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A note on the proper mean indexes of subdivided stars

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Abstract

For an edge coloring c of a connected graph G of order 3 or more with positive integers, the chromatic mean of a vertex v of G is the sum of the colors of the edges incident with v divided by the degree of v. If the chromatic mean of every vertex of G is a positive integer, then c is a mean coloring of G. If adjacent vertices have distinct chromatic means, then c is a proper mean coloring of G. The maximum vertex color in a proper mean coloring c of G is the proper mean index of c and the proper mean index of G is the minimum proper mean index among all proper mean colorings of G. It is conjectured that the proper mean index of every tree of order 3 or more is either 3 or 4. In this note, we verify this conjecture for all trees having exactly one vertex of degree 3 or more, that is, trees that are obtained by subdividing the edges of a star of order 4 or more and characterize those subdivided stars having proper mean index 3 (equivalently 4).

Keywords: proper mean colorings; proper mean index; subdivided stars.

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1. Introduction

For every connected graph G of order 3 or more, there are edge colorings c with positive integers that induce an integer vertex coloring cm defined for each vertex v of G by

$$\mathrm{cm}(v) = \frac{\sum_{e\in E_v} c(e)}{\deg v}, \text{ where } E_v \text{ is the set of edges incident with } v.$$

Edge colorings with this property are called *mean colorings*. The induced vertex color $\mathrm{cm}(v)$ of a vertex v of G is called the *chromatic mean* of v. Consequently, only edge colorings c are considered for which $\mathrm{cm}(v)$ is a positive integer for every vertex v of G. If distinct vertices have distinct chromatic means, then the edge coloring c is called a *rainbow mean coloring* of G. This concept was introduced and studied in [1], where it was shown that every connected graph of order 3 or more has a rainbow mean coloring. The maximum vertex color in a rainbow mean coloring c is the *rainbow mean index* $\mathrm{rm}(c)$ of c and the minimum rainbow mean index among all rainbow mean colorings of G is the *rainbow mean index* $\mathrm{rm}(G)$ of G. A mean coloring of a connected graph G of order 3 or more is a *proper mean coloring* of G if every two adjacent vertices of G have distinct chromatic means. The maximum vertex color in a proper mean coloring c is the *proper mean index* $\mu(c)$ of c and the minimum proper mean index among all proper mean colorings of G is the *proper mean index* $\mu(G)$ of G. This concept was introduced and studied in [2].

Since every connected graph of order 3 or more has a rainbow mean coloring, each such graph has a proper mean coloring as well. Furthermore, the proper mean index of a graph G is at least its chromatic number $\chi(G)$. Therefore, $\chi(G) \le \mu(G) \le \mathrm{rm}(G)$ for every connected graph G of order at least 3. The proper mean index was determined for all graphs belonging to some well-known classes of graphs in [2], including complete graphs, cycles, stars, double stars, and paths. For all connected graphs that have been studied, the proper mean index of a graph has not exceeded its chromatic number by more than 2. This led to the following conjecture stated in [2].

Conjecture 1.1. *For every connected graph G of order 3 or more,*

$$\chi(G) \le \mu(G) \le \chi(G) + 2.$$

In the case of trees, Conjecture 1.1 is stated as follows.

Conjecture 1.2. *For every tree T of order 3 or more, $\mu(T) = 3$ or $\mu(T) = 4$.*

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In [2], Conjecture 1.2 was verified for several classes of trees. In particular, those paths or stars having proper mean index 3 (equivalently 4) were characterized.

Theorem 1.1. *For each integer $n \geq 3$, $\mu(P_n) = \begin{cases} 3 & \text{if } n \text{ is odd} \\ 4 & \text{if } n \text{ is even.} \end{cases}$*

Theorem 1.2. *For each integer $k \geq 2$, $\mu(K_{1,k}) = \begin{cases} 3 & \text{if } k \text{ is even} \\ 4 & \text{if } k \text{ is odd.} \end{cases}$*

In this note, we first show that Conjecture 1.2 holds for all trees having exactly one vertex of degree 3 or more, that is, trees that are obtained by subdividing the edges of a star of order 4 or more and then present a characterization of those subdivided stars having proper mean index 3 (equivalently 4). We refer to the book [3] for graph theory notation and terminology not described in this paper.

2. Subdivided stars

A *subdivided star* is a tree obtained from a star by subdividing the edges of the star in any manner. First, we show that Conjecture 1.2 is true for subdivided stars of order 5 or more. In order to present this fact, we present two preliminary results, the first of which appeared in [2]. For a positive integer n, it is useful to let $[n] = \{1, 2, \dots, n\}$. We also write $\mathbb{N}$ for the set of all positive integers.

Proposition 2.1. *If G is a connected graph of order at least 3, then $\mu(G) \geq 3$. Furthermore, if c is a proper mean coloring of G with $\mu(c) = 3$, then $\{\mathrm{cm}(v) : v \in V(G)\} = [3]$.*

Lemma 2.1. *If P_n is a path of order $n \geq 3$, then there is a proper mean coloring c of P_n with $\mu(c) \leq 4$ and the chromatic mean of at least one end-vertex of P_n is 3.*

Proof. For an edge coloring c of a path $P_n = (v_1, v_2, \dots, v_n)$ of order $n \geq 3$, let

$$\mathcal{S}_c(P_n) = (c(v_1v_2), c(v_2v_3), \dots, c(v_{n-1}v_n))$$

be the color sequence of c and let

$$\mathcal{S}_{\mathrm{cm}}(P_n) = (\mathrm{cm}(v_1), \mathrm{cm}(v_2), \dots, \mathrm{cm}(v_n))$$

be the color sequence of the vertex coloring cm induced by c. We consider two cases based on the parity of n.

Case 1. *n is odd.* We consider two subcases, according to whether $n \equiv 1 \pmod 4$ or $n \equiv 3 \pmod 4$.

Subcase 1.1. $n \equiv 1 \pmod 4$. Thus, $n \geq 5$ and $|E(P_n)| \equiv 0 \pmod 4$. Let a proper mean coloring $c : E(P_n) \to \mathbb{N}$ be given by the color sequence

$$\mathcal{S}_c(P_n) = (3, 1, 1, 3, 3, 1, 1, 3, \dots, 3, 1, 1, 3).$$

The vertex coloring cm induced by c is given by the sequence

$$\mathcal{S}_{\mathrm{cm}}(P_n) = (3, 2, 1, 2, 3, 2, 1, 2, 3, \dots, 2, 1, 2, 3).$$

Therefore, $\mu(c) = 3$ and $\mathrm{cm}(v_1) = 3$ as desired.

Subcase 1.2. $n \equiv 3 \pmod 4$. Thus, $n \geq 3$ and $|E(P_n)| \equiv 2 \pmod 4$. Let a proper mean coloring $c : E(P_n) \to \mathbb{N}$ be given by the color sequence

$$\mathcal{S}_c(P_n) = (3, 1, 1, 3, 3, 1, 1, 3, 3, 1 \dots, 1, 3, 3, 1).$$

The vertex coloring cm induced by c is given by the sequence

$$\mathcal{S}_{\mathrm{cm}}(P_n) = (3, 2, 1, 2, 3, 2, 1, 2, 3, 2, 1 \dots, 2, 3, 2, 1).$$

Therefore, $\mu(c) = 3$ and $\mathrm{cm}(v_1) = 3$ as desired.

Case 2. *n is even.* We consider two subcases, according to whether $n \equiv 0 \pmod 4$ or $n \equiv 2 \pmod 4$.

Subcase 2.1. $n \equiv 0 \pmod 4$. Thus, $n \geq 4$ and $|E(P_n)| \equiv 3 \pmod 4$. Let a proper mean coloring $c : E(P_n) \to \mathbb{N}$ be given by the color sequence

$$\begin{aligned} \mathcal{S}_c(P_4) &= (3, 5, 1) \\ \mathcal{S}_c(P_n) &= (3, 5, 1, 1, 3, 3, 1, 1, 3, 3, 1 \ldots, 1, 3, 3, 1) \text{ if } n \geq 8. \end{aligned}$$

The vertex coloring cm induced by c is given by the sequence

$$\begin{aligned} \mathcal{S}_{\mathrm{cm}}(P_4) &= (3, 4, 3, 1) \\ \mathcal{S}_{\mathrm{cm}}(P_n) &= (3, 4, 3, 1, 2, 3, 2, 1, \ldots, 2, 3, 2, 1) \text{ if } n \geq 8. \end{aligned}$$

Therefore, $\mu(c) = 4$ and $\mathrm{cm}(v_1) = 3$ as desired.

Subcase 2.2. $n \equiv 2 \pmod 4$. Thus, $n \geq 6$ and $|E(P_n)| \equiv 1 \pmod 4$. Let a proper mean coloring $c : E(P_n) \to \mathbb{N}$ be given by the color sequence

$$\begin{aligned} \mathcal{S}_c(P_6) &= (3, 5, 1, 1, 3) \\ \mathcal{S}_c(P_n) &= (3, 5, 1, 1, 3, 3, 1, 1, 3, \ldots, 3, 1, 1, 3) \text{ if } n \geq 10. \end{aligned}$$

The vertex coloring cm induced by c is given by the sequence

$$\begin{aligned} \mathcal{S}_{\mathrm{cm}}(P_6) &= (3, 4, 3, 1, 2, 3) \\ \mathcal{S}_{\mathrm{cm}}(P_n) &= (3, 4, 3, 1, 2, 3, 2, 1, 2, 3 \ldots, 2, 1, 2, 3) \text{ if } n \geq 10. \end{aligned}$$

Therefore, $\mu(c) = 4$ and $\mathrm{cm}(v_1) = 3$ as desired. □

Let T be a subdivision of a star $K_{1,k}$ of order $k + 1 \geq 4$ where v is the central vertex of $K_{1,k}$. Therefore, $\deg_T(v) = k$. A path P in T is a *v-path* if P is a $v - w$ path for some end-vertex w of T. Then there are k paths $P_1, P_2, \ldots, P_k$ that are v-paths of T. We are now prepared to verify Conjecture 1.2 for all subdivided stars of order 3 or more.

Theorem 2.1. *If T is a subdivided star of order 5 or more, then $\mu(T) = 3$ or $\mu(T) = 4$.*

Proof. Let T be the tree obtained from the star $K_{1,k}$, $k \geq 3$, by subdividing at least one edge of $K_{1,k}$. Suppose, in constructing the tree T, that r edges of $K_{1,k}$ are subdivided and s edges of $K_{1,k}$ are not subdivided, where then $r \geq 1$, $s \geq 0$, and $r + s = k$. We show that there is a proper mean coloring c of T with $\mu(c) \leq 4$.

Let v be the central vertex of $K_{1,k}$, let $U = \{v_1, v_2, \ldots, v_r\}$ be the set of vertices adjacent to v with degree at least 2 in T, and let $W = \{w_1, w_2, \ldots, w_s\}$ be the set of end-vertices of T adjacent to v. We consider two cases.

Case 1. *s is even.* We define a proper mean coloring $c : E(T) \to \mathbb{N}$ by $c(vw_i) = 2$ if i is even, $c(vw_i) = 4$ if i is odd, and color the edges of each v-path of length at least 2 using the coloring defined in Lemma 2.1, with $c(vv_i) = 3$ for all i where $1 \leq i \leq r$. It follows that $\mathrm{cm}(v) = 3$ and $\mathrm{cm}(w_i) \in \{2, 4\}$, which implies that cm is a proper vertex coloring of T using colors in the set $[4]$ and so $\mu(c) \leq 4$.

Case 2. *s is odd.* There are two subcases, according to whether $s = 1$ or $s \geq 3$.

Subcase 2.1. $s = 1$. A proper mean coloring $c : E(T) \to \mathbb{N}$ is defined as follows. First, let $c(vw_1) = 1$. Consider a v-path P of length at least 2 where $vv_1 \in E(P)$. Let $c(vv_1) = 5$. If P has even length, then let $c(v_1v_2) = 3$ and iteratively color the remaining edges of this v-path by alternating between the color sequences $(3, 1)$ and $(1, 3)$. If P has odd length, then iteratively color the remaining edges starting with $e = v_1v_2$ by alternating between the color sequences $(3, 1)$ and $(1, 3)$. Color the edges of any remaining v-paths using the coloring described in Lemma 2.1 so that $c(vv_i) = 3$ for $2 \leq i \leq r$. Then $\mathrm{cm}(v) = 3$, $\mathrm{cm}(w_1) = 1$, $\mathrm{cm}(v_1) = 4$, and the vertices of each v-path are colored properly by the vertex coloring cm induced by c. It follows that cm is a proper vertex coloring of T using colors in the set $[4]$, implying that $\mu(c) \leq 4$.

Subcase 2.2. $s \geq 3$. A proper mean coloring $c : E(T) \to \mathbb{N}$ is defined by $c(vw_2) = 1$, $c(vw_i) = 4$ if i is odd, $c(vw_i) = 2$ if i is even for $i \geq 4$, and coloring the edges of each v-path of length at least 2 using the coloring defined in Lemma 2.1, with $c(vv_i) = 3$ for all i where $1 \leq i \leq r$. It follows that $\mathrm{cm}(v) = 3$ and $\mathrm{cm}(w_i) \in \{1, 2, 4\}$, which implies that cm is a proper vertex coloring of T using colors in the set $[4]$ and so $\mu(c) \leq 4$. □

3. A characterization

In this section, we characterize those subdivided stars with central vertex v having proper mean index 3 (equivalently 4) by examining the lengths of the v-paths. First, we present some additional definitions. A path is an *even path* if its length is even; while a path is an *odd path* if its length is odd. An edge e is a *central edge* of T if e is incident to the central vertex v. The following lemma will be useful.

Lemma 3.1. *Let T be a subdivided star whose central vertex v has degree at least 3. If c is a proper mean coloring of T with $\mu(c) = 3$, then $c(e) \in [5]$ for every edge e of T. Furthermore, the chromatic mean $\mathrm{cm}(v)$ of v satisfies the following conditions.*

(*a*) *If $\mathrm{cm}(v) = 1$, then $c(e) = 1$ for each central edge e and every v-path is even.*

(*b*) *If $\mathrm{cm}(v) = 3$, then $c(e) = 3$ for each central edge e and every v-path is even.*

(*c*) *Let $\mathrm{cm}(v) = 2$ and e the central edge on a v-path P of T.*

- $\star$ *If $c(e) \in \{1, 3\}$, then P is odd.*
- $\star$ *If $c(e) = 4$, then P has length 2.*
- $\star$ *If $c(e) = 5$, then there is no restriction on the length of P.*

Proof. First, we show that if c is a proper mean coloring of T with $\mu(c) = 3$, then $c(e) \in [5]$ for every edge e of T. Assume, to the contrary, that $c(e) \geq 6$ for some edge e of T. Let $e = uv$, where $\deg u \geq 2$. If v is an end-vertex, then $\mathrm{cm}(v) = c(e) \geq 6$, which is impossible. Thus, we may assume that e is adjacent to the edge $f = vw$. Since $c(f) \geq 2$, it follows that $\mathrm{cm}(v) \geq 4$, which again is impossible. Therefore, $c(e) \in [5]$ for every edge e of T.

For the statements in (a), (b), and (c), we only consider the case (c) where $\mathrm{cm}(v) = 2$ and $c(e) = 4$ and $e = vx$ is the central edge of a v-path P since the arguments for other cases are straightforward. Since $c(e) = 4$, it follows that x cannot be an end-vertex of T. Thus, P contains an edge $f = xy$ adjacent to e. Necessarily, $c(f) = 2$ and $\mathrm{cm}(x) = 3$. Suppose that P contains an edge $g = yz$ adjacent to f. Since $\mathrm{cm}(x) = 3$, it follows that $c(g) = 2$. Because $2 = \mathrm{cm}(y) \neq \mathrm{cm}(z)$, there is an edge $h = zw$ on P adjacent to g. Thus, $c(h) \geq 4$. Since $\mu(c) = 3$, it follows that $c(h) = 4$ and $\mathrm{cm}(z) = 3$. If w is an end-vertex, then $\mathrm{cm}(w) = 4$, which is impossible. If w is not an end-vertex, then w is adjacent to another vertex w' and so $c(ww') \geq 2$. However, $3 = \mathrm{cm}(z) \neq \mathrm{cm}(w)$ and $c(h) = 4$, it follows that $\mathrm{cm}(w) \geq 4$, which again is impossible. Therefore, the v-path P containing e has length 2. □

We are now prepared to characterize all subdivided stars having proper mean index 3.

Theorem 3.1. *Let T be a subdivision of the star $K_{1,k}$ of order $k+1 \geq 4$ and let v be the central vertex of $K_{1,k}$. Furthermore, let q be the number of odd v-paths, r the number of v-paths of length 2, and s the number of even v-paths of length at least 4. Then $\mu(T) = 3$ if and only if $q = 0$ or there exists an integer z with $2r + 3s \leq z \leq 3r + 3s$ such that $q - z$ is a nonnegative even number.*

Proof. First, assume that $\mu(T) = 3$. Consider a proper mean coloring $c : E(T) \to \mathbb{N}$ with $\mu(c) = 3$. We consider three cases, according to the chromatic mean of the central vertex v of T.

Case 1. $\mathrm{cm}(v) = 1$. Then $c(e) = 1$ for every central edge e of T and so every v-path in T is of even length by Lemma 3.1. Thus, $q = 0$.

Case 2. $\mathrm{cm}(v) = 3$. Here, we claim that $c(e) = 3$ for every central edge e of T. Suppose that there exists a central edge $f = uv$ with $c(f) \geq 4$. If u is an end-vertex, then $\mathrm{cm}(u) \geq 4$, which is impossible. Thus, the v-path P containing f has length 2 or more. Let g be the edge adjacent to f on P. Since $c(f)$ and $c(g)$ are of the same parity and $c(f) \geq 4$, it follows that $\mathrm{cm}(u) \geq 3$, which is impossible. Hence, $c(e) \leq 3$ for every central edge e of T. Since $\mathrm{cm}(v) = 3$, we have $c(e) = 3$ for every central edge e of T, as claimed. Consequently, every v-path in T is of even length by Lemma 3.1. Thus, $q = 0$.

Case 3. $\mathrm{cm}(v) = 2$. First, we claim that there is no central edge e of T with $c(e) = 2$, for suppose that T contains such a central edge $f = vx$ of T. Then the vertex x cannot be an end-vertex for otherwise $\mathrm{cm}(x) = \mathrm{cm}(v) = 2$. Hence, the v-path containing f has an edge $g = xy$ adjacent to f. Necessarily, $c(g)$ is even and $c(g) \geq 4$. Since $\mathrm{cm}(x) \leq 3$, it follows that $c(g) = 4$, which implies that $\mathrm{cm}(x) = 3$. This, in turn, implies that $\mathrm{cm}(y) \geq 3$, which is impossible. Therefore, no central edge of T is colored 2, as claimed.

Consequently, $c(e) \in \{1, 3, 4, 5\}$ for every central edge e of T. Since $\mathrm{cm}(v) = 2$ and $c(e) \neq 2$ for every central edge e of T, some central edges of T must be colored 1. It then follows by Lemma 3.1 that there are odd v-paths in T and so $q \neq 0$. Hence, it remains to show that there exists an integer z with $2r + 3s \leq z \leq 3r + 3s$ such that $q - z$ is a nonnegative even number.

Assume, to the contrary, for every integer z with $2r + 3s \leq z \leq 3r + 3s$, that either (1) $q - z < 0$ or (2) $q - z > 0$ is an odd integer. Observe that for every central edge colored 4, there are two central edges colored 1; while for every central edge colored 5, there are three central edges colored 1. Furthermore, by Lemma 3.1, if a central edge e is colored 4, then the

v-path containing e has length 2; while if a central edge e is colored 5, then the v-path containing e is an even path. Next, let $Q_1, Q_2, \ldots, Q_a$ be all even v-paths of T and let e_i be the central edge of Q_i for $1 \le i \le a$. Consequently, if

$$z = \sum_{i=1}^{a} [c(e_i) - 2] = \left[\sum_{i=1}^{a} c(e_i)\right] - 2a,$$

then $2r + 3s \le z \le 3r + 3s$.

(1) If $q - z < 0$ or $q < z$, then $\mathrm{cm}(v) > 2$, which is a contradiction.

(2) If $q - z$ is a positive odd integer, then the number of the central edges colored $1, 3, 5$ is odd. This implies that the sum of the colors of all central edges is odd and so $\mathrm{cm}(v) \neq 2$, which is a contradiction.

To verify the converse, suppose that either $q = 0$ or there exists an integer z with $2r + 3s \le z \le 3r + 3s$ such that $q - z$ is a nonnegative even number. We show that there is a proper mean coloring c of T with $\mu(c) = 3$. We consider two cases.

Case 1. $q = 0$. Then every v-path in T has even length. A proper mean coloring c of T with $\mu(c) = 3$ and $\mathrm{cm}(v) = 1$ can be obtained by iteratively coloring the edges of every v-path by alternating between the color sequences $(1, 3)$ and $(3, 1)$. Therefore, $\mu(T) = 3$.

Case 2. $q \neq 0$. Then there exists an integer z with $2r + 3s \le z \le 3r + 3s$ such that $q - z$ is a nonnegative even number. A proper mean coloring of T is constructed by assigning the color sequence $(4, 2)$ to the edges of $3r + 3s - z$ v-paths of length 2 so that the central edges of these paths are colored 4. Next, the edges of every remaining even v-path are colored starting from v with the color sequence $(5, 1)$ (where the central edge is colored 5) and then alternating between the color sequences $(1, 3)$ and $(3, 1)$, respectively, for the remaining edges. Now, the edges of the z odd v-paths are colored starting with 1 on the central edge and then coloring the remaining edges by alternating the color sequences $(1, 3)$ and $(3, 1)$. Consequently, there are $q - z$ uncolored paths remaining. By assumption, $q - z = 2\ell$ for some integer $\ell \ge 0$. The edges of ℓ of the remaining odd v-paths are colored starting with 1 on the central edge and alternating the color sequences $(1, 3)$ and $(3, 1)$ for the remaining edges. The edges of the remaining ℓ odd v-paths are colored starting with 3 on the central edge and alternating the color sequences $(3, 1)$ and $(1, 3)$ for the remaining edges. Consequently,

$$\begin{aligned}
\mathrm{cm}(v) &= \frac{4(3r + 3s - z) + 5(r + s - 3r - 3s + z) + z + 2(q - z)}{q + r + s} \\
&= \frac{12r + 12s - 4z - 10r - 10s + 5z + z + 2q - 2z}{q + r + s} \\
&= \frac{2q + 2r + 2s}{q + r + s} = 2
\end{aligned}$$

and $\mathrm{cm}(w) \in \{1, 3\}$ for all $w \in N(v)$. Furthermore, using this construction, all v-paths are properly colored such that no vertex color exceeds 3. Thus, $\mu(c) = 3$ and so $\mu(G) = 3$. □

By Theorem 2.1, if T is a subdivided star of order 5 or more such that $\mu(T) \neq 3$, then $\mu(T) = 4$. Consequently, Theorem 3.1 is equivalent to the following result.

Theorem 3.2. *Let T be a subdivision of the star $K_{1,k}$ of order $k + 1 \ge 4$ and let v be the central vertex of $K_{1,k}$. Furthermore, let q be the number of odd v-paths, r the number of v-paths of length 2, and s the number of even v-paths of length at least 4. Then $\mu(T) = 4$ if and only if $q \ge 1$ and for each integer z with $2r + 3s \le z \le 3r + 3s$ either $q - z < 0$ or $q - z$ is a positive odd number.*

To illustrate Theorems 3.1 and 3.2, we consider the three subdivided stars in Figure 1.

⋆ For the subdivided star T_1 in Figure 1(a), there are ten odd paths, two paths of length 2, and one even path of length 4. Thus, $q = 10$, $r = 2$, and $s = 1$. So, $2r + 3s = 7$ and $3r + 3s = 9$. If $z = 8$, then $q - z = 2$ is a positive even integer. Consequently, $\mu(T_1) = 3$. We now apply the proper mean coloring described in the proof of Theorem 3.1. We assign the color 1 to $\frac{q+z}{2} = 9$ central edges on odd paths, the color 3 to $\frac{q-z}{2} = 1$ central edge on one odd path, the color 4 to $3r + 3s - z = 1$ central edge on one path of length 2, and the color 5 to $z - 2r - 2s = 2$ central edges on the remaining two even paths. Hence, for each of the two v-paths of length 3, its edge colors are $1, 1, 3$ and the central edge is colored 1; for one v-path of length 2, its edge colors are $4, 2$ and the central edge is colored 4; for the other v-path of length 2, its edge colors are $5, 1$ and the central edge is colored 5; and for the v-path of length 4, its edge colors are $5, 1, 1, 3$ and the central edge is colored 5. This produces a proper mean coloring c of T_1 with $\mu(c) = 3$.

⋆ For the subdivided star T_2 in Figure 1(b), there are seven odd paths, one path of length 2, and two even paths of length 4. Thus, $q = 7$, $r = 1$, and $s = 2$. So, $2r + 3s = 8$ and $3r + 3s = 9$. Thus, either $z = 8$ or $z = 9$. Thus, $q - z = -1$ or $q - z = -2$. Therefore, $\mu(T_3) = 4$ by Theorem 3.2.

⋆ For the subdivided star T_3 in Figure 1(c), there are ten odd paths, no path of length 2, and three even paths of length 4. Thus, $q = 10$, $r = 0$, and $s = 3$. So, $2r + 3s = 9$ and $3r + 3s = 9$. Thus, $z = 9$ and $q - z = 1$ is an odd positive integer. Therefore, $\mu(T_3) = 4$ by Theorem 3.2.

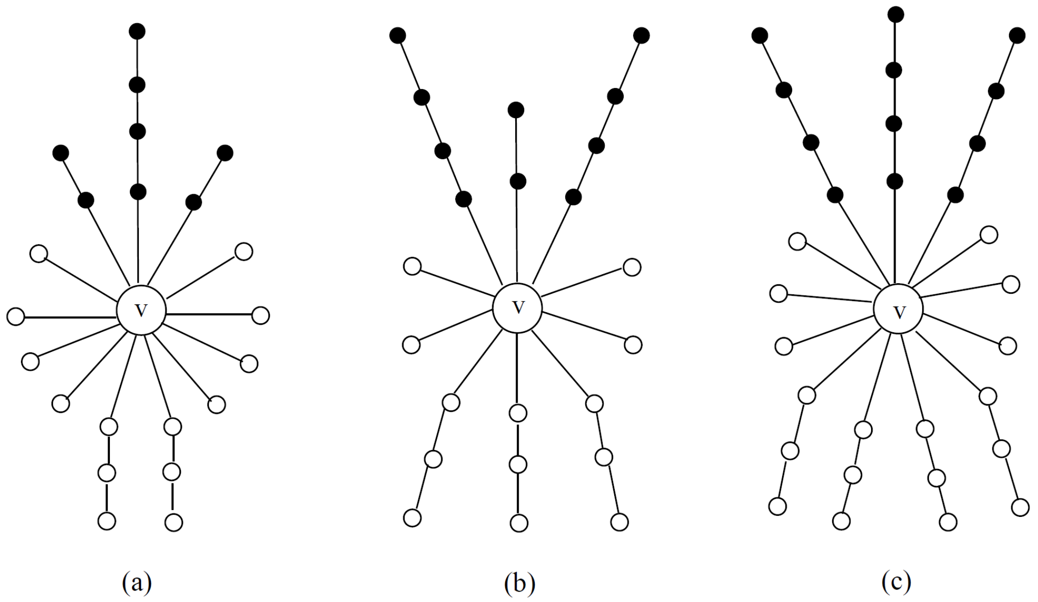

Figure 1: Three subdivided stars.

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