Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **3** (2020) 44–49

The edge-strength of graphs*

Rikio Ichishima[1,†], Akito Oshima[2], Yukio Takahashi[3]

[1]*Department of Sport and Physical Education, Faculty of Physical Education, Kokushikan University, Tama-shi, Tokyo 206-8515, Japan*
[2]*Graph Theory and Applications Research Group, School of Electrical Engineering and Computer Science, Faculty of Engineering and Built Environment, The University of Newcastle, NSW 2308 Australia*
[3]*Department of Science and Engineering, Faculty of Electronics and Informations, Kokushikan University, Setagaya-ku, Tokyo 154-8515, Japan*

(Received: 24 February 2020. Received in revised form: 27 April 2020. Accepted: 27 April 2020. Published online: 1 May 2020.)

Abstract

The optimization problem studied in this paper is, for any graph G of size q, how labels $1, 2, \ldots, q$ can be assigned to the edges of G to minimize the maximum of the additions between the labels of all pairs of adjacent edges. If it is possible for a graph G, then we say that the minimum such number is the edge-strength of G. We provide formulas for the edge-strength of some classes of graphs whose line graphs are defined in terms of various graph operations.

Keywords: edge-strength; strength; line graph; graph operation; graph labeling; combinatorial optimization.

2020 Mathematics Subject Classification: 05C76, 05C78, 90C27.

1. Introduction

Only graphs without loops or multiple edges are considered in this paper. Undefined graph theoretical notation and terminology can be found in [2]. The *vertex set* of a graph G is denoted by $V(G)$, while the *edge set* of G is denoted by $E(G)$. The *complete bipartite graph* with partite sets U and V, where $|U| = m$ and $|V| = n$, is denoted by $K_{m,n}$. The graph $K_{1,n}$ is called a *star*. The graph with n vertices and no edges is referred to as the *empty graph*. The *degree of a vertex* v in a graph G is the number of edges of G incident with v, which is denoted by $\deg v$. A vertex of degree 1 is called an *end-vertex* of G. The *minimum degree* of G is the minimum degree among the vertices of G and is denoted by $\delta(G)$. A graph G is *regular* of degree r if $\deg v = r$ for each $v \in V(G)$. Such graphs are called r-*regular*. An r-regular subgraph H of a graph G is an r-*factor* of G if $V(H) = V(G)$.

An extensive survey on graph labeling problems as well as their applications has been given by Gallian [8]. Among all graph labeling problems, bandwidth numbering of graphs has perhaps attracted the most attention in the literature. The bandwidth numbering was independently proposed by Harary [9] and Harper [10]. The motivation for these numberings came from the study of sparse matrix computations, representing data structures by linear arrays, VLSI layouts and mutual simulations of interconnection networks (see [3, 4, 15]).

For the sake of notational convenience, we will denote the interval of integers k such that $i \le k \le j$ by simply writing $[i, j]$. A *numbering* f of a graph G of order p is a labeling that assigns distinct elements of the set $[1, p]$ to the vertices of G. The *bandwidth* band(G) of G is

$$\text{band}(G) = \min\{\text{band}_f(G) \,|\, f \text{ is a numbering of } G\},$$

where

$$\text{band}_f(G) = \max\left\{|f(u) - f(v)| \,\middle|\, uv \in E(G)\right\}.$$

Additive analogues for bandwidth numberings of graphs have been introduced in [11] as a generalization of the problem of finding whether a graph is super edge-magic or not (see [6] for the definition of a super edge-magic graph, and also consult [1, 7] for alternative and often more useful definitions of the same concept). The *strength* str(G) of G is

$$\text{str}(G) = \min\{\text{str}_f(G) \,|\, f \text{ is a numbering of } G\},$$

where

$$\text{str}_f(G) = \max\{f(u) + f(v) \,|\, uv \in E(G)\}.$$

*Dedicated to the memory of Emeritus Professor Haruo Kawasaki
†Corresponding author (ichishim@kokushikan.ac.jp)

If G is an empty graph, then $\mathrm{str}\,(G)$ is undefined (or we could define $\mathrm{str}\,(G) = +\infty$).

Several bounds for the strength have been found in terms of other invariants defined on graphs (see [11, 14]). Among others, the following result that provides a lower bound for the strength of a graph in terms of its order and minimum degree has been proven to be useful (see [11–14]).

Lemma 1.1. *For every graph G of order p with $\delta(G) \geq 1$,*

$$\mathrm{str}\,(G) \geq p + \delta\,(G)\,.$$

The strengths for the paths P_n, the cycles C_n, the complete graphs K_n of order n and the 1-regular graphs nP_2 were determined in [11]. These classes of graphs illustrate the sharpness of the bound given in Lemma 1.1.

Lemma 1.2.

(i) *For every integer $n \geq 2$,* $\mathrm{str}\,(P_n) = n + 1$.
(ii) *For every integer $n \geq 3$,* $\mathrm{str}\,(C_n) = n + 2$.
(iii) *For every integer $n \geq 2$,* $\mathrm{str}\,(K_n) = 2n - 1$.
(iv) *For every positive integer n,* $\mathrm{str}\,(nP_2) = 2n + 1$.

In this paper, we study the following problem that concerns labeling the edges of a graph in terms of its size rather than its order. An *edge numbering* f of a graph G of size q is a labeling that assigns distinct elements of the set $[1, q]$ to the edges of G. The *edge-strength* $\mathrm{estr}(G)$ of G is defined by

$$\mathrm{estr}\,(G) = \min\left\{\mathrm{estr}_f\,(G)\,|f \text{ is an edge numbering of } G\right\},$$

where

$$\mathrm{estr}_f\,(G) = \max\left\{f\,(e_1) + f\,(e_2)\,|e_1, e_2 \text{ are adjacent edges of } G\right\}.$$

The determination of $\mathrm{estr}\,(G)$ can be transformed into a problem dealing with strengths, namely, from the definitions it is immediate that

$$\mathrm{estr}\,(G) = \mathrm{str}\,(L\,(G)),$$

where $L\,(G)$ is the line graph of G. The *line graph* $L\,(G)$ of a graph G is that graph whose vertices can be put in one-to-one correspondence with the edges of G in such a way that two vertices of $L\,(G)$ are adjacent if and only if the corresponding edges of G are adjacent. The line graph $L\,(G)$ is empty if and only if every component of G is either K_1 or K_2. In such a case, $\mathrm{estr}\,(G)$ is undefined (or we could define $\mathrm{estr}\,(G) = +\infty$).

2. The edge-strength of graphs

In this section, we present formulas for the edge-strength of some classes of graphs whose line graphs are defined in terms of graph operations described next. In the following definitions, we assume that G_1 and G_2 are two graphs with disjoint vertex sets.

The *union* $G \cong G_1 \cup G_2$ has $V\,(G) = V\,(G_1) \cup V\,(G_2)$ and $E\,(G) = E\,(G_1) \cup E\,(G_2)$. If a graph G consists of k disjoint copies of a graph H, then we write $G \cong kH$, where $k \geq 2$.

If G_1 has order p, the *corona* $G_1 \odot G_2$ is the graph obtained by taking one copy of G_1 and p copies of G_2 and joining the ith vertex of G_1 with an edge to every vertex in the ith copy of G_2.

The *join* $G \cong G_1 + G_2$ has $V\,(G) = V\,(G_1) \cup V\,(G_2)$ and

$$E\,(G) = E\,(G_1) \cup E\,(G_2) \cup \{uv\,|u \in V\,(G_1) \text{ and } v \in V\,(G_2)\}\,.$$

The *cartesian product* $G \cong G_1 \times G_2$ has $V\,(G) = V\,(G_1) \times V\,(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E\,(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E\,(G_1)$.

It is evident that $L\,(P_n) \cong P_{n-1}$, $L\,(K_{1,n}) \cong K_n$, $L\,(nP_3) \cong nP_2$ and $L\,(C_n) \cong C_n$. From these facts and Lemma 1.2, the next result is obtained rather easily.

Theorem 2.1.

(i) *For every integer $n \geq 3$,* $\mathrm{estr}\,(P_n) = n$.
(ii) *For every integer $n \geq 2$,* $\mathrm{estr}\,(K_{1,n}) = 2n - 1$.
(iii) *For every positive integer n,* $\mathrm{estr}\,(nP_3) = 2n + 1$.
(iv) *For every integer $n \geq 3$,* $\mathrm{estr}\,(C_n) = n + 2$.

For a corona of two graphs, the following result was established in [12].

Theorem 2.2. *If G is a graph of order p with $\delta(G) \geq 1$ and $\operatorname{str}(G) = p + \delta(G)$, then*

$$\operatorname{str}(G \odot mK_1) = (m+1)\,p + 1$$

for every positive integer m.

The *subdivision graph* $S(G)$ of a graph G is that graph obtained from G by replacing each $uv \in E(G)$ by a vertex w and edges uw and vw. For example, if $G \cong K_{1,n}$ $(n \geq 2)$, then $L(S(G)) \cong K_n \odot K_1$. Since $|V(K_n)| = n$ and $\delta(K_n) = n-1$, it follows from Lemma 1.2 that K_n satisfies the hypothesis of Theorem 2.2. Hence, we have the next result.

Theorem 2.3. *For every integer $n \geq 2$,*

$$\operatorname{estr}(S(K_{1,n})) = 2n + 1.$$

The *complement* $\overline{G}$ of a graph G is that graph with vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if these vertices are not adjacent in G.

To settle the problem, proposed in [11], of finding sufficient conditions for a graph G of order p with $\delta(G) \geq 1$ to satisfy $\operatorname{str}(G) = p + \delta(G)$, the following class of graphs was recursively defined in [13]. Let F_2 be a graph with $V(F_2) = \{v_1, v_2\}$ and $E(F_2) = \{v_1v_2\}$. For $k \geq 2$, let F_{k+1} be the graph with $V(F_{k+1}) = V(F_k) \cup \{v_{k+1}\}$ and

$$E(F_{k+1}) = E(F_k) \cup \{v_iv_j \,|i \in [1, \lceil k/2 \rceil] \text{ and } i + j = k + 2\}.$$

It is now possible to state the next result that is an equivalent version of the characterization found in [13]. This will prove to be useful in our study of the edge-strength of graphs.

Theorem 2.4. *Let G be a graph of order p. Then $\operatorname{str}(G) \geq 2p - k + 1$ if and only if $\overline{G}$ does not contain F_k as a subgraph, where $k \in [2, p-1]$.*

The *double star* $S_{m,n}$ is a tree obtained by joining the centers of two disjoint stars $K_{1,m}$ and $K_{1,n}$ with an edge. A formula for the strength of $S_{m,n}$ was found in [12]. With the aid of Theorem 2.4, it is now possible to determine a formula for $\operatorname{estr}(S_{m,n})$.

Theorem 2.5. *For every two positive integers m and n,*

$$\operatorname{estr}(S_{m,n}) = 2(m+n) - 1.$$

Proof. Let $G \cong S_{m,n}$ and $H \cong L(G)$. Then $H \cong (K_m \cup K_n) + K_1$ so that $\overline{H} \cong K_{m,n} \cup K_1$. Since $\overline{H}$ is a bipartite graph, it follows that $\overline{H}$ cannot contain a cycle of odd length. However, F_4 contains C_3 as a subgraph. Consequently, $\overline{H}$ does not contain F_4 as a subgraph. From Theorem 2.4, it is now immediate that $\operatorname{estr}(G) = \operatorname{str}(H) \geq 2(m+n) - 1$.

To complete the proof, it suffices to show the existence of a numbering f of H with

$$\operatorname{str}_f(H) = 2(m+n) - 1.$$

Define the graph H with

$$V(H) = \{x\} \cup \{y_i \,|i \in [1,m]\} \cup \{z_i \,|i \in [1,n]\}$$

and

$$E(H) = \{xy_i \,|i \in [1,m]\} \cup \{xz_i \,|i \in [1,n]\} \cup \{y_iy_j \,|1 \leq i < j \leq m\} \cup \{z_iz_j \,|1 \leq i < j \leq n\}.$$

Then the labeling $f : V(H) \to [1, m+n+1]$ such that

$$f(v) = \begin{cases} 1 & \text{if } v = x, \\ n + i & \text{if } v = y_i \text{ and } i \in [1,m], \\ m + n + 1 & \text{if } v = z_1, \\ i & \text{if } v = z_i \text{ and } i \in [2,n] \end{cases}$$

has the property that

$$\begin{aligned} \operatorname{str}_f(H) &= \max\{f(u) + f(v) \,|uv \in E(H)\} \\ &= f(y_{m-1}) + f(y_m) \\ &= (m+n-1) + (m+n) = 2(m+n) - 1. \end{aligned}$$

Thus, $\operatorname{str}(H) = 2(m+n) - 1$, implying that $\operatorname{estr}(G) = 2(m+n) - 1$. □

The following corollary is obtained from the proof of the preceding theorem.

Corollary 2.1. *For every two positive integers m and n,*

$$\operatorname{str}((K_m \cup K_n) + K_1) = 2(m+n) - 1.$$

Let $T_{m,n}$ denote the tree obtained from the double star $S_{m,n}$ by subdividing the edge joining the centers of two disjoint stars $K_{1,m}$ and $K_{1,n}$. With this definition in hand, we have the next result.

Theorem 2.6. *For every two positive integers m and n,*

$$\operatorname{estr}(T_{m,n}) = 2(m+n) + 1.$$

Proof. Let $G \cong T_{m,n}$ and $H \cong L(G)$. Then H is the graph obtained by joining a vertex of K_{m+1} and a vertex of K_{n+1} with an edge. This gives a bipartite graph $\overline{H} \cong K_{m+1,n+1} - e$, where $e \in E(K_{m+1,n+1})$. Thus, $\overline{H}$ cannot contain a cycle of odd length. However, F_4 contains C_3 as a subgraph. Consequently, $\overline{H}$ does not contain F_4 as a subgraph. It follows from Theorem 2.4 that $\operatorname{estr}(G) = \operatorname{str}(H) \geq 2(m+n)+1$.

To show that $\operatorname{estr}(G) = 2(m+n)+1$, it suffices to verify the existence of a numbering f of H for which

$$\operatorname{str}_f(H) = 2(m+n)+1.$$

Let

$$V(H) = \{x_i \,|i \in [1, m+1]\} \cup \{y_i \,|i \in [1, n+1]\}$$

and

$$E(H) = \{x_i x_j \,|1 \leq i < j \leq m+1\} \cup \{y_i y_j \,|1 \leq i < j \leq n+1\} \cup \{x_1 y_1\}.$$

Then the labeling $f : V(H) \to [1, m+n+2]$ such that

$$f(v) = \begin{cases} 1 & \text{if } v = x_1, \\ m+n+3-i & \text{if } v = x_i \text{ and } i \in [2, m+1], \\ n+2-i & \text{if } v = y_i \text{ and } i \in [1, n], \\ m+n+2 & \text{if } v = y_{n+1} \end{cases}$$

has the property that

$$\begin{aligned} \operatorname{str}_f(H) &= \max\{f(u) + f(v) \,|uv \in E(H)\} \\ &= f(x_2) + f(x_3) \\ &= (m+n+1) + (m+n) = 2(m+n)+1. \end{aligned}$$

Thus, $\operatorname{str}(H) = 2(m+n)+1$, which implies that $\operatorname{estr}(T_{m,n}) = 2(m+n)+1$. □

The *theta graph* $\Theta(k_1, k_2 \ldots, k_m)$ is the graph consisting of m pairwise internally disjoint paths $P_{k_1}, P_{k_2}, \ldots, P_{k_m}$ with common end-vertices. Theta graphs in which $k_1 = k_2 = \cdots = k_m = n$ are denoted by $\Theta(m; n)$. For $m = 1$ and $m = 2$, the theta graph reduces to a path P_n and a cycle C_{2n-2}, respectively. A formula for the bandwidth of $\Theta(m; 3)$ was determined by Chvatalova and Optrny [5]. In light of their result, it seems natural to explore the edge-strength of $\Theta(m; 3)$ next.

Theorem 2.7. *For every integer $m \geq 3$,*

$$\operatorname{estr}(\Theta(m; 3)) = 4m - 3.$$

Proof. Let $G \cong \Theta(m; 3)$ and $H \cong L(G)$. Then $H \cong K_m \times K_2$, which gives a bipartite graph $\overline{H} \cong K_{m,m} - F$, where F is a 1-factor of $K_{m,m}$. Thus, $\overline{H}$ cannot contain a cycle of odd length. However, F_4 contains C_3 as a subgraph. Consequently, $\overline{H}$ does not contain F_4 as a subgraph. It follows from Theorem 2.4 that $\operatorname{estr}(G) = \operatorname{str}(H) \geq 4m - 3$.

To show that $\operatorname{estr}(G) = 4m - 3$, it suffices to verify the existence of a numbering f of H with $\operatorname{str}_f(H) = 4m - 3$. Define the graph H with

$$V(H) = \{x_i \,|i \in [1, m]\} \cup \{y_i \,|i \in [1, m]\}$$

and

$$E(H) = \{x_i x_j \,|1 \leq i < j \leq m\} \cup \{y_i y_j \,|1 \leq i < j \leq m\} \cup \{x_i y_i \,|i \in [1, m]\}.$$

Then the labeling $f : V(H) \to [1, 2m]$ such that

$$f(v) = \begin{cases} 2m & \text{if } v = x_1, \\ i-1 & \text{if } v = x_i \text{ and } i \in [2, m], \\ m+i-1 & \text{if } v = y_i \text{ and } i \in [1, m] \end{cases}$$

has the property that

$$\begin{aligned} \mathrm{str}_f(H) &= \max\{f(u) + f(v) \,|uv \in E(H)\} \\ &= f(y_{m-1}) + f(y_m) \\ &= (2m-2) + (2m-1) = 4m - 3. \end{aligned}$$

Thus, $\mathrm{str}(H) = 4m - 3$, completing the proof. □

This result also has a rather immediate corollary.

Corollary 2.2. *For every integer $m \geq 3$,*

$$\mathrm{str}(K_m \times K_2) = 4m - 3.$$

The next result provides a formula for $\mathrm{estr}(K_{1,m} \cup K_{1,n})$.

Theorem 2.8. *For every two integers $m \geq 2$ and $n \geq 2$,*

$$\mathrm{estr}(K_{1,m} \cup K_{1,n}) = 2(m+n) - 3.$$

Proof. Let $G \cong K_{1,m} \cup K_{1,n}$ and $H \cong L(G)$. Then $H \cong K_m \cup K_n$. Note that for any two graphs G_1 and G_2, we have $\overline{G_1 \cup G_2} \cong \overline{G_1} + \overline{G_2}$. Applying this with $G_1 \cong K_m$ and $G_2 \cong K_n$, we obtain a complete bipartite graph $\overline{H} \cong \overline{K_m} + \overline{K_n} \cong K_{m,n}$. Thus, $\overline{H}$ cannot contain a cycle of odd length. However, F_4 contains C_3 as a subgraph. Consequently, $\overline{H}$ does not contain F_4 as a subgraph. It follows from Theorem 2.4 that $\mathrm{estr}(G) = \mathrm{str}(H) \geq 2(m+n) - 3$. It remains to show that $\mathrm{str}(H) \leq 2(m+n) - 3$. This can be completed by finding a numbering f of H for which $\mathrm{str}_f(H) = 2(m+n) - 3$. Let H be the graph with

$$V(H) = \{x_i \,|i \in [1, m]\} \cup \{y_i \,|i \in [1, n]\}$$

and

$$E(H) = \{x_i x_j \,|1 \leq i < j \leq m\} \cup \{y_i y_j \,|1 \leq i < j \leq n\}.$$

Then the labeling $f : V(H) \to [1, m+n]$ such that

$$f(v) = \begin{cases} m+n & \text{if } v = x_1, \\ i-1 & \text{if } v = x_i \text{ and } i \in [2, m], \\ m+i-1 & \text{if } v = y_i \text{ and } i \in [1, n] \end{cases}$$

has the property that

$$\begin{aligned} \mathrm{str}_f(H) &= \max\{f(u) + f(v) \,|uv \in E(H)\} \\ &= f(y_{n-1}) + f(y_n) \\ &= (m+n-2) + (m+n-1) = 2(m+n) - 3. \end{aligned}$$

Thus, $\mathrm{str}(H) = 2(m+n) - 3$, which implies that $\mathrm{estr}(G) = 2(m+n) - 3$. □

The following corollary is obtained from the proof of the preceding theorem.

Corollary 2.3. *For every two integers $m \geq 2$ and $n \geq 2$,*

$$\mathrm{str}(K_m \cup K_n) = 2(m+n) - 3.$$

Acknowledgments

The authors are very grateful to Francesc A. Muntaner-Batle whose valuable suggestions resulted in an improved paper. The authors are also greatly indebted to the anonymous referees of this paper for reviewing it carefully.

References

[1] B. D. Acharya, S. M. Hegde, Strongly indexable graphs, *Discrete Math.* **93** (1991) 123–129.

[2] G. Chartrand, L. Lesniak, *Graphs & Digraphs*, Wadsworth & Brooks/Cole Advanced Books and Software, Monterey, 1986.

[3] P. Z. Chinn, J. Chvátalová, A. K. Dewdney, N. E. Gibbs, The bandwidth problem for graphs and matrices - a survey, *J. Graph Theory* **6** (1982) 223–254.

[4] F. R. K. Chung, Labelings of graphs, In: L. W. Beineke, R. J. Wilson (Eds.), *Selected Topics in Graph Theory*, Vol. 3, Elsevier, San Diego, 1988, pp. 151–168.

[5] J. Chvatalova, J. Opatrny, The bandwidth of theta graphs, *Util. Math.* **33** (1988) 9–22.

[6] H. Enomoto, A. Lladó, T. Nakamigawa, G. Ringel, Super edge-magic graphs, *SUT J. Math.* **34** (1998) 105–109.

[7] R. M. Figueroa-Centeno, R. Ichishima, F. A. Muntaner-Batle, The place of super edge-magic labelings among other classes of labelings, *Discrete Math.* **231** (2001) 153–168.

[8] J. A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.* (2019) Art# DS6.

[9] F. Harary, Problem 16, In: M. Ficdler (Ed.), *Theory of Graphs and its Applications*, Czech. Acad. Sci., Prague, 1967, pp. 161–161.

[10] L. H. Harper, Optimal assignments of numbers to vertices, *J. Soc. Indust. Appl. Math.* **12** (1964) 131–135.

[11] R. Ichishima, F. A. Muntaner-Batle, A. Oshima, Bounds for the strength of graphs, *Australas. J. Combin.* **72** (2018) 492–508.

[12] R. Ichishima, F. A. Muntaner-Batle, A. Oshima, The strength of some trees, *AKCE Int. J. Graphs Comb.*, DOI: 10.1016/j.akcej.2019.06.002, In press.

[13] R. Ichishima, F. A. Muntaner-Batle, A. Oshima, Minimum degree conditions for the strength and bandwidth of graphs, *J. Discrete Math. Sci. Cryptogr.*, To appear.

[14] R. Ichishima, F. A. Muntaner-Batle, A. Oshima, Y. Takahashi, The strength of graphs and related invariants, *Memoirs Kokushikan Univ. Inf. Sci.* **41** (2020) 1–8.

[15] Y. L. Lai, K. Williams, A survey of solved problems and applications on bandwidth, edgesum, and profile of graphs, *J. Graph Theory* **31(2)** (1999) 75–94.