text{ of } n \text{ with } k < \lambda_i < n, & \text{if } 0 < k < n, \\ 0, & \text{otherwise.} \end{cases}$$

For $n \geq 4$ and $k \geq 1$, it was proved in [4] that

$$\sum_{i=0}^{\infty} a_{k,i}\, p(n-i) = P(n,k) - P(n-1,k),$$

where the coefficients $a_{k,i}$ are obtained from the expansion

$$\left(x^n - 2x^{n-1} + x^{n-2}\right) \prod_{i=2}^{k} \left(1 - x^{-i}\right) = \sum_{i=0}^{\infty} a_{k,i} x^{n-i},$$

where the empty product is taken to be 1. This provided several corollaries to get the value of $p(n)$. In the present paper, the same idea is used to find the recurrence formula for the numbers of partitions of n into distinct parts.

Denote by Q_n the set of partitions of n into distinct parts. For $k \in \mathbb{N}$, denote by $Q^*_{n,k}$ the set of partitions of n into distinct parts whose least part is $k+1$. Also, let $Q_{n,k}$ be the set of partitions of n into distinct parts whose least part is $k+1$ and not equal to n. Furthermore, let $q(n)$, $q^*(n,k)$, and $q(n,k)$ be the number of elements in Q_n , $Q^*_{n,k}$, and $Q_{n,k}$, respectively. The numbers of partitions of n into distinct parts is a classical problem in the theory of partitions; for example, see [1]. Erdős, Nicolas, and Szalay [3] gave an asymptotic relation of $q^*(n,k)$; namely, for all $n \geq 1$ and $0 \leq k \leq n-1$,

$$\frac{q(n)}{2^k} \leq q^*(n,k+1) \leq q\Big(n + \frac{k(k+1)}{2}\Big).$$

Moreover, at the end of the paper [3], the authors calculated the value of $q^*(n,k)$ by using the recurrence relation

$$q^*(n,k) = q^*(n,k+1) + q^*(n-k,k+1),$$

*E-mail address: fscitrp@ku.ac.th

and $q^*(n,k) = 1$ for $k \geq n/2 - 1$. Thus, finding a new recurrence formula for the numbers of partitions of n into distinct parts with a restriction is still interesting. Following theorems are proved in the present paper.

Theorem 1.1. *For any positive integers $n \geq 3$, one has*

$$q(n) = q(n,1) + q(n-1,1) + 2.$$

Since there are ten partitions of 10 into distinct parts: $10, 9+1, 8+2, 7+3, 6+4, 7+2+1, 6+3+1, 5+4+1, 5+3+2, 4+3+2+1$, one has

$$q(10) = q(10,1) + q(9,1) + 2 = 4 + 4 + 2 = 10,$$

which gives an illustration of Theorem 1.1.

Theorem 1.2. *For any positive integers $n \geq 3$ and $k \in \mathbb{N}$ with $k < \lfloor \frac{n-1}{2} \rfloor$, one has*

$$q(n,k) = q(n,k+1) + q(n-k-1,k+1) + 1.$$

Note that there are four partitions of 10 into distinct parts whose least part is 2 and not equal to 10:

$$8+2, 7+3, 6+4, 5+3+2.$$

Thus, as an illustration of Theorem 1.2, one has

$$q(10,1) = q(10,2) + q(8,2) + 1 = 2 + 1 + 1 = 4.$$

Theorem 1.3. *For any positive integers $n \geq 3$ and $l \leq n-4$, the following equation holds*

$$\sum_{i=0}^{l} (-1)^i q(n-i) = q(n,1) + (-1)^l q(n-l-1,1) + (-1)^l + 1.$$

Theorem 1.4. *For any positive integers $k, l \in \mathbb{N}$ and $n \geq 3$ satisfying $k < \lfloor \frac{n-1}{2} \rfloor$ and $l \leq \frac{n}{k+1} - 4$, one has*

$$\sum_{i=0}^{l} (-1)^i q(n-i(k+1),k) = q(n,k+1) + (-1)^l q(n-(l+1)(k+1),k+1) + \frac{(-1)^l + 1}{2}.$$

Theorem 1.5. *For any positive integers $n \geq 3$ and $d \leq \frac{n}{4}$, the following equation holds*

$$q(n) = q(n,d) + \sum_{j=1}^{d} q(n-j,j) + d + 1.$$

2. Proofs of Theorems 1.1, 1.2, 1.3, 1.4, and 1.5

Proof of Theorem 1.1. Let $Q_n^{(2)}$ and $Q_n^{(>2)}$ be the set of partitions of n into distinct parts with two parts and having at least three parts, respectively. Denote by $q^{(2)}(n)$ and $q^{(>2)}(n)$ the number of elements in $Q_n^{(2)}$ and $Q_n^{(>2)}$, respectively. Then, one has

$$Q_n = Q_n^{(2)} \cup Q_n^{(>2)} \cup \{n\}.$$

Thus, for $n \geq 3$, it holds that

$$q(n) = q^{(2)}(n) + q^{(>2)}(n) + 1. \tag{1}$$

Note that

$$Q_n^{(2)} = \left\{ (1,n-1), (2,n-2), \ldots, \left(\left\lfloor \frac{n-1}{2} \right\rfloor, n - \left\lfloor \frac{n-1}{2} \right\rfloor \right) \right\}.$$

Thus, for $n \geq 3$,

$$q^{(2)}(n) = \left\lfloor \frac{n-1}{2} \right\rfloor. \tag{2}$$

Also, note that

$$Q_n^{(>2)} = \bigcup_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} Q_{n-j,j}.$$

Since the sets $Q_{n-j,j}$ are disjoint for all $j = 1, \dots, \lfloor \frac{n-1}{2} \rfloor$, it holds that

$$q^{(>2)}(n) = \sum_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} q(n-j,j). \tag{3}$$

In view of (1)–(3), one has

$$q(n) = \left\lfloor \frac{n-1}{2} \right\rfloor + \sum_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} q(n-j,j) + 1. \tag{4}$$

Thus, Theorem 1.1 follows from (4) and following relation

$$q(n,1) = \left\lfloor \frac{n-3}{2} \right\rfloor + \sum_{j=2}^{\lfloor \frac{n-1}{2} \rfloor} q(n-j,j). \tag{5}$$

To prove (5), let $Q^{(2)}_{n,1}$ be the set of partitions of n into distinct parts whose least part is 2 and having two parts. Also, let $Q^{(>2)}_{n,1}$ be the set of partitions of n into distinct parts whose least part is 2 and having at least three parts. Denote by $q^{(2)}(n,1)$ and $q^{(>2)}(n,1)$ the number of elements in $Q^{(2)}_{n,1}$ and $Q^{(>2)}_{n,1}$, respectively. Then, one has

$$Q_{n,1} = Q^{(2)}_{n,1} \cup Q^{(>2)}_{n,1}.$$

Thus, for $n \geq 3$, it holds that

$$q(n,1) = q^{(2)}(n,1) + q^{(>2)}(n,1). \tag{6}$$

Also, note that

$$Q^{(2)}_{n,1} = \left\{ (2, n-2), \dots, \left(\left\lfloor \frac{n-1}{2} \right\rfloor, n - \left\lfloor \frac{n-1}{2} \right\rfloor \right) \right\}.$$

Thus, for $n \geq 3$, one has

$$q^{(2)}(n,1) = \left\lfloor \frac{n-3}{2} \right\rfloor. \tag{7}$$

Note that

$$Q^{(>2)}_{n,1} = \bigcup_{j=2}^{\lfloor \frac{n-1}{2} \rfloor} Q_{n-j,j}.$$

Since the sets $Q_{n-j,j}$ are disjoint for all $j = 2, \dots, \lfloor \frac{n-1}{2} \rfloor$, it holds that

$$q^{(>2)}(n,1) = \sum_{j=2}^{\lfloor \frac{n-1}{2} \rfloor} q(n-j,j). \tag{8}$$

Thus, (5) follows from (6)–(8). □

Proof of Theorem 1.2. Let $n \geq 3$ and $k \in \mathbb{N}$ with $k < \lfloor \frac{n-1}{2} \rfloor$. Theorem 1.2 can be proved by the reasoning similar to the one used in the proof of (5) by replacing 1 with k. Particularly, let $Q^{(2)}_{n,k}$ be the set of partitions of n into distinct parts whose least part is $k+1$ and having two parts. Denote by $Q^{(>2)}_{n,k}$ the set of partitions of n into distinct parts whose least part is $k+1$ and having at least three parts. Let $q^{(2)}(n,k)$ and $q^{(>2)}(n,k)$ be the number of elements in $Q^{(2)}_{n,k}$ and $Q^{(>2)}_{n,k}$, respectively. Then, one has

$$Q_{n,k} = Q^{(2)}_{n,k} \cup Q^{(>2)}_{n,k}.$$

For $n \geq 3$ and $k < \lfloor \frac{n-1}{2} \rfloor$, it holds that

$$q(n,k) = q^{(2)}(n,k) + q^{(>2)}(n,k). \tag{9}$$

Note that

$$Q^{(2)}_{n,k} = \left\{ (k+1, n-k-1), \dots, \left(\left\lfloor \frac{n-1}{2} \right\rfloor, n - \left\lfloor \frac{n-1}{2} \right\rfloor \right) \right\}.$$

Thus,

$$q^{(2)}(n,k) = \left\lfloor \frac{n-1}{2} \right\rfloor - k. \tag{10}$$

Also, note that

$$Q^{(>2)}_{n,k} = \bigcup_{j=k+1}^{\lfloor \frac{n-1}{2} \rfloor} Q_{n-j,j}.$$

Since the sets $Q_{n-j,j}$ are disjoint for all $j = k+1, \ldots, \lfloor \frac{n-1}{2} \rfloor$, it holds that

$$q^{(>2)}(n,k) = \sum_{j=k+1}^{\lfloor \frac{n-1}{2} \rfloor} q(n-j,j). \tag{11}$$

From (9)–(11), it follows that

$$q(n,k) = \left\lfloor \frac{n-1}{2} \right\rfloor - k + \sum_{j=k+1}^{\lfloor \frac{n-1}{2} \rfloor} q(n-j,j). \tag{12}$$

Replacing k by $k+1$ in (12) gives the desired result. □

Proof of Theorem 1.3. Let $n \geq 4$ and $l \leq n-4$. Using Theorem 1.1 iteratively in l-steps, one has

$$\begin{aligned} q(n) &= q(n,1) + q(n-1,1) + 2 \\ q(n-1) &= q(n-1,1) + q(n-2,1) + 2 \\ q(n-2) &= q(n-2,1) + q(n-3,1) + 2 \\ &\vdots \\ q(n-l-1) &= q(n-l-1,1) + q(n-l-2,1) + 2 \\ q(n-l) &= q(n-l,1) + q(n-l-1,1) + 2. \end{aligned}$$

By alternatively summing these equations, one gets the conclusion of Theorem 1.3 . □

Proof of Theorem 1.4. The proof is similar to the proof of Theorem 1.3. □

Proof of Theorem 1.5. It can be proved by the reasoning similar to the one used in the proof of Theorem 1.1. □

As an application of the obtained results, the values of the partition function $q(n,k)$ are given in the following table for $1 \leq n \leq 20$ and $1 \leq k \leq 8$:

The values of the partition function $q(n,k)$									
n	$q(n)$	$q(n,1)$	$q(n,2)$	$q(n,3)$	$q(n,4)$	$q(n,5)$	$q(n,6)$	$q(n,7)$	$q(n,8)$
1	1	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0
3	2	0	0	0	0	0	0	0	0
4	2	0	0	0	0	0	0	0	0
5	3	1	0	0	0	0	0	0	0
6	4	1	0	0	0	0	0	0	0
7	5	2	1	0	0	0	0	0	0
8	6	2	1	0	0	0	0	0	0
9	8	4	2	1	0	0	0	0	0
10	10	4	2	1	0	0	0	0	0
11	12	6	3	2	1	0	0	0	0
12	15	7	4	2	1	0	0	0	0
13	18	9	5	3	2	1	0	0	0
14	22	11	6	3	2	1	0	0	0
15	27	14	8	5	3	2	1	0	0
16	32	16	9	5	3	2	1	0	0
17	38	20	11	7	4	3	2	1	0
18	46	24	14	8	5	3	2	1	0
19	54	28	16	10	6	4	3	2	1
20	64	34	19	11	7	4	3	2	1

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