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Proper mean colorings of graphs

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Abstract

For an edge coloring of a connected graph G of order 3 or more with positive integers, the chromatic mean of a vertex v of G is the sum of the colors of the edges incident with v divided by the degree of v. Only edge colorings c are considered for which the chromatic mean of every vertex is a positive integer. If adjacent vertices have distinct chromatic means, then c is a proper mean coloring of G. The maximum vertex color in a proper mean coloring c of G is the proper mean index of c and the proper mean index $\mu(G)$ of G is the minimum proper mean index among all proper mean colorings of G. The proper mean index is determined for complete graphs, cycles, stars, double stars, and paths. The non-leaf minimum degree $\delta^*(T)$ of a tree T is the minimum degree among the non-leaves of T. It is shown that if T is tree with $\delta^*(T) \geq 10$ or a caterpillar with $\delta^*(T) \geq 6$, then $\mu(T) \leq 4$. Furthermore, it is conjectured that $\chi(G) \leq \mu(G) \leq \chi(G) + 2$ for every connected graph G of order 3 or more.

Keywords: chromatic mean; proper mean colorings; proper mean index.

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1. Introduction

For every connected graph G of order 3 or more, there are edge colorings c with positive integers that induce an integer vertex coloring cm defined for each vertex v of G by

$$\mathrm{cm}(v) = \frac{\sum_{e\in E_v} c(e)}{\deg v}, \text{ where } E_v \text{ is the set of edges incident with } v.$$

Edge colorings with this property are called *mean colorings*. The induced vertex color $\mathrm{cm}(v)$ of a vertex v of G is called the *chromatic mean* of v. Consequently, only edge colorings c are considered for which $\mathrm{cm}(v)$ is a positive integer for every vertex v of G. For a vertex v in a graph G with a mean coloring c, the *chromatic sum* of v is defined as $\mathrm{cs}(v) = \sum_{e\in E_v} c(e)$. Therefore, $\mathrm{cs}(v) = \deg v \cdot \mathrm{cm}(v)$ and the sum of the chromatic sums of all vertices in a graph satisfies the identity below.

Observation 1.1. *If c is a mean coloring of a connected graph G, then $\sum_{v\in V(G)} \mathrm{cs}(v) = 2\sum_{e\in E(G)} c(e)$.*

If distinct vertices have distinct chromatic means, then the edge coloring c is called a *rainbow mean coloring* of G. This concept was introduced and studied in [2] where it was shown that every connected graph of order 3 or more has a rainbow mean coloring. For a rainbow mean coloring c of a graph G, the maximum vertex color is the *rainbow mean index* $\mathrm{rm}(c)$ of c. That is, $\mathrm{rm}(c) = \max\{\mathrm{cm}(v) : v \in V(G)\}$. The *rainbow mean index* $\mathrm{rm}(G)$ of G itself is defined as

$$\mathrm{rm}(G) = \min\{\mathrm{rm}(c) : \ c \text{ is a rainbow mean coloring of } G\}.$$

The following is an immediate observation.

Observation 1.2. *If G is a connected graph of order $n \geq 3$, then $\mathrm{rm}(G) \geq n$.*

A mean coloring of a connected graph G of order 3 or more is defined to be a *proper mean coloring* of G if every two adjacent vertices of G have distinct chromatic means. The maximum vertex color in a proper mean coloring c is the *proper mean index* $\mu(c)$ of c and the minimum proper mean index among all proper mean colorings of G is the *proper mean index* $\mu(G)$ of G. Since every such graph has a rainbow mean coloring, each such graph has a proper mean coloring as well. Furthermore, the proper mean index of a graph G is at least its chromatic number $\chi(G)$. Therefore, $\chi(G) \leq \mu(G) \leq \mathrm{rm}(G)$ for every connected graph G of order at least 3. As an illustration of these concepts, Figure 1 shows proper meaning colorings of the cycles C_5 and C_6. In fact, $\mu(C_5) = 4 = \chi(C_5) + 1$ and $\mu(C_6) = 4 = \chi(C_6) + 2$.

While $\chi(G) = 2$ for every nontrivial connected bipartite graph G, $\mu(G) \neq \chi(G)$ for every bipartite graph G. Indeed, $\mu(G) \neq 2$ for every connected graph G of order at least 3. In order to verify this fact, we first present a useful observation.

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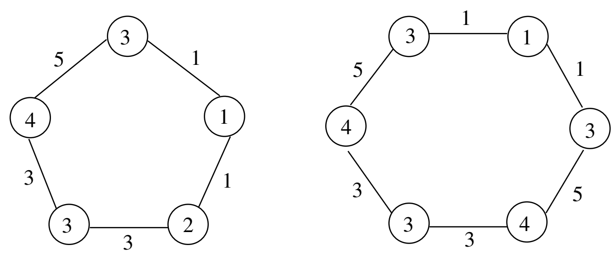

Figure 1: Proper mean colorings of C_5 and C_6.

Observation 1.3. *Let G be a connected bipartite graph with partite sets U and W. If c is an edge coloring of G, then*

$$\sum_{u\in U} \mathrm{cs}(u) = \sum_{w\in W} \mathrm{cs}(w).$$

For a positive integer n, it is useful to let $[n] = \{1, 2, \ldots, n\}$.

Proposition 1.1. *If G is a connected graph of order at least 3, then $\mu(G) \geq 3$. Furthermore, if c is a proper mean coloring of G with $\mu(c) = 3$, then $\{\mathrm{cm}(v) : v \in V(G)\} = [3]$.*

Proof. We show that if c is a proper mean coloring of G, then the resulting vertex coloring cm uses at least three distinct colors. Assume, to the contrary, that cm uses only two distinct colors $a, b \in [3]$. Then G is a connected bipartite graph. Let U and W be the partite sets of G. We may assume that $\mathrm{cm}(u) = a$ for each $u \in U$ and $\mathrm{cm}(w) = b$ for each $w \in W$. Since $\sum_{u\in U} \mathrm{cs}(u) = a \sum_{u\in U} \deg u = a|E(G)|$ and $\sum_{w\in W} \mathrm{cs}(w) = b \sum_{w\in W} \deg w = b|E(G)|$, it follows by Observation 1.3 that $a = b$, which is a contradiction. Consequently, $\mu(G) \geq \mu(c) \geq 3$. Furthermore, if $\mu(c) = 3$, then cm must use all three colors in $[3]$. □

2. The proper mean index of some well-known graphs

In [2] the rainbow mean indexes of complete graphs were determined. Since $\mu(K_n) = \mathrm{rm}(K_n)$ for each integer $n \geq 3$, we have the following result.

Theorem 2.1. *For each integer $n \geq 3$,* $\mu(K_n) = \mathrm{rm}(K_n) = \begin{cases} n & \text{if } n \geq 4 \text{ and } n \not\equiv 2 \pmod 4 \\ n+1 & \text{if } n = 3 \text{ or } n \equiv 2 \pmod 4. \end{cases}$

Theorem 2.1 also shows that there are graphs G for which $\mu(G) = \chi(G)$. We now determine the proper mean index of all paths and cycles, beginning with paths.

Theorem 2.2. *For each integer $n \geq 3$,* $\mu(P_n) = \begin{cases} 3 & \text{if } n \text{ is odd} \\ 4 & \text{if } n \text{ is even.} \end{cases}$

Proof. First, suppose that n is odd. By Proposition 1.1 , it suffices to show that there is a proper mean coloring c such that $\mu(c) = 3$. Let $P = (u_1, u_2, \ldots u_n)$. For each integer i with $1 \leq i \leq n$, define

$$c(e) = \begin{cases} 1 & \text{if } e \text{ is incident with } u_i \text{ for } i \equiv 1 \pmod 4 \\ 3 & \text{if } e \text{ is incident with } u_i \text{ for } i \equiv 3 \pmod 4. \end{cases}$$

Then the vertex color $\mathrm{cm}(u_i)$, $1 \leq i \leq n$, is given by

$$\mathrm{cm}(u_i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod 4 \\ 2 & \text{if } i \equiv 0, 2 \pmod 4 \\ 3 & \text{if } i \equiv 3 \pmod 4. \end{cases}$$

Since cm is a proper coloring of P_n, it follows that $\mu(c) = 3$. Therefore, $\mu(P_n) = 3$ for every odd integer $n \geq 3$.

Next, suppose that n is even. We show that $\mu(P_n) = 4$. First, we show that $\mu(P_n) \geq 4$. Assume, to the contrary, that there is a proper mean coloring c of P_n with $\mu(c) = 3$. By Proposition 1.1, there is a vertex v in P_n such that $\mathrm{cm}(v) = 1$. Hence, each edge incident with v is colored 1, which implies that $c(e)$ is odd for all $e \in E(P_n)$. Suppose that there is an edge e such that $c(e) = 5$. If e is a pendant edge of P_n, then $\mu(c) \geq 5$, which is a contradiction. Thus, we may assume that e is adjacent to two edges e_1 and e_2 of P_n. Since $c(e_1) \neq c(e_2)$ and $c(e_1)$ and $c(e_2)$ are both odd, at least one of $c(e_1)$ and $c(e_2)$ is 3 or more. However then, $\mathrm{cm}(u) \geq 4$ for at least one vertex u incident with e, which is a contradiction. Thus, all edges are colored 1 or 3. We may assume, without loss of generality, that $c(u_1u_2) = 1$. This implies that $c(e) = 1$ if e is incident with u_i where $i \equiv 1 \pmod 4$ and $c(e) = 3$ if e is incident with u_i where $i \equiv 3 \pmod 4$. Since $n \geq 4$ is even, it follows that $\mathrm{cm}(u_{n-1}) = \mathrm{cm}(u_n) \in \{1, 3\}$, which is a contradiction. Therefore, $\mu(P_n) \geq 4$.

To verify that $\mu(P_n) \leq 4$, it remains to show that there is a proper mean coloring c with $\mu(c) = 4$ for each even integer $n \geq 4$. For the path $P_{n-1} = (v_1, v_2, v_3, \ldots, v_{n-1})$, where $n-1 \geq 3$ is odd, let c_0 be the proper mean coloring of P_{n-1} with $\mu(c_0) = 3$ defined in Case 1. Subdividing the edge v_2v_3 of P_{n-1}, we obtain the path $P_n = (v_1, v_2, w, v_3, \ldots, v_{n-1})$ of order n. Now, define the edge coloring c of P_n by $c(v_2w) = 5$, $c(wv_3) = 3$, and $c(e) = c_0(e)$ if e is not incident with w. If we denote P_n by $(u_1, u_2, \ldots, u_n)$, then the vertex color $\mathrm{cm}(u_i)$, $1 \leq i \leq n$, is given by

$$\mathrm{cm}(u_i) = \begin{cases} 1 & \text{if } i = 1 \text{ or } i \equiv 2 \pmod 4 \text{ for } i \neq 2 \\ 2 & \text{if } i \equiv 1, 3 \pmod 4 \text{ for } 5 \leq i \leq n-1 \\ 3 & \text{if } i = 2 \text{ or } i \equiv 0 \pmod 4 \\ 4 & \text{if } i = 3. \end{cases}$$

Since cm is a proper coloring of P_n, it follows that $\mu(c) = 4$. Therefore, $\mu(P_n) = 4$ for each even integer $n \geq 4$. □

Theorem 2.3. *For each integer $n \geq 4$,* $\mu(C_n) = \begin{cases} 3 & \text{if } n \equiv 0 \pmod 4 \\ 4 & \text{if } n \not\equiv 0 \pmod 4. \end{cases}$

Proof. Let $C_n = (u_1, u_2, \ldots, u_n, u_{n+1} = u_1)$ be a cycle of order $n \geq 3$, where $e_i = u_iu_{i+1}$ for $1 \leq i \leq n$. First, suppose that $n \geq 4$ and $n \equiv 0 \pmod 4$. By Proposition 1.1, it suffices to show that there exists a proper mean coloring c with $\mu(c) = 3$. Define the edge coloring c by

$$c(e) = \begin{cases} 1 & \text{if } e \text{ is incident with } u_i \text{ where } i \equiv 2 \pmod 4 \\ 3 & \text{if } e \text{ is incident with } u_i \text{ where } i \equiv 0 \pmod 4. \end{cases} \tag{1}$$

Then the vertex color $\mathrm{cm}(u_i)$, $1 \leq i \leq n$, is given by

$$\mathrm{cm}(u_i) = \begin{cases} 1 & \text{if } i \equiv 2 \pmod 4 \\ 2 & \text{if } i \text{ is odd} \\ 3 & \text{if } i \equiv 0 \pmod 4. \end{cases}$$

Since the vertex coloring cm is proper, it follows that $\mu(c) = 3$ and so $\mu(C_n) = 3$ if $n \equiv 0 \pmod 4$.

Next, suppose that $n \not\equiv 0 \pmod 4$. First, we show that $\mu(C_n) \geq 4$. Assume, to the contrary, that there exists a proper mean coloring c of C_n with $\mu(c) = 3$. By Proposition 1.1, there is a vertex v such that $\mathrm{cm}(v) = 1$. Hence, each edge incident with v of C_n is colored 1, which implies that $c(e)$ is odd for all $e \in E(C_n)$. First, suppose that there is an edge e such that $c(e) = 5$. Let e_1 and e_2 be the two edges adjacent to e. Since $c(e_1) \neq c(e_2)$ and $c(e_1)$ and $c(e_2)$ are both odd, at least one of $c(e_1)$ and $c(e_2)$ is 3 or more. However then, $\mathrm{cm}(u) \geq 4$ for at least one vertex u incident with e, which is a contradiction. Hence, we may assume that each edge of C_n is colored 1 or 3. Since the resulting vertex coloring *cm* is proper, no edge is adjacent to two edges having the same color. Without loss of generality, we may conclude that the edges incident with u_i with $i \equiv 1 \pmod 4$ are colored 1 and the edges incident with u_i with $i \equiv 3 \pmod 4$ are colored 3. This, in turn, implies that $n \equiv 0 \pmod 4$, a contradiction.

It remains then to show that there is a proper mean coloring c of C_n with $\mu(c) = 4$. We consider three cases.

Case 1. $n \equiv 1 \pmod 4$. Then $n - 1 \equiv 0 \pmod 4$ and so $\mu(C_{n-1}) = 3$. Let c_0 be the proper mean coloring of $C_{n-1} = (u_1, u_2, \ldots, u_{n-1}, u_1)$ defined in (1) with $\mu(c_0) = 3$. Now, let C_n be obtained from C_{n-1} by subdividing the edge $u_{n-1}u_1$ with the vertex u_n. We now extend the coloring c_0 to a proper mean coloring c of C_n by defining $c(u_{n-1}u_n) = 3$ and $c(u_nu_1) = 5$.

Case 2. $n \equiv 2 \pmod 4$. Then $n - 1 \equiv 1 \pmod 4$. Let c_1 be the proper mean coloring of $C_{n-1} = (u_1, u_2, \ldots, u_{n-1}, u_1)$ defined in Case 1 and let C_n be obtained from C_{n-1} by subdividing the edge $u_{n-2}u_{n-3}$ with the vertex w. We now extend the coloring c_1 to a proper mean coloring c of C_n with $\mu(c) = 4$ by defining $c(u_{n-2}w) = 3$ and $c(wu_{n-3}) = 5$.

Case 3: $n \equiv 3 \pmod 4$. Then $n - 3 \equiv 0 \pmod 4$ and so $\mu(C_{n-3}) = 3$. Let c_0 be the proper mean coloring of $C_{n-3} = (u_1, u_2, \ldots, u_{n-3}, u_1)$ defined in (1) with $\mu(c_0) = 3$. Now, let C_n be obtained from C_{n-3} by replacing the edge u_1u_{n-3} (a 2-path) by the 5-path $(u_1, u_n, u_{n-1}, u_{n-2}, u_{n-3})$. We now extend the coloring c_0 to a proper mean coloring c of C_n with $\mu(c) = 4$ by defining $c(u_1u_n) = 3$, $c(u_nu_{n-1}) = 5$, $c(u_{n-1}u_{n-2}) = 1$ and $c(u_{n-2}u_{n-3}) = 3$. □

Next, we determine the proper mean index of all complete bipartite graphs.

Theorem 2.4. *For positive integers s and t with $s + t \geq 3$,* $\mu(K_{s,t}) = \begin{cases} 3 & \textit{if } st \textit{ is even} \\ 4 & \textit{if } st \textit{ is odd.} \end{cases}$

Proof. Let $G = K_{s,t}$ with partite sets $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_t\}$. First, suppose that st is even. We may assume that s is even. Then $s = 2a$ for some positive integer a. By Proposition 1.1, it suffices to show that there is a proper mean coloring $c : E(G) \to \mathbb{N}$ such that $\mu(c) = 3$. For each $w \in W$, define

$$c(u_i w) = \begin{cases} 1 & \text{if } 1 \le i \le a \\ 3 & \text{if } a+1 \le i \le 2a. \end{cases}$$

Then $\mathrm{cm}(u_i) = 1$ for $1 \le i \le a$, $\mathrm{cm}(u_i) = 3$ for $a+1 \le i \le 2a$, and $\mathrm{cm}(w) = 2$ for each $w \in W$. Since cm is a proper coloring of G, it follows that $\mu(c) = 3$. Therefore, $\mu(G) = 3$ if st is even.

Next, suppose that st is odd. We may assume that $1 \le s \le t$. Then $s = 2a+1$ and $t = 2b+1$ for some integers a and b with $0 \le a \le b$ and $b \ge 1$. First, we show that there is a proper mean coloring c of G with $\mu(c) = 4$. If $a = 0$, then define

$$c(u_1 w_i) = \begin{cases} 1 & \text{if } 1 \le i \le b+1 \\ 3 & \text{if } b+2 \le i \le 2b \\ 4 & \text{if } i = 2b+1. \end{cases}$$

Then $\mathrm{cm}(u_1) = 2$, $\mathrm{cm}(w_i) = 1$ for $1 \le i \le b+1$, $\mathrm{cm}(w_i) = 3$ for $b+2 \le i \le 2b$, and $\mathrm{cm}(w_{2b}) = 4$. If $a = 1$, then define

$$c(u_i w) = \begin{cases} 1 & \text{if } i = 1, 2 \\ 4 & \text{if } i = 3. \end{cases}$$

Then $\mathrm{cm}(u_1) = \mathrm{cm}(u_2) = 1$, $\mathrm{cm}(u_3) = 4$, and $\mathrm{cm}(w) = 2$ for each $w \in W$. If $a \ge 2$, then define

$$c(u_i w) = \begin{cases} 1 & \text{if } 1 \le i \le a+1 \\ 3 & \text{if } a+2 \le i \le 2a \\ 4 & \text{if } i = 2a+1. \end{cases}$$

Then $\mathrm{cm}(u_i) = 1$ for $1 \le i \le a+1$, $\mathrm{cm}(u_i) = 3$ for $a+2 \le i \le 2a$, $\mathrm{cm}(u_{2a+1}) = 4$, and $\mathrm{cm}(w) = 2$ for each $w \in W$. Therefore, $\mu(G) \le 4$.

It remains to show that $\mu(G) \ne 3$. Assume, to the contrary, that there is a proper mean coloring c of G with $\mu(G) = 3$. Thus, $\{\mathrm{cm}(v) : v \in V(G)\} = \{1, 2, 3\}$ by Observation 1.1. First, suppose that $s = 1$. Since $\mathrm{cm}(u_1) \ne 1$, it follows that $\mathrm{cm}(u_1) = 2$ or $\mathrm{cm}(u_1) = 3$.

- ⋆ First, suppose that $\mathrm{cm}(u_1) = 2$. Thus, $\mathrm{cm}(w) \in \{1, 3\}$ for each $w \in W$. Let x be the number of the vertices $w \in W$ such that $\mathrm{cm}(w) = 1$. Then there are $2b+1-x$ vertices $w \in W$ such that $\mathrm{cm}(w) = 3$. By Observation 1.3, $x \cdot 1 + (2b+1-x) \cdot 3 = 2(2b+1)$. However then, $2x = 2b+1$, which is impossible.

- ⋆ Next, suppose that $\mathrm{cm}(u_1) = 3$. Thus, $\mathrm{cm}(w) \in \{1, 2\}$ for each $w \in W$. Let x be the number of the vertices $w \in W$ such that $\mathrm{cm}(w) = 1$. Then there are $2b+1-x$ vertices $w \in W$ such that $\mathrm{cm}(w) = 2$. By Observation 1.3, $x \cdot 1 + (2b+1-x) \cdot 2 = 3(2b+1)$. However then, $2b+1+x = 0$, which is impossible.

Next, suppose that $s \ge 3$. We may assume that $\mathrm{cm}(u_1) = 1$ (as the argument for $\mathrm{cm}(w_1) = 1$ is similar). Hence, there is $u \in U$ such that $\mathrm{cm}(u) \ne 1$; for otherwise, $c(e) = 1$ for every edge e of G and so $\mathrm{cm}(v) = 1$ for every vertex v of G. Since G is a complete bipartite graph, it follows that $\mathrm{cm}(u) \ne \mathrm{cm}(w)$ for every $u \in U$ and $w \in W$. Thus, either $\{\mathrm{cm}(u) : u \in U\} = \{1, 2\}$ or $\{\mathrm{cm}(u) : u \in U\} = \{1, 3\}$.

- ⋆ If $\{\mathrm{cm}(u) : u \in U\} = \{1, 2\}$, then $\mathrm{cm}(w) = 3$ for each $w \in W$. Let x be the number of the vertices $u \in U$ such that $\mathrm{cm}(u) = 1$. Then there are $2a+1-x$ vertices $u \in U$ such that $\mathrm{cm}(u) = 2$. By Observation 1.3,

  $$x(2b+1) \cdot 1 + (2a+1-x)(2b+1) \cdot 2 = (2b+1)(2a+1) \cdot 3.$$

  However then, $2a+x+1 = 0$, which is impossible.

- ⋆ If $\{\mathrm{cm}(u) : u \in U\} = \{1, 3\}$, then $\mathrm{cm}(w) = 2$ for each $w \in W$. Let x be the number of vertices $u \in U$ such that $\mathrm{cm}(u) = 1$. Then there are $2a+1-x$ vertices $u \in U$ such that $\mathrm{cm}(u) = 3$. By Observation 1.3,

  $$x(2b+1) \cdot 1 + (2a+1-x)(2b+1) \cdot 3 = (2b+1)(2a+1) \cdot 2.$$

  However then, $2a+1 = 2x$, which is impossible. □

In each of the examples we've seen, the proper mean index of a graph has not exceeded its chromatic number by more than 2. This leads to the following conjecture.

Conjecture 2.1. *For every connected graph G of order 3 or more, $\chi(G) \le \mu(G) \le \chi(G) + 2$.*

3. Trees

In the case of trees, Conjecture 2.1 states that $\mu(T) = 3$ or $\mu(T) = 4$ for every tree T of order at least 3. We thus turn our attention to investigate this conjecture for various classes of trees. By Theorems 2.2 and 2.4, Conjecture 2.1 is true for paths and stars. It can also be shown that if the edges of a star are subdivided in any manner, then the proper mean index of the resulting tree is at most 4. Hence, Conjecture 2.1 is true for all trees having at most one vertex of degree greater than 2. In fact, those subdivided stars having proper mean index 3 (equivalently 4) have been characterized in [4]. We now show that Conjecture 2.1 is true as well for trees all of whose non-leaves have sufficiently large degree. The *non-leaf minimum degree* $\delta^*(T)$ of a tree T of order 3 or more is the minimum degree among the non-leaves of T.

Lemma 3.1. *Let x be a vertex in a tree T such that $\deg x \geq 10$.*

(a) *There exists a coloring of the edges of T incident with x using colors from $[4]$ such that $\mathrm{cm}(x) = 2$, where (i) exactly one edge incident with x is colored 2 or (ii) no edges incident with x are colored 2.*

(b) *There exists a coloring of the edges of T incident with x using colors from $[4]$ such that $\mathrm{cm}(x) = 3$, where (i) exactly one edge incident with x is colored 3 or (ii) no edges incident with x are colored 3.*

Proof. We begin with (a). First, suppose that x has even degree. Then $\deg x = 10 + 2k$ where $k \geq 0$. For (i), we color $k+5$ edges incident with x by 1, one edge by 2, $k+3$ edges by 3, and one edge by 4. For (ii), we color $k+6$ edges incident with x by 1, $k+2$ edges by 3, and two edges by 4. If (i) occurs, then x is referred to as a *Type* 2.1 *vertex*; while if (ii) occurs, then x is referred to as a *Type* 2.2 *vertex*. It is convenient to represent these colorings of the edges incident with x as follows:

Type 2.1:

$k+5$	1	$k+3$	1
1	2	3	4

Type 2.2:

$k+6$	0	$k+2$	2
1	2	3	4

Next, suppose that x has odd degree. Then $\deg x = 11 + 2k$ where $k \geq 0$. The following colorings of the edges incident with x have the desired properties (i) or (ii). Here, the vertex x is referred to as a *Type* 2.3 *vertex* if (i) occurs or as a *Type* 2.4 *vertex* if (ii) occurs.

Type 2.3:

$k+6$	1	$k+2$	2
1	2	3	4

Type 2.4:

$k+6$	0	$k+4$	1
1	2	3	4

Next, we verify (b). First, suppose that x has even degree. Then $\deg x = 10 + 2k$ where $k \geq 0$. The following colorings of the edges incident with x have the desired properties (i) or (ii) and the vertex x is referred to as a *Type* 3.1 *vertex* if (i) occurs or as a *Type* 3.2 *vertex* if (ii) occurs.

Type 3.1:

1	$k+3$	1	$k+5$
1	2	3	4

Type 3.2:

2	$k+2$	0	$k+6$
1	2	3	4

Next, suppose that x has odd degree. Then $\deg x = 11+2k$ where $k \geq 0$. The following colorings of the edges incident with x have the desired properties (i) or (ii). Similarly, the vertex x is called a *Type* 3.3 *vertex* if (i) occurs or a *Type* 3.4 *vertex* if (ii) occurs.

Type 3.3:

2	$k+2$	1	$k+6$
1	2	3	4

Type 3.4:

1	$k+4$	0	$k+6$
1	2	3	4

Therefore, (a) and (b) both hold. $\square$

Theorem 3.1. *If T is a tree with $\delta^*(T) \geq 10$, then $\mu(T) = 3$ or $\mu(T) = 4$.*

Proof. Since $\mu(T) \geq 3$ by Proposition 1.1, it suffices to show that $\mu(T) \leq 4$. By Theorem 2.4, the statement is true if T is a star. Hence, we may assume that T is not a star. Let v be a vertex of T that is not a leaf. Thus, $\deg v = d \geq 10$. Let T be a tree rooted at v, where $V_i = \{u \in V(T) : d(u, v) = i\}$ for $i = 0, 1, \ldots, e(v)$, where $e(v)$ is the eccentricity of v. Hence, $V_0 = \{v\}$, $V_1 = N(v)$, and $V_i \neq \emptyset$ for $0 \leq i \leq e(v)$. Furthermore, for each vertex $x \in V_i$, where $1 \leq i \leq e(v)$, there is exactly one vertex $y \in V_{i-1}$ such that $xy \in E(T)$. Next, we construct a proper mean coloring c of T recursively such that $\mu(c) = 4$.

Let $V_1 = \{v_1, v_2, \ldots, v_d\}$. Since T is not a star, at least one vertex of V_1 has degree 10 or more. By Lemma 3.1, we can color the edges incident with v so that v is a Type 2.2 vertex if v has even degree or a Type 2.4 vertex if v has odd degree. Thus, $\mathrm{cm}(v) = 2$ and no edge incident with v is colored 2. Hence, if $v_i \in V_1$, $1 \leq i \leq d$, is a leaf, then $\mathrm{cm}(v_i) \neq 2$. On the other hand, one or more vertices in V_1 has degree 10 or more. Let $v_j \in V_1$, $1 \leq j \leq d$, such that $\deg v_j \geq 10$. Then $c(vv_j) \in \{1, 3, 4\}$. If $c(vv_j) \in \{1, 4\}$, then we color the edges incident with v_j so that v_j is a Type 3.2 vertex if $\deg v_j$ is even or color these edges so that v_j is a Type 3.4 vertex if $\deg v_j$ is odd. If $c(vv_j) = 3$, then we color the edges incident with v_j so that v_j is a Type 3.1 vertex if $\deg v_j$ is even or color these edges so that v_j is a Type 3.3 vertex if $\deg v_j$ is odd. In either case, $\mathrm{cm}(v_j) = 3$ and for any leaf x (necessarily in V_2) adjacent to v_j, it follows that $\mathrm{cm}(x) \neq 3$. We perform such a coloring for each vertex $v_j \in V_1$ of degree 10 or more such that $\mathrm{cm}(v_j) = 3$ where no edge joining v_j and a vertex in V_2 is colored 3.

Next, suppose that y is a vertex in V_2 such that $\deg y \geq 10$. Let x be the vertex of V_1 such that $xy \in E(T)$. Then $c(xy) \in \{1, 2, 4\}$. If $c(xy) \in \{1, 4\}$, then we color the remaining $\deg y - 1$ edges incident with y so that y is a Type 2.2 vertex if $\deg y$ is even or color these edges so that y is a Type 2.4 vertex if $\deg y$ is odd. If $c(xy) = 2$, then we color the remaining $\deg y - 1$ edges incident with y so that v_j is a Type 2.1 vertex if $\deg y$ is even or color these edges so that y is a Type 2.3 vertex if $\deg y$ is odd. In either case, $\mathrm{cm}(y) = 2$ and $\mathrm{cm}(z) \neq 2$ for all leaves $z \in V_3$ adjacent to y. We perform such a coloring for each vertex $y \in V_2$ of degree 10 or more such that $\mathrm{cm}(y) = 2$ where no edge joining y and a vertex in V_3 is colored 2.

Proceeding in this manner for each vertex x in V_i for $3 \leq i \leq e(v) - 1$ with $\deg x \geq 10$, we arrive at a proper mean coloring c of T with $\mu(c) = 4$. Therefore, $\mu(T) \leq 4$. $\square$

If the tree T being considered is a caterpillar (the removal of all leaves produces a path, called the *spine* of T), then a result similar to Theorem 3.1 can be obtained with a weaker hypothesis. Once again, we begin with a lemma.

Lemma 3.2. *Let x be a vertex in a caterpillar T such that $\deg x \geq 6$.*

(*a*) *There exists a coloring of the edges of T incident with x with colors from $[4]$ such that $\mathrm{cm}(x) = 2$, where (i) exactly one edge incident with x is colored 2 or (ii) exactly two edges incident with x are colored 2.*

(*b*) *There exists a coloring of the edges of T incident with x with colors from $[4]$ for which $\mathrm{cm}(x) = 3$ such that no edges incident with x are colored 3.*

Proof. We begin with (a). First, suppose that x has even degree. Then $\deg x = 6 + 2k$ where $k \geq 0$. For (i), we color $k + 3$ edges incident with x by 1, one edge by 2, $k + 1$ edges by 3, and one edge by 4. For (ii), we color $k + 2$ edges incident with x by 1, two edges by 2, and $k + 2$ edges by 3. If (i) occurs, then x is referred to as a *Type* $2a$ *vertex*; while if (ii) occurs, then x is referred to as a *Type* $2b$ *vertex*.

Type 2a:

$k+3$	1	$k+1$	1
1	2	3	4

Type 2b:

$k+2$	2	$k+2$	0
1	2	3	4

Next, suppose that x has odd degree. Then $\deg x = 7 + 2k$ where $k \geq 0$. The following colorings of the edges incident with x have the desired properties (i) or (ii). Here, the vertex x is referred to as a *Type* $2c$ *vertex* if (i) occurs or as a *Type* $2d$ *vertex* if (ii) occurs.

Type 2c:

$k+3$	1	$k+3$	0
1	2	3	4

Type 2d:

$k+3$	2	$k+1$	1
1	2	3	4

Next, we verify (b). If x has even degree, then $\deg x = 6 + 2k$ where $k \geq 0$. The following coloring of the edges incident with x (labeled Type 3a) has the desired properties and the vertex x is referred to as a *Type* $3a$ *vertex*. If x has odd degree, then $\deg x = 7 + 2k$ where $k \geq 0$. The following coloring of the edges incident with x (labeled Type 3b) has the desired properties and the vertex x is referred to as a *Type* $3b$ *vertex*.

Type 3a:

0	$k+3$	0	$k+3$
1	2	3	4

Type 3b:

1	$k+2$	0	$k+4$
1	2	3	4

Therefore, (a) and (b) hold. $\square$

Theorem 3.2. *If T is a caterpillar with $\delta^*(T) \geq 6$, then $\mu(T) = 3$ or $\mu(T) = 4$.*

Proof. By Proposition 1.1, it suffices to show that $\mu(T) \leq 4$. Let $(v_1, v_2, \ldots, v_d)$ be the spine of T. Since the statement is true if T is a star, we may assume that $d \geq 2$. With the aid of Lemma 3.2, we construct a proper mean coloring c of T such that $\mu(c) = 4$.

First, we color the edges incident with v_1 so that v_1 is a Type 2a vertex if v_1 has even degree or a Type 2c vertex if v_1 has odd degree where v_1v_2 is colored 2. Thus, $\mathrm{cm}(v_1) = 2$ and no leaf incident with v_1 is colored 2. Next, we color the remaining $\deg v_2 - 1$ edges incident with v_2 so that v_2 is a Type 3a vertex if $\deg v_2$ is even or color these edges so that v_2 is a Type 3b vertex if $\deg v_2$ is odd. If $d \geq 3$, then v_2v_3 is colored 2. Thus, $\mathrm{cm}(v_2) = 3$ and no leaf incident with v_2 is colored 3.

We now proceed to v_3 if $d \geq 3$. First, suppose that $d = 3$. Since $c(v_2v_3) = 2$ and v_3 is adjacent to $\deg v_3 - 1$ leaves, we color the edges incident with v_3 so that v_3 is a Type 2a vertex if v_3 has even degree or a Type 2c vertex if v_3 has odd degree. Thus, $\mathrm{cm}(v_3) = 2$ and no leaf incident with v_3 is colored 2. Next, suppose that $d \geq 4$. We color the remaining $\deg v_3 - 1$ edges incident with v_3 so that v_3 is a Type 2b vertex if v_3 has even degree or a Type 2d vertex if v_3 has odd degree. If $d \geq 4$, then v_3v_4 is colored 2. Thus, $\mathrm{cm}(v_3) = 2$ and no leaf incident with v_3 is colored 2.

We now proceed to v_4 if $d \geq 4$. Since $c(v_3v_4) = 2$ and $\mathrm{cm}(v_3) = 2$, we color the remaining $\deg v_4 - 1$ edges incident with v_4 so that v_2 is a Type 3a vertex if $\deg v_2$ is even or color these edges so that v_2 is a Type 3b vertex if $\deg v_2$ is odd so that $\mathrm{cm}(v_4) = 3$ and no leaf incident with v_4 is colored 3. Furthermore, if $d \geq 5$, we color the edge v_4v_5 by 2.

In general, if i is odd and $5 \leq i \leq d$, then we color the remaining $\deg v_i - 1$ edges incident v_i in the same manner as the coloring of the edges incident with v_3; while if i is even and $6 \leq i \leq d$, then we color the remaining $\deg v_i - 1$ edges incident v_i in the same manner as the coloring of the edges incident with v_4. Proceeding in this manner, we arrive at a proper mean coloring c of T with $\mu(c) = 4$. Therefore, $\mu(T) \leq 4$. □

If the caterpillar T being considered has small diameter, then it can be shown that $\mu(T) \leq 4$ regardless of the non-leaf minimum degree of T.

Theorem 3.3. *If T is a caterpillar of diameter 4, then $\mu(T) = 3$ or $\mu(T) = 4$.*

Proof. Let T be a caterpillar of diameter 4 whose spine is (u, v, w). Since $\mu(T) \geq 3$ by Proposition 1.1, we show that $\mu(T) \leq 4$. We may assume that $2 \leq \deg u \leq \deg w$. We consider two cases, according to the parities of the degrees of u and w.

Case 1. Both $\deg u$ and $\deg w$ are odd. First, suppose that T is one of the trees T' and T'' shown in Figure 2. Since each of T' and T'' has a proper mean coloring with proper mean index 3 (as shown in Figure 2), it follows that $\mu(T) = 3$ if $T \in \{T', T''\}$. Next, suppose that $T \notin \{T', T''\}$. Then T contains either T' or T'' as a subtree. We show that the proper mean coloring of T' or of T'' in Figure 2 can be extended to a proper mean coloring c of T such that $\mu(c) = 3$.

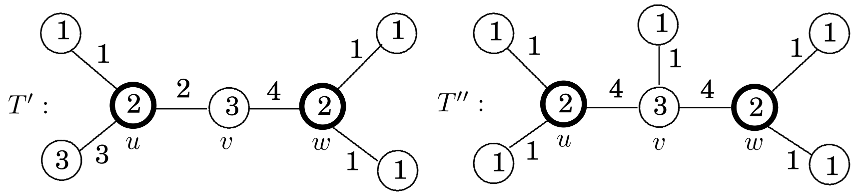

Figure 2: Proper mean colorings of T' and T''.

- ⋆ If $\deg v \geq 4$ is even, say $\deg v = 2 + 2k$ for some positive integer k, then we begin with the coloring of T' and color each of the additional k pairs of pendant edges at v by 2 and 4. If $\deg v \geq 5$ is odd, say $\deg v = 3 + 2k$ for some positive integer k, then we begin with the coloring of T'' and color each of the additional k pairs of pendant edges at v by 2 and 4.

- ⋆ If $\deg u \geq 5$ or $\deg w \geq 5$, say $\deg u = 3 + 2\ell$ for some positive integer ℓ, then we begin with the coloring of T' (if $\deg v$ is even) or the coloring of T'' (if $\deg v$ is odd) and color each of the additional ℓ pairs of pendant edges at u by 1 and 3.

Since the resulting coloring c of T is a proper mean coloring with $\mu(c) = 3$, it follows that $\mu(T) = 3$ if both $\deg u$ and $\deg w$ are odd.

Case 2. At least one of $\deg u$ and $\deg w$ is even, say $\deg u$ is even. There are two subcases, according to whether $\deg u = 2$ or $\deg u \geq 4$.

Subcase 2.1. $\deg u = 2$. First, suppose that T is one of the seven caterpillars $T_1, T_2, \ldots, T_7$ of diameter 4 shown in Figure 3. Since each of these seven caterpillars has a proper mean coloring with proper mean index at most 4 (as shown in Figure 3), it follows that $\mu(T_i) \leq 4$ for $1 \leq i \leq 7$.

Next, suppose that $T \neq T_i$ for $1 \leq i \leq 7$. Then T contains T_i as a subtree for some $i \in [7]$. We show that the proper mean coloring c_i of T_i in Figure 3 can be extended to a proper mean coloring c of T such that $\mu(c) = \mu(c_i)$.

- ⋆ Suppose that $\deg v$ and $\deg w$ are both even. Then $\deg v = 2 + 2k$ and $\deg w = 2 + 2\ell$ for some nonnegative integers k and ℓ. Since $T \neq T_1$, it follows that $\max\{k, \ell\} \geq 1$. Beginning with the coloring of T_1, we color each of the additional k pairs of pendant edges at v (not in T_1) by 2 and 4 and color each of the additional ℓ pairs of pendant edges at w (not in T_1) by 1 and 3.

- ⋆ Suppose that $\deg v$ and $\deg w$ are both odd. Then $\deg v = 3 + 2k$ and $\deg w = 3 + 2\ell$ for some nonnegative integers k and ℓ with $\max\{k, \ell\} \geq 1$ (since $T \neq T_6$). Beginning with the coloring of T_6, we color each of the additional k pairs of pendant edges at v (not in T_6) by 2 and 4 and color each of the additional ℓ pairs of pendant edges at w (not in T_6) by 1 and 3.

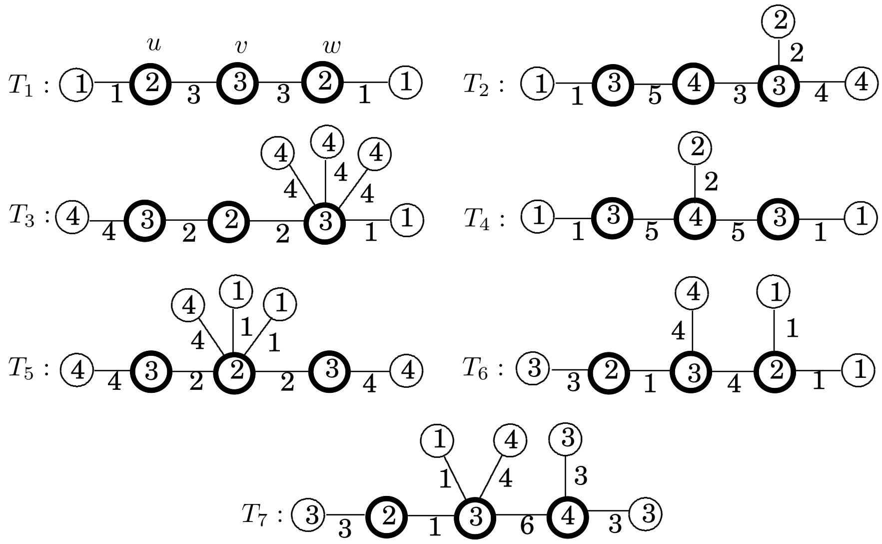

Figure 3: Proper mean colorings of T_i for $1 \le i \le 7$.

- ⋆ Suppose that $\deg v$ is even and $\deg w$ is odd. Then $\deg v = 2 + 2k$ and $\deg w = 3 + 2\ell$, where $k \ge 1$ and $\ell \ge 0$. If $\ell = 0$, then we may assume that $k \ge 2$ (since $T \ne T_2, T_7$). Beginning with the coloring of T_7, we color each of the additional $k-1$ pairs of pendant edges at v (not in T_7) by 2 and 4. If $\ell \ge 1$, then we begin with the coloring of T_3, color each of the additional k pairs of pendant edges at v (not in T_3) by 1 and 3, and color each of the additional $\ell - 1$ pairs of pendant edges at w (not in T_3) by 2 and 4.

- ⋆ Suppose that $\deg v$ is odd and $\deg w$ is even. Then $\deg v = 3 + 2k$ and $\deg w = 2 + 2\ell$ where $k, \ell \ge 0$. If $k = 0$, then we may assume that $\ell \ge 1$ (since $T \ne T_4$). Beginning with the coloring of T_4, we color each of the additional ℓ pairs of pendant edges at w (not in T_4) by 2 and 4. If $k \ge 1$, then we begin with the coloring of T_5, color each of the additional $k-1$ pairs of pendant edges at v (not in T_5) by 1 and 3, and color each of the additional ℓ pairs of pendant edges at w (not in T_5) by 2 and 4.

In each situation, the resulting coloring c is a proper mean coloring of T with $\mu(c) \le 4$.

Subcase 1.2. $\deg u \ge 4$ *is even*. Let $\deg u = 2 + 2p$ for some positive integer p. Then T is obtained from a caterpillar T_0 of diameter 4 of Subcase 2.1 by adding $2p$ pendant edges at u. We begin with the coloring c_0 of T_0 as described in Subcase 2.1. Then $\mathrm{cm}_{c_0}(u) \in \{2, 3\}$. If $\mathrm{cm}_{c_0}(u) = 2$, then we color each of the additional p pairs of the pendant edges at u (not in T_0) by 1 and 3; while if $\mathrm{cm}_{c_0}(u) = 3$, then we color each of the additional p pairs of the pendant edges at u (not in T_0) by 2 and 4. In each case, the resulting coloring c is a proper mean coloring of T with $\mathrm{cm}(c) = \mathrm{cm}(c_0) \le 4$. □

For caterpillars of diameter 3 (that is double stars), the proper mean index has been determined exactly. Since the proof uses an approach similar to that employed above, we state this result without proof. For integers a and b with $2 \le a \le b$, let $S_{a,b}$ denote the double star of order $a + b$ whose central vertices have degrees a and b.

Theorem 3.4. *If a and b are integers with $2 \le a \le b$, then* $\mu(S_{a,b}) = \begin{cases} 3 & \text{if } a \ne b \\ 4 & \text{if } a = b. \end{cases}$

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