DML Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **3** (2020) 61–66

A note on a problem of Henning and Yeo about the transversal number of uniform linear systems whose 2-packing number is fixed

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(Received: 20 April 2020. Received in revised form: 29 May 2020. Accepted: 29 May 2020. Published online: 1 June 2020.)

Abstract

A linear system is a pair $(P,\mathcal{L})$ where $\mathcal{L}$ is a family of subsets of a ground finite set P such that $|l\cap l'|\leq 1$, for every pair of different sets $l,l'\in\mathcal{L}$. If every member of $\mathcal{L}$ has r elements, then the linear system $(P,\mathcal{L})$ is called r-uniform linear system. The transversal number $\tau(P,\mathcal{L})$ of a linear system $(P,\mathcal{L})$ is the minimum cardinality of a subset $\hat{P}\subseteq P$ satisfying $l\cap\hat{P}\neq\emptyset$, for every $l\in\mathcal{L}$. The 2-packing number $\nu_2(P,\mathcal{L})$ of a linear system $(P,\mathcal{L})$ is the maximum cardinality of a subset $R\subseteq\mathcal{L}$ such that every triplet of different elements of R do not have a common point. Henning and Yeo [*Discrete Math.* **313** (2013) 959–966] state the following question: is it true that if $(P,\mathcal{L})$ is an r-uniform linear system then $\tau(P,\mathcal{L})\leq(|P|+|\mathcal{L}|)/(r+1)$ holds for every $r\geq 2$? In this note, we prove that the mentioned inequality holds for several classes of r-uniform linear systems having a fixed 2-packing number.

Keywords: linear systems; 2-packing number; transversal number; finite projective plane.

2020 Mathematics Subject Classification: 05C65, 05C69.

1. Introduction

A linear system is a pair $(P,\mathcal{L})$ where $\mathcal{L}$ is a family of subsets of a ground finite set P such that $|l\cap l'|\leq 1$, for every pair of different sets $l,l'\in\mathcal{L}$. The linear system $(P,\mathcal{L})$ is *intersecting* if $|l\cap l'|=1$ for every pair of different sets $l,l'\in\mathcal{L}$. The elements of P and $\mathcal{L}$ are called *points* and *lines*, respectively. An r-*line* is a line containing exactly r points. An r-*uniform* linear system $(P,\mathcal{L})$ is a linear system such that all lines of $\mathcal{L}$ are r-lines. In this context, a *simple graph* is a 2-uniform linear system. Throughout this paper, we will consider linear systems of rank $r\geq 2$.

Let $(P,\mathcal{L})$ be a linear system and consider a point $p\in P$. The set of lines incident to p is denoted by $\mathcal{L}_p$. The *degree* of p is defined as $deg(p)=|\mathcal{L}_p|$ and the *maximum degree* over all points of the linear system $(P,\mathcal{L})$ is denoted by $\Delta=\Delta(P,\mathcal{L})$. Two points $p,q\in P$ are *adjacent* if there is a line $l\in\mathcal{L}$ such that $\{p,q\}\subseteq l$.

A *linear subsystem* $(P',\mathcal{L}')$ of a linear system $(P,\mathcal{L})$ is a linear system such that for every line $l'\in\mathcal{L}'$ there exists a line $l\in\mathcal{L}$ satisfying $l'=l\cap P'$. The *linear subsystem induced* by a set of lines $\mathcal{L}'\subseteq\mathcal{L}$ is the linear system $(P',\mathcal{L}')$, where $P'=\bigcup_{l\in\mathcal{L}'}l$. The linear subsystem $(P',\mathcal{L}')$ of $(P,\mathcal{L})$ is called *spanning linear subsystem* if $P'=P$. Given a linear system $(P,\mathcal{L})$ and a point $p\in P$, the linear system obtained from $(P,\mathcal{L})$ by *deleting point* p is the linear subsystem $(P',\mathcal{L}')$ induced by $\mathcal{L}'=\{l\setminus\{p\}:l\in\mathcal{L}\}$. On the other hand, given a linear system $(P,\mathcal{L})$ and a line $l\in\mathcal{L}$, the linear system obtained from $(P,\mathcal{L})$ by *deleting the line* l is the linear subsystem $(P',\mathcal{L}')$ induced by $\mathcal{L}'=\mathcal{L}\setminus\{l\}$. Finally, let $(P',\mathcal{L}')$ and $(P,\mathcal{L})$ be two linear systems. The linear systems $(P',\mathcal{L}')$ and $(P,\mathcal{L})$ are *isomorphic*, $(P',\mathcal{L}')\simeq(P,\mathcal{L})$, if after of deleting points of degree 1 or 0 from both, the linear systems $(P',\mathcal{L}')$ and $(P,\mathcal{L})$ are isomorphic as hypergraphs, see [5].

Let $(P,\mathcal{L})$ be a linear system. A subset of points T of P is a *transversal* of $(P,\mathcal{L})$ (also called *vertex cover* or *hitting set*) if $T\cap l\neq\emptyset$, for every line $l\in\mathcal{L}$. The minimum cardinality of a transversal of a linear system $(P,\mathcal{L})$, $\tau=\tau(P,\mathcal{L})$, is called *transversal number* of $(P,\mathcal{L})$. On the other hand, a subset of lines R of $\mathcal{L}$ is called a *2-packing* of $(P,\mathcal{L})$ if every triplet of different elements of R do not have a common point. The maximum cardinality of a 2-packing of $(P,\mathcal{L})$, $\nu_2=\nu_2(P,\mathcal{L})$, is called *2-packing number* of $(P,\mathcal{L})$. This new parameter has been studied in some papers, see for example [3–5, 18–20].

Araujo-Pardo *et al.* in [5] proved a relationship between the transversal and the 2-packing numbers

$$\lceil\nu_2/2\rceil\leq\tau\leq\frac{\nu_2(\nu_2-1)}{2}. \tag{1}$$

Hence, the transversal number of any linear system is upper bounded by a quadratic function of their 2-packing number. For some linear systems the transversal number is bounded above by a linear function of their 2-packing number, see [3–5].

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Eustis and Verstraëte in [13] proved, using probabilistic methods, the existence of k-uniform linear systems $(P, \mathcal{L})$ for infinitely many k's and $n = |P|$ large enough, which transversal number is $\tau = n - o(n)$. This k-uniform linear systems has 2-packing number upper bounded by $\frac{2n}{k}$.

There are works which the transversal number of an r-uniform linear system is bounded above by a function of their number points and lines, see for example [12, 14]. Henning and Yeo in [14] stated the following question: is it true that if $(P, \mathcal{L})$ is an r-uniform linear system then

$$\tau \leq \frac{|P| + |\mathcal{L}|}{r+1}, \tag{2}$$

for every $r \geq 2$? Chvátal and McDiarmid in [10] proved (2) when $r \in \{2, 3\}$. Dorfling and Henning in [12] proved (2) when $\Delta \leq 2$ and there are only two families of r-uniform linear systems that achieve equality in the bound. Also, Dorfling and Henning in [12] gave a better upper bound for the transversal number in terms of the number of points and the number of lines, namely, they proved that if $(P, \mathcal{L})$ is an r-uniform linear system with $\Delta \leq 2$ and if $r \geq 3$ is an odd integer, then $r(r^2 - 3)\tau \leq (r-2)(r+1)n + (r-1)^2 m + r - 1$; similar bounds were also proved when $r \geq 2$ is an even integer.

This note is organized as follows. In Section 2, we present an infinite family of r-uniform linear systems $(P, \mathcal{L})$ for which equality in (2) holds, where $r \geq 3$ is an odd integer. This family of linear systems was defined in [3]. In Section 3, we prove that if $(P, \mathcal{L})$ is an intersecting r-uniform linear system with $\tau = r$, then (2) holds. In Section 4, we prove if $(P, \mathcal{L})$ is an r-uniform linear system with $\nu_2 \in \{2, 3, 4\}$, then (2) holds. Finally, in Section 5, we prove that if $(P, \mathcal{L})$ is an r-uniform linear system with $\Delta = 2$, then (2) holds, and equality in (2) holds if and only if $(P, \mathcal{L})$ is an $(\nu_2 - 1)$-uniform linear system with $\nu_2 \geq 2$ an even integer. This result was obtained first by Dorfling and Henning in [12].

2. Examples of linear systems $(P, \mathcal{L})$ with $\tau = \dfrac{|P| + |\mathcal{L}|}{r+1}$

Let $(\Gamma, +)$ be an additive Abelian group, with neutral element e, satisfying $\sum_{g \in \Gamma} g = e$ and $2g \neq e$, for all $g \in \Gamma \setminus \{e\}$. An example of this groups is $(\mathbb{Z}_n, +)$, with $n \geq 3$ an odd integer.

Let $n = 2k + 1$, with k a positive integer; and let $(\Gamma, +)$ be an additive Abelian group of order n as above. Alfaro *et al.* in [3] defined the following linear system $\mathcal{C}_{n,n+1} = (P_n, \mathcal{L}_n)$, where

$$P_n = (\Gamma \times \Gamma \setminus \{e\}) \cup \{p, q\} \text{ and } \mathcal{L}_n = \mathcal{L} \cup \mathcal{L}_p \cup \mathcal{L}_q,$$

with

$$\mathcal{L} = \{L_g : g \in \Gamma \setminus \{e\}\}, \text{ and } L_g = \{(h, g) : h \in \Gamma\},$$

for $g \in \Gamma \setminus \{e\}$, and

$$\mathcal{L}_p = \{l_{p_g} : g \in \Gamma\}, \text{ with } l_{p_g} = \{(g, h) : h \in \Gamma \setminus \{e\}\} \cup \{p\},$$

for $g \in \Gamma$, and $\mathcal{L}_q = \{l_{q_g} : g \in \Gamma\}$, and

$$l_{q_g} = \{(h, f_g(h)) : h \in \Gamma, f_g(h) = h + g \text{ with } f_g(h) \neq e\} \cup \{q\},$$

for $g \in \Gamma$.

The set of lines $\mathcal{L}$ is a set of pairwise disjoint lines with $|\mathcal{L}| = n - 1$ and each line of $\mathcal{L}$ has n points. The set of lines $\mathcal{L}_p$ and $\mathcal{L}_q$ are lines incidents to p and q, respectively, with $|\mathcal{L}_p| = |\mathcal{L}_p| = n$, and each line of $\mathcal{L}_p \cup \mathcal{L}_q$ has n points. This linear system is an n-uniform linear system with $n(n-1) + 2$ points and $3n - 1$ lines. Moreover, this linear system has 2 points of degree n (points p and q) and $n(n-1)$ points of degree 3.

Alfaro *et al.* in [3] proved the following:

Theorem 2.1. [3] *The linear system $\mathcal{C}_{n,n+1}$ satisfies $\tau(\mathcal{C}_{n,n+1}) = \nu_2(\mathcal{C}_{n,n+1}) = n + 1$.*

A consequence of Theorem 2.1 is the following corollary.

Corollary 2.1. *Let $(P, \mathcal{L})$ be an r-uniform linear system such that $(P, \mathcal{L}) \simeq \mathcal{C}_{n,n+1}$, where $r \geq n$, then $\tau \leq \dfrac{|P| + |\mathcal{L}|}{r+1}$. Moreover, the equality holds if and only if $(P, \mathcal{L}) = \mathcal{C}_{n,n+1}$.*

Proof. Let $(P, \mathcal{L})$ be an r-uniform linear system such that $(P, \mathcal{L}) \simeq \mathcal{C}_{n,n+1}$. Then $|P| = n(n-1) + 2 + k|\mathcal{L}|$, where $n + k = r$ with $k \geq 0$, and $|\mathcal{L}| = 3n - 1$. Hence

$$\frac{|P| + |\mathcal{L}|}{r+1} = \frac{n(n-1) + 2 + (3n-1)(k+1)}{n+k+1} = \frac{(n-1)(n+k+1) + 2(n(k+1)+1)}{n+k+1} = (n+1) + \frac{2k(n-1)}{n+k+1} \geq n + 1.$$

Hence, by Theorem 2.1

$$\tau \leq \frac{|P|+|\mathcal{L}|}{n+1}.$$

The equality holds if and only if $k=0$, that is, if and only if $(P,\mathcal{L}) = \mathcal{C}_{n,n+1}$. □

Theorem 2.2. *If $r \geq 2$ is an positive integer and $(P,\mathcal{L})$ is an r-uniform linear system with $\Delta \geq \nu_2 - 1$, $|\mathcal{L}| \geq \nu_2 + \Delta - 2$ and $\Delta \geq 3$, then*

$$\nu_2 - 1 \leq \frac{|P|+|\mathcal{L}|}{r+1}.$$

Proof. Since $|P| \geq \Delta(r-1)+1$, we have

$$\frac{|P|+|\mathcal{L}|}{r+1} \geq \frac{\Delta(r-1)+1+\nu_2+\Delta-2}{r+1} = \frac{r\Delta+\nu_2-1}{r+1} \geq \frac{(\nu_2-1)(r+1)}{r+1} = \nu_2 - 1.$$

□

Corollary 2.2. *If $r \geq 2$ is an positive integer and $(P,\mathcal{L})$ is an r-uniform linear system with $\Delta \geq \nu_2$, $|\mathcal{L}| \geq \nu_2 + \Delta - 1$ and $\Delta \geq 3$, then*

$$\nu_2 \leq \frac{|P|+|\mathcal{L}|}{r+1}.$$

Proof. The proof of this corollary is analogous to the proof of Theorem 2.2. □

3. Intersecting r-uniform linear systems

Through this paper, all linear systems $(P,\mathcal{L})$ satisfy $|\mathcal{L}| > \nu_2$, due to the fact that $|\mathcal{L}| = \nu_2$ if and only if $\Delta \leq 2$.

The proofs of Lemma 3.1 and Lemma 3.2 are analogous to the proofs of Lemma 2.4 and Lemma 2.5 of [11], respectively.

Lemma 3.1. [11] *Let $(P,\mathcal{L})$ be an intersecting r-uniform linear system, with $r \geq 3$. If $\tau = r$, then every line of $(P,\mathcal{L})$ has at most one point of degree two and $\Delta = r$.*

Lemma 3.2. [11] *Let $(P,\mathcal{L})$ be an intersecting r-uniform linear system, with $r \geq 3$. If $\tau = r$, then $3(r-1)| \leq |\mathcal{L}| \leq r^2 - r + 1$ and $|P| = r^2 - r + 1$.*

The proof of Lemma 3.3 is analogous to the proof of Lemma 4.1 of [18].

Lemma 3.3. [18] *Let $(P,\mathcal{L})$ be an intersecting r-uniform linear system, with $r \geq 3$ be an odd integer. If $\tau = r$, then $\nu_2 = r+1$.*

Corollary 3.1. *Let $r \geq 3$ be an odd integer. If $(P,\mathcal{L})$ is an intersecting r-uniform linear system with $\tau = r$, then*

$$\tau \leq \frac{|P|+|\mathcal{L}|}{r+1}.$$

Proof. By Lemma 3.1 and Lemma 3.3 then $\Delta = \nu_2 - 1$. On the other hand, by Theorem 2.2 and $\nu_2 = r+1$ which implies $\tau \leq \dfrac{|P|+|\mathcal{L}|}{r+1}$. □

Let us consider the case when r is an even integer. If $(P,\mathcal{L})$ is an intersecting r-uniform linear system, then $\nu_2 \leq r+1$. However, if r is an even integer, then by the next lemma, it holds that $\nu_2 \leq r$.

Lemma 3.4. [19] *Let $(P,\mathcal{L})$ be an r-uniform intersecting linear system where $r \geq 2$ is an even integer. If $\nu_2 = r+1$, then $\tau = \frac{r+2}{2}$.*

Corollary 3.2. *Let $(P,\mathcal{L})$ be an r-uniform intersecting linear system with $r \geq 2$ be an even integer. If $\nu_2 = r+1$, then $\tau \leq \dfrac{|P|+|\mathcal{L}|}{r+1}$.*

Proof. It is not difficult to prove $\Delta = 2$, see [19]. Hence, by Corollary 5.4 (see in Section 5), $\tau \leq \dfrac{|P|+|\mathcal{L}|}{r+1}$. □

Therefore, if $\tau > \frac{r+2}{2}$, then $\nu_2 \leq r$ and $r \geq 4$ being an even integer. The proof of Lemma 3.5 is analogous to the proof of Lemma 6 of [19].

Lemma 3.5. [19] *Let $(P,\mathcal{L})$ be an intersecting r-uniform linear system, with $r \geq 4$ be an even integer. If $\tau = r$, then $\nu_2 = r$.*

Corollary 3.3. *Let $r \geq 4$ be an even integer. If $(P,\mathcal{L})$ is an intersecting r-uniform linear system with $\tau = r$, then $\tau \leq \dfrac{|P|+|\mathcal{L}|}{r+1}$.*

Proof. By Lemma 3.1 and Lemma 3.5, $\Delta = \nu_2 = r$. Hence, by Corollary 2.2, it holds that $\tau \leq \dfrac{|P| + |\mathcal{L}|}{r+1}$. $\square$

By Corollary 3.1 and Corollary 3.3, we have

Theorem 3.1. *Let $r \geq 3$ be an integer. If $(P, \mathcal{L})$ is an intersecting r-uniform linear system with $\tau = r$, then $\tau \leq \dfrac{|P| + |\mathcal{L}|}{r+1}$.*

A *finite projective plane*, or merely *projective plane*, is an intersecting linear system satisfying the following conditions:

- any pair of points have a common line,
- any pair of lines have a common point, and
- there exist four points in general position (there are not three collinear points).

It is well known that if $(P, \mathcal{L})$ is a projective plane then there exists a number $q \in \mathbb{N}$, called *order of projective plane*, such that every point (line, respectively) of $(P, \mathcal{L})$ is incident to exactly $q+1$ lines (points, respectively), and $(P, \mathcal{L})$ contains exactly $q^2 + q + 1$ points (lines, respectively). Also, it is well known that projective planes of order q, denoted by Π_q, exist when q is a power prime. For more information about the existence and the unicity of projective planes see, for instance, [7, 8].

In relation to the transversal number of projective planes, it is well known that every line in Π_q is a minimum transversal, then $\tau(\Pi_q) = q + 1$. On the other hand, related to the 2-packing number of a projective planes, since projective planes are dual systems, this parameter coincides with the cardinality of an *oval*, which is the maximum number of points in general position (no three of them collinear), and it is equal to $q + 1$ when q is odd integer, and it is equal to $q + 2$ when q is even integer (see for example [8]).

Consequently, for the projective planes Π_q of odd order q we have $\tau(\Pi_q) = \nu_2(\Pi_q) = q + 1$, and for the projective planes Π_q of even order q we have $\tau(\Pi_q) = \nu_2(\Pi_q) - 1 = q + 1$, see [5].

Corollary 3.4. *Let q be a prime power, and let $(P, \mathcal{L})$ be an $(q + 1)$-uniform linear system such that $(P, \mathcal{L}) \simeq \Pi_q$, then $\tau \leq \dfrac{|P| + |\mathcal{L}|}{q+2}$.*

Proof. The proof is a simple consequence of Theorem 2.2 and Corollary 2.2, since $\Delta = q + 1$ and $\nu_2 = q + 2$ if q is an even integer, and $\nu_2 = q + 1$ if q is an odd integer. $\square$

4. The r-uniform linear systems with $\nu_2 \in \{2, 3, 4\}$

Let $(P, \mathcal{L})$ be an r-uniform linear system with $\nu_2 \in \{2, 3\}$. It is not difficult to prove (see [5]) that $\nu_2 = 2$ if and only if $\tau = 1$. Also, if $\nu_2 = 3$, then $\tau = 2$, see [5].

Lemma 4.1. [5] *Any linear system $(P, \mathcal{L})$ with $\nu_2 = 4$ and $\Delta \geq 5$ satisfies $\tau \leq \nu_2 - 1$.*

Corollary 4.1. *Any linear system $(P, \mathcal{L})$ with $\nu_2 = 4$ and $\Delta \geq 5$ satisfies $\tau \leq \dfrac{|P| + |\mathcal{L}|}{r+1}$.*

Proof. The required result follows from Theorem 2.2. $\square$

Lemma 4.2. [5] *Let $(P, \mathcal{L})$ be a linear system with $\nu_2 = 4$ and $\Delta = 3$. If $(P, \mathcal{L}) \simeq \mathcal{C}_{3,4}$, then $\tau = \nu_2$, otherwise $\tau \leq \nu_2 - 1$.*

Corollary 4.2. *Let $r \geq 2$ be an integer and let $(P, \mathcal{L})$ be an r-uniform linear system. If $\nu_2 = 4$ and $\Delta = 3$, then*

$$\tau \leq \frac{|P| + |\mathcal{L}|}{r+1}.$$

The equality holds if and only if $(P, \mathcal{L}) = \mathcal{C}_{3,4}$.

Proof. Let $(P, \mathcal{L})$ be an r-uniform linear system with $\nu_2 = 4$ and $\Delta = 3$ such that $(P, \mathcal{L}) \not\simeq \mathcal{C}_{3,4}$. By Theorem 2.2 and Lemma 4.2, we have

$$\tau \leq \frac{|P| + |\mathcal{L}|}{r+1}.$$

On the other hand, if $(P, \mathcal{L}) \simeq \mathcal{C}_{3,4}$, then $|P| = 8 + 8k$ and $|\mathcal{L}| = 8$, where $k + 3 = r$ and $k \geq 0$. Hence

$$\frac{|P| + |\mathcal{L}|}{r+1} = \frac{8(k+2)}{k+4} \geq \frac{16}{4} = 4 = \tau.$$

Therefore, $\tau \leq \dfrac{|P| + |\mathcal{L}|}{r+1}$, where the equality holds if and only if $k = 0$, that is, if and only if $(P, \mathcal{L}) = \mathcal{C}_{3,4}$. $\square$

Lemma 4.3. [5] *If $(P,\mathcal{L})$ is a linear system with $\nu_2 = 4$ and $\Delta = 4$, then $\tau \le \nu_2$.*

Corollary 4.3. *Let $r \ge 2$ be an integer and $(P,\mathcal{L})$ be an r-uniform linear system. If $\nu_2 = 4$ and $\Delta = 4$, then $\tau \le \dfrac{|P|+|\mathcal{L}|}{r+1}$.*

Proof. The required result follows from Corollary 2.2 and Lemma 4.3. □

Therefore, the main result of this section states as:

Theorem 4.1. *Let $r \ge 2$ be an integer and $(P,\mathcal{L})$ be an r-uniform linear system with $|\mathcal{L}| > \nu_2$. If $\nu_2 = 4$, then $\tau \le \dfrac{|P|+|\mathcal{L}|}{r+1}$.*

Proof. The required result follows from Corollary 4.1, Corollary 4.2 and Corollary 4.3. □

5. The r-uniform linear systems with $\Delta = 2$

In this section, we present some results regarding r-uniform linear systems $(P,\mathcal{L})$ with $\Delta = 2$ satisfying $\tau \le \dfrac{|P|+|\mathcal{L}|}{r+1}$.

Proposition 5.1. *If $(P,\mathcal{L})$ is a linear system with $\Delta = 2$, then $\tau \le \nu_2 - 1$.*

Proof. Let A be a maximum subset of P such that every $p \in A$ satisfies $deg(p) = 2$, and $\{p,q\} \not\subseteq l$, for every $p,q \in A$. Since $\Delta = 2$, then $A \ne \emptyset$. Let $\mathcal{L}_A = \bigcup_{p\in A}\mathcal{L}_p$ and $\mathcal{L}' = \mathcal{L} \setminus \mathcal{L}_A$. Hence, if $\mathcal{L}' \ne \emptyset$, then the set of lines of $\mathcal{L}'$ is pairwise disjoint. Therefore, the following set $T = A \cup B$, where $B = \{p_l : l \in \mathcal{L}' \text{ and } p_l \in l\}$, is a transversal of $(P,\mathcal{L})$. Hence, $\tau \le |T| = |A| + |B| \le |\mathcal{L}| - 1 = \nu_2 - 1$. □

Corollary 5.1. *Let $(P,\mathcal{L})$ be a linear system with $\Delta = 2$ and let $\mathcal{L}'$ as above. If $|\mathcal{L}'| \le 1$, then $\tau = \lceil \nu_2/2 \rceil$. Moreover, if $|\mathcal{L}'| = \nu_2 - 2$, then $\tau = \nu_2 - 1$.*

Corollary 5.2. *If $(P,\mathcal{L})$ is an r-uniform linear system with $\Delta = 2$ and $\nu_2 \ge 4$, then $\lceil \nu_2/2 \rceil \le \left\lfloor \dfrac{|P|+|\mathcal{L}|}{r+1} \right\rfloor \le \nu_2 - 1$.*

Proof. Let A as in the proof of Proposition 5.1. If $|A| = k$, where $1 \le k \le \nu_2(\nu_2-1)/2$, then $r\nu_2 - k \le |P| \le r\nu_2 - 1$. Hence

$$\left\lfloor \frac{|P|+|\mathcal{L}|}{r+1} \right\rfloor \le \left\lfloor \nu_2 - \frac{1}{r+1} \right\rfloor = \nu_2 - 1.$$

On the other hand, since $|P| \ge r\nu_2 - k$ then

$$\left\lfloor \frac{|P|+|\mathcal{L}|}{r+1} \right\rfloor \ge \left\lfloor \nu_2 - \frac{k}{r+1} \right\rfloor \ge \left\lfloor \nu_2 - \frac{\nu_2(\nu_2-1)/2}{r+1} \right\rfloor \ge \lceil \nu_2/2 \rceil,$$

and the statement holds. □

In [12], the following result was proved.

Theorem 5.1. [12] *If $(P,\mathcal{L})$ is a linear system with $\Delta = 2$, then $\tau \le \dfrac{|P|+|\mathcal{L}|}{r+1}$.*

As a simple consequence, since $\tau \in \mathbb{N}$, we have $\tau \le \left\lfloor \frac{|P|+|\mathcal{L}|}{r+1} \right\rfloor$.

Theorem 5.2. *If $(P,\mathcal{L})$ is an r-uniform linear system with $\nu_2 - 1 \le r$, then*

$$\lceil \nu_2/2 \rceil \le \frac{|P|+|\mathcal{L}|}{r+1}.$$

Proof. Since $|P| \ge r\nu_2 - \frac{\nu_2(\nu_2-1)}{2}$ and $|\mathcal{L}| \ge \Delta + \nu_2 - 2$, then

$$\frac{|P|+|\mathcal{L}|}{r+1} \ge \frac{r\nu_2 - \frac{\nu_2(\nu_2-1)}{2} + \nu_2 + \Delta - 2}{r+1} = \nu_2\left[1 - \frac{\nu_2-1}{2(r+1)}\right] + \frac{\Delta-2}{r+1}.$$

Since $\nu_2 - 1 \le r$ then

$$\frac{|P|+|\mathcal{L}|}{r+1} \ge \nu_2\left[1 - \frac{\nu_2-1}{2\nu_2}\right] + \frac{\Delta-2}{r+1} = \frac{\nu_2+1}{2} + \frac{\Delta-2}{r+1} \ge \lceil \nu_2/2 \rceil.$$

Hence, the theorem holds. □

Corollary 5.3. *Let $(P,\mathcal{L})$ be an r-uniform linear system with $\nu_2 - 1 \le r$ and $\tau = \lceil \nu_2/2 \rceil$, then $\tau \le \dfrac{|P|+|\mathcal{L}|}{r+1}$.*

Corollary 5.4. *Let $(P,\mathcal{L})$ be an r-uniform intersecting linear system with $\Delta = 2$, then $\tau \le \dfrac{|P|+|\mathcal{L}|}{r+1}$.*

Corollary 5.5. *Let $(P,\mathcal{L})$ be an r-uniform intersecting linear system with $\Delta = 2$ and $r \ge 2$ an even integer. Then*

$$\tau = \frac{|P|+|\mathcal{L}|}{r+1}$$

if and only if $r = \nu_2 - 1$.

Acknowledgments

The authors would like to thank the anonymous referees and chief editor for their kind comments and suggestions for the improvement of the paper. This research work is partially supported by SNI (Sistema Nacional de Investigadores) and CONACyT (Consejo Nacional de Ciencia y Tecnología).

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