Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **3** (2020) 37–43

On upper dimension of graphs and their bases sets

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(Received: 25 February 2020. Received in revised form: 27 March 2020. Accepted: 11 April 2020. Published online: 14 April 2020.)

Abstract

The metric representation of a vertex v with respect to an ordered subset $W = \{w_1, w_2, \cdots, w_n\} \subseteq V(G)$ is an ordered $k-$tuple defined by $r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_n))$, where $d(u, v)$ denotes the distance between the vertices u and v. A subset $W \subseteq V(G)$ is a resolving set if all vertices of G have distinct representations with respect to W. A resolving set of the largest order whose no proper subset resolves all vertices of G is called the upper basis of G and the cardinality of the upper basis is called the upper dimension of G. A vertex v having at least one pendent edge incident on it is called a star vertex and the number of pendent edges incident on a vertex v is called the star degree of v. We determine the upper dimension of certain families of graphs and characterize the cases in which upper dimension equals the metric dimension. For instance, it is shown that metric dimension equals upper dimension for the graphs defined by the Cartesian product of K_n and K_2 and for trees having no star vertices of star degree 1. Further, it is also shown that the upper dimension of a graph equals its metric dimension if the vertex set of G can be partitioned into distance similar equivalence classes.

Keywords: graph; upper dimension; metric dimension; resolving number.

2020 Mathematics Subject Classification: 05C12, 05C38, 05C76, 05C90.

1. Introduction

A graph G consists of a vertex set $V(G)$ and an edge set $E(G)$, where an edge is an unordered pair of distinct vertices of G. The concept of finding the metric dimension of a graph first appeared in 1970's introduced by Slater [15] and independently by Harary and Melter [7]. Since then metric dimension appeared in various applications of graph theory, as diverse as, robot navigation [8], pharmaceutical chemistry [2], combinatorial optimization [14] and sonar and coast guard Loran [15], to name a few. A basic problem in chemistry is to give unique mathematical representations for a set of chemical compounds. The structure of a chemical compound can be represented by a labeled graph whose vertex labels represent the atoms and edge labels specify bond types, respectively [3]. Thus, a graph- theoretic interpretation of this problem is to provide representations for the vertices of a graph in such a way that distinct vertices have distinct representations. This is the subject of the papers [3, 6–8, 14, 15].

Navigation can be studied in a graph-structured framework in which the navigation agent moves from vertex to vertex of a "graph space". The robot can locate itself by the presence of distinctly labeled vertices in the graph space. For a robot navigating in Euclidean space, visual detection of a distinctive vertices provides information about the direction to the vertex, and allows the robot to determine its position by triangulation. Evidently, if the robot knows its distances to a sufficiently large number of vertices, its position on the graph is uniquely determined. This suggests the problem of finding the smallest subset of vertices of a given graph and where they should be located, so that the distances to the vertices uniquely determines the position of robot which actually amounts to a classical problem about metric spaces. A minimum set of vertices which uniquely determines the robot's position is called a "metric basis", and the order of metric basis is called the "metric dimension" of a graph.

In a connected graph G, the distance $d(u, v)$ between two vertices $u, v \in G$ is the length of a shortest path between them. Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The representation $r(v|W)$ of v with respect to W is the $k-$tuple $(d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$. W is called a resolving set if every vertex of G is uniquely identified by a $k-$tuple with respect to W, that is, if all vertices of G have distinct representations with respect to W. A set W is called a minimal resolving set if no proper subset of W is a resolving set. A minimal resolving set of smallest cardinality is called metric basis of G and the one with the largest cardinality is called the upper basis of G. The cardinality of metric basis is called the metric dimension and cardinality of upper basis is called the upper dimension which are denoted by $dim(G)$ and $dim^+(G)$, respectively. Clearly, every set of $n-1$ vertices forms a resolving set and a connected graph with at

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least one edge has dimension at least 1, therefore we have $1 \le dim(G) \le dim^+(G) \le n-1$. The resolving number $res(G)$ of a connected graph G is the minimum k such that every $k-$set W of vertices of G is also a resolving set of G.

The concept of *upper dimension* of graphs was introduced by Chartrand et al. [3], where they defined the upper dimension to be the order of the largest minimal resolving set. Among the important results, they showed that $dim^+(G) = res(G) = n-1$ if and only if $G = K_n$. They also proved that for any positive integer N, there exists a graph G with $res(G) - dim^+(G) \ge N$ and $dim^+(G) - dim(G) \ge N$. Further, they conjectured that for every pair a, b of integers with $2 \le a \le b$, there exists a connected graph G with $dim(G) = a$ and $dim^+(G) = b$. The conjecture was later solved in affirmative by Garijo et al. [6], where they also proved that no integer $a \ge 2$ is realizable as the resolving number of an infinite family of graphs. Recently, the concept of upper dimension of graphs was extended to zero divisor graphs of rings by the authors [11]. For other notations and definitions, we refer the reader to [5, 9, 16].

2. Main results

2.1 Basis sets and upper dimension of simple graphs

We note the following observations.

Lemma 2.1. [Lemma 2.2 [10]] *If G is a connected graph and $D \subseteq V(G)$ is a subset of the distance similar vertices, then every resolving set of G contains exactly $|D|-1$ vertices of D.*

Theorem 2.1. [Theorem 4.5 [13]] *Let G be a connected graph. Then there exists a minimal resolving set of G containing no cut vertices.*

We now determine the upper dimension of some well known graphs. The upper dimension of paths is given in [3]. We give a new proof of this result and the importance of this proof is that it produces all possible basis sets of P_n.

Lemma 2.2. *For a connected graph G of order $n \ge 1$, $dim^+(G) = 1$ if and only if $G \cong P_2$ or P_3 and for $n \ge 4$, $dim^+(P_n) = 2$, where P_n denotes the path on n vertices.*

Proof. First we consider P_2 and P_3. Then any end vertex forms a resolving set. Also, any subset of two vertices contains an end vertex and so any subset of two vertices is not a minimal resolving set. Thus, $dim^+(P_2) = dim^+(P_3) = 1$.

Conversely, let G be a graph with n vertices and $dim^+(G) = 1$. Let the basis set be $W = \{w\}$. Since all vertices have different representations with respect to W, there exists a vertex, say $v \in V(G)$, such that $d(v, w) = n-1$. Consequently, G is an n vertex graph with diameter equal to $n-1$, and thus G is a path $P_n = v_1 v_2 \dots v_i v_{i+1} \dots v_n$.

Now, if $n \ge 4$, we observe that any set of two vertices say $\{v_i, v_{i+1}\}$ not containing the end vertex forms a resolving set, since $r(v_j) = (i-j, i-j+1)$, $1 \le j \le i$ and $r(v_k) = (k-i, k-i-1)$, $k \le i+1 \le n$. Clearly, these representations are distinct, since otherwise $(i-j, i-j+1) = (k-i, k-i-1)$ which gives $k+j = 2i,\ k+j-2 = 2i$, a contradiction. Also, clearly no proper subset of $\{v_i, v_{i+1}\}$ forms a resolving set. Thus, $dim^+(P_n) \ge 2$ for all $n \ge 4$.

Finally, let $W = \{v_i, v_{i+k}\}$, $k \ge 1$ be a two vertex subset of $P_n,\ n \ge 4$. Then $r(v_a) = (i-a, i+k-a)$, for $1 \le a \le i$; $r(v_b) = (b-i, i+k-b)$, for $i < b \le i+k$ and $r(v_c) = (c-i, c-k-i)$, for $i+k < c \le n$.

Clearly, each of these representations are distinct, for if $r(v_a) = r(v_b)$, then $a = b$, a contradiction. If $r(v_a) = r(v_c)$, then $a + c = 2(k+i) = 2i$, which gives $k = 0$, a contradiction. If $r(v_b) = r(v_c)$, then $b = c$, a contradiction. Hence, we conclude that $dim^+(P_n) = 2$ for all $n \ge 4$. □

Lemma 2.3. *A connected graph G of order n has upper dimension equal to $n-1$ if and only if $G \cong K_n$.*

Proof. Since any resolving set contains a minimal resolving set, so $dim(G) \le dim^+(G)$. As $dim(K_n) = n-1$, it follows that $dim^+(K_n) = n-1$.

Now, assume that G is a graph of order n with $dim^+(G) = n-1$, but $G \ncong K_n$. This implies that there is some minimal resolving set $W \subseteq V(G)$ of order $n-1$. Without loss of generality, let $V(G) = \{v_1, v_2, \dots, v_n\}$ and $W = V(G) - \{v_1\}$. We will contradict that W is a minimal resolving set by looking at the following two cases.

Case 1. Suppose $d(v_1, w) = 1$ for all $w \in W$. Since $G \ncong K_n$, there exist $v_i, v_j \in W$ with $d(v_i, v_j) > 1$. Define $W^* = W - \{v_i\}$. To establish that W^* is a resolving set, we need only to show that $r(v_1, W^*) \ne r(v_i, W^*)$, which is indeed the case, since coordinate j of $r(v_1, W^*)$ equals 1 but coordinate j of $r(v_i, W^*)$ is greater than 1.

Case 2. There exists some $v_m \in W$ with $d(v_1, v_m) > 1$. Let there exist some $v_t \in W$ with $d(v_m, v_t) = 1$. Define $W^{**} = W - \{v_t\}$. To establish that W^{**} is a resolving set, we need only to show that $r(v_1, W^{**}) \ne r(v_t, W^{**})$, which is indeed the case, since coordinate m of $r(v_t, W^{**})$ equals 1 but coordinate m of $r(v_1, W^{**})$ is greater than 1.

Thus in each case, W is not a minimal resolving set. Hence, if G is a graph of order n and $G \ncong K_n$, then $dim^+(G) < n-1$. □

Corollary 2.1. [3] *For a finite graph G of order n, $res(G) = n-1$ if and only if $G \cong K_n$.*

Lemma 2.4. *The upper dimension of a cycle C_n is 2, where $n \geq 3$ is a positive integer.*

Proof. Let $C_n = v_1v_2 \dots v_nv_1$ be a cycle of order n. Let $W = \{v_1, v_i\}$, where $v_1 \neq v_i$ and v_1 and v_i are not opposite if n is even. We show that W forms a resolving set for C_n. Assume there exist $v_a, v_b \in V(C_n)$ such that $r(v_a|W) = r(v_b|W)$. Then $d(v_a, v_1) = d(v_b, v_1) = m_1$ and $d(v_a, v_i) = d(v_b, v_i) = m_2$. Without loss of generality, $a < b$ and $m_1 \leq m_2$. We claim that $v_1 - v_2 - \cdots - v_{a-1} - v_a - v_{a+1} - \cdots - v_i - \cdots - v_b - \cdots - v_n$ is a path in C_n with no repeated edges (that is, a path of length n passing through each vertex of C_n only once). Clearly, we can create the path $v_1 - \cdots - v_a - \cdots - v_i$ in C_n without repeating any edges as a path of length $m_1 + m_2$. Note that if v_b lies on the path $v_1 - v_2 - \cdots - v_{a-1}$, then $d(v_b, v_1) < d(v_a, v_1)$. Similarly, if v_b lies on the path $v_a - v_{a+1} - \cdots - v_i$, then $d(v_b, v_i) < d(v_a, v_i)$. Hence, $v_1 - \cdots - v_a - \cdots - v_i - \cdots - v_b$ is a path in C_n not repeating any edges. Clearly, this path can be extended to a path with no repeated edges by adding edges v_b to v_{b+1}, v_{b+1} to v_{b+2}, $\dots$, and v_{n-1} to v_n. By our assumptions, this path must have length $n = 2m_1 + 2m_2$, which is a contradiction if m is odd. If n even, then $\frac{n}{2} = m_1 + m_2 = d(v_1, v_a) + d(v_a, v_i) = d(v_1, v_i)$. However, this is a contradiction, since v_1 and v_i are not opposite. Hence, W is a resolving set. Thus, $dim^+(C_n) = 2$, for all $n \geq 3$. □

Theorem 2.2. *If G is a path or a cycle or a complete graph or bipartite graph (not a path), then $dim^+(G) \geq cl(G) - 1$, where $cl(G)$ denotes the clique number of G.*

Proof. In view of Lemma 2.2, $dim^+(G) = 1 = cl(G) - 1$ if $G = P_2$ or P_3 and $dim^+(G) = 2 > cl(G) - 1$ if $G = P_n,\ n \geq 4$.

By Lemma 2.4, $dim^+(C_n) = 2 \geq cl(G) - 1$ if $G = C_n$ and by Lemma 2.3, $dim^+(G) = n-1 = cl(G) - 1$ if $G = K_n$.

Now, if G is bipartite, by [Theorem 1.9 [9]], G does not contain a triangle so $cl(G) = 2$. Thus, $dim^+(G) \geq 2$ since paths are the only graphs with upper dimension equal to 1.

□

In a graph G, the vertices x and y are *distance similar* if $d(x, z) = d(y, z)$, for every $z \in V(G) - \{x, y\}$. As seen in [12], distance similarity defines an equivalence relation on $V(G)$.

Theorem 2.3. *Let G be a finite connected graph such that every $x \in V(G)$ is distance similar to some vertex $y \neq x$. Then $dim^+(G) = dim(G)$.*

Proof. Assume that $dim^+(G) > dim(G)$. This implies that there exist minimal resolving sets W_1 and W_2 such that $|W_1| > |W_2|$. Let $K_1, K_2, \dots, K_n$ be the partition of G into distance similar classes, (that is, $K_1 \cup K_2 \cup \cdots \cup K_n = V(G)$, $K_i \cap K_j = \emptyset$ if $i \neq j$; if $x, y \in K_i$, then x and y are distance similar, and if $z \in K_i$ and $w \in K_j$ with $i \neq j$, then z and w are not distance similar). By the assumptions of this theorem, $|K_i| \geq 2$ for each i. Since $|W_1| > |W_2|$, there exists some i such that $|W_1 \cap K_i| > |W_2 \cap K_i|$. Thus, either $|W_1 \cap K_i| \neq |K_i| - 1$ or $|W_2 \cap K_i| \neq |K_i| - 1$. However, by Theorem 2.1, $|W \cap K_i| \geq |K_i| - 1$ for any minimal resolving set W. Thus $|W_1 \cap K_i| = |K_i|$, implying $K_i \subseteq W_1$.

Let $x_1 \in K_i$ and let $W^* = W_1 - \{x_1\}$. We will show that W_1 is not a minimal resolving set by showing that W^* is a resolving set. Let $a, b \in V(G) - W_1$. Then $r(a|W_1) \neq r(b|W_1)$, implying there is some $c \in W_1$ with $d(a, c) \neq d(b, c)$. If $c \neq x_1$, then $c \in W^*$ and $r(a|W^*) \neq r(b|W^*)$. If $c = x_1$, then let $v \in K_i$ with $v \neq x_1$. Therefore, v and x_1 are distance similar and $d(a, v) = d(a, x_1) \neq d(b, x_1) = d(b, v)$. Hence, $r(a|W^*) \neq r(b|W^*)$. Finally, if $t \in V(G) - W^*$ with $t \neq x_1$, then t is not distance similar to x_1. Thus, there is some vertex $z \in V(G) - \{t, x_1\}$ such that $d(t, z) \neq d(x_1, z)$. Since $W_1 \cap K_j \neq \emptyset$ for all j, there is some $z^* \in W^*$ such that $z = z^*$ or z is distance similar to z^*. Thus, $d(t, z^*) = d(t, z) \neq d(x_1, z) = d(x_1, z^*)$. Hence, $r(t|W^*) \neq r(x_1|W^*)$.

□

2.2 Upper dimension of Cartesian product of graphs

In this section, we determine the upper dimension of Cartesian product of some graphs and provide their basis sets.

Definition 2.1. The Cartesian product of two graphs G_1 and G_2, denoted by $G = G_1 \times G_2$, is the graph whose vertex set is $V = V(G_1) \times V(G_2)$ and for any two vertices $w_1 = (u_1, v_1)$ and $w_2 = (u_2, v_2)$ in V with $u_1, u_2 \in V(G_1)$ and $v_1, v_2 \in V(G_2)$, there is an edge $w_1, w_2 \in E(G)$ if and only if

(a) either $u_1 = u_2$ and $v_1v_2 \in E(G_2)$ or (b) $v_1 = v_2$ and $u_1u_2 \in E(G_1)$, where $E(G)$, $E(G_1)$, $E(G_2)$ denote the edge sets of G, G_1 and G_2 respectively. Figure 1 illustrates the Cartesian product of $K_{1,3}$ and K_2.

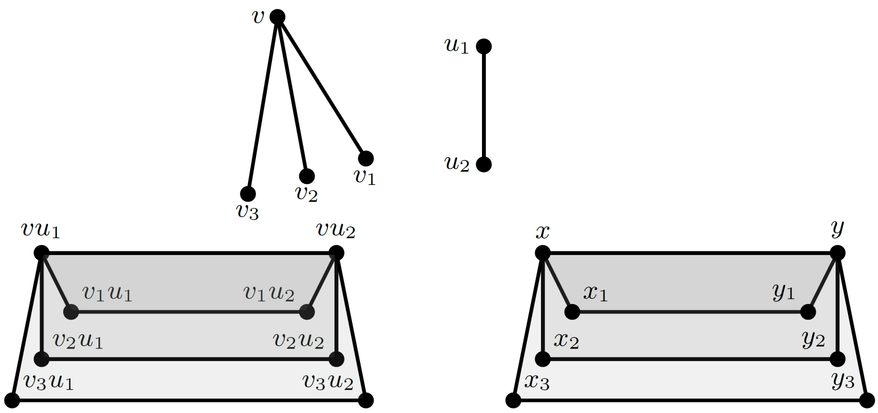

Figure 1: Cartesian product of $K_{1,3}$ and K_2 with names of vertices in sync to the symbols used in Theorem 2.4.

The following result gives the upper dimension for the Cartesian product of a star graph with a complete graph on two vertices.

Theorem 2.4. *For* $n \geq 3$, $dim^+(K_{1,n} \times K_2) = dim^+(K_{1,n}) + 1 = n$.

Proof. Let $V(K_{1,n}) = \{v, v_1, v_2, \ldots, v_n\}$ and $V(K_2) = \{u_1, u_2\}$. Let $G = K_{1,n} \times K_2$, and let the two copies of $K_{1,n}$ in G be denoted by G_1 and G_2 with $V(G_1) = \{x, x_1, x_2, \ldots, x_n\}$ and $V(G_2) = \{y, y_1, y_2, \ldots, y_n\}$. Further, let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. By properly adjusting the vertices and edges of G, the adjacencies are as follows:
$x \sim x_i$, $y \sim y_i$ for all $i = 1, 2, \ldots, n$, $x \sim y$ and $x_i \sim y_j$ if and only if $i = j$.

Let W be a subset of $v(K_{1,n} \times K_2)$ with $|W| \geq n+1$. We will show that W is not a minimal resolving set. First, note that if there are two distinct indices $1 \leq r, s \leq n$ with $\{x_r, y_r\} \cap W = \emptyset$ and $\{x_s, y_s\} \cap W = \emptyset$, then W is not a resolving set as $r(x_r \mid W) = r(x_s \mid W)$. Thus, $n - 1 \leq dim(K_{1,n} \times K_2) \leq dim^+(K_{1,n} \times K_2)$ The proof will proceed in two cases.

Case 1. Suppose that $\{x_i, y_i\} \cap W \neq \emptyset$, for each $1 \leq i \leq n$. For each i, choose $z_i \in \{x_i, y_i\} \cap W$ and define $W^* = \{z_1, z_2, \ldots, z_n\}$. Then $r(x \mid W)$ is the vector whose i^{th} coordinate is 1 or 2 according as $z_i \in X$ or $z_i \in Y$, and $r(y \mid W)$ is the vector whose i^{th} coordinate is 2 or 1 according as $z_i \in X$ or $z_i \in Y$. Also, if $x_k \notin W^*$, $r(x_k \mid W)$ is the vector whose k^{th} coordinate is 1 and whose i^{th} coordinate is 2 or 3 according as $z_i \in X$ or $z_i \in Y$ when $i \neq k$. Similarly, if $y_k \notin W^*$, $r(y_k \mid W)$ is the vector whose k^{th} coordinate is 1 and whose i^{th} coordinate is 3 or 2 according as $z_i \in X$ or $z_i \in Y$ when $i \neq k$. Thus, W^* is a resolving set.

The fact that W^* is a minimal resolving set shows that $dim^+(K_{1,n} \times K_2) \geq n$, when considering either case.

Case 2. (Without loss of generality) Suppose that $\{x_n, y_n\} \cap W = \emptyset$ and $\{x_i, y_i\} \cap W \neq \emptyset$, for each $1 \leq i \leq n-1$. Again, choose $z_i \in \{x_i, y_i\} \cap W$, for $i = 1, \ldots, n-1$. The proof now proceeds into subcases.

Subcase 2.1. Suppose there is some $j \in \{1, \ldots, n-1\}$ such that $\{x_j, y_j\} \subseteq W$. Define $W^* = \{x_j, y_j\} \cup \{z_1, z_2, \ldots, z_{n-1}\}$. Using an analysis similar to that used in Case 1, it is routine to verify that W^* is a resolving set.

Subcase 2.2. Suppose that $|\{x_i, y_i\} \cap W| = 1$, for each $i = 1, \ldots, n-1$. Since $|W| \geq n+1$, we have $W = \{x, y, z_1, z_2, \ldots, z_{n-1}\}$. If $|W \cap X| \geq 2$, define $W_1 = \{x, z_1, \ldots, z_{n-1}\}$. If $|W \cap Y| \geq 2$, define $W_2 = \{y, z_1, \ldots, z_{n-1}\}$. If $|W \cap X| = |W \cap Y| = 1$, then $n = 3$ and (without loss of generality) $W = \{x, y, x_1, y_2\}$ and we define $W_3 = \{x, x_1, y_2\}$. Using an analysis similar to that used in Case 1, it is routine to verify that W_j is a resolving set in each scenario.

Hence, in all cases, W contains a proper subset that is a resolving set. Since we found at least one resolving set consisting of n vertices, $dim^+(K_{1,n} \times K_2) = n$. □

Note. For $n = 1$, $dim^+(K_{1,1} \times K_2) = 2$, given by Lemma 2.4, as $K_{1,1} \times K_2 \cong C_4$. For the case when $n = 2$, consider the set $W = \{x, y, x_1\}$, which is easily verified to be a resolving set, but each proper subset of W is not a resolving set. The sets $\{x, y, x_2\}$, $\{x, y, y_1\}$, $\{x, y, y_2\}$, $\{x_1, x_2, y_1\}$, $\{x_1, x_2, y_2\}$, $\{x_1, y_1, y_2\}$, and $\{x_2, y_1, y_2\}$ are also easily verified to be resolving sets. Further, since $\{x_i, y_i\} \cap W = \emptyset$ for at most one index i for any resolving set W, any subset of $v(K_{1,2} \times K_2)$ with more than 3 elements will contain one of these subsets. Hence, $dim^+(K_{1,2} \times K_2) = 3$.

Corollary 2.2. *For $n \geq 5$, $dim(K_{1,n} \times K_2) = dim^+(K_{1,n}) = n - 1$.*

Proof. Using the notation of Theorem 2.4, define $W = \{x_1, x_2, y_3, y_4, \dots, y_{n-1}\}$. Then $r(x \mid W) = (1, 1, 2, \dots, 2)$, $r(y \mid W) = (2, 2, 1, \dots, 1)$, $r(y_1 \mid W) = (1, 3, 2, \dots, 2)$, $r(y_2 \mid W) = (3, 1, 2, \dots, 2)$, $r(x_n \mid W) = (2, 2, 3, \dots, 3)$, $r(y_n \mid W) = (3, 3, 2, \dots, 2)$, and $r(x_i \mid W) = (1, 1, 2, \dots, 2)$ with 1 in coordinate i if $i \geq 3$. Thus W is a resolving set. As in Theorem 2.4, there can be at most one index $j \in \{1, 2, \dots, n\}$ such that $\{x_j, y_j\} \cap W_0 = \emptyset$ for any resolving set W_0. Hence, W is a minimal resolving set. □

Note. As in the previous note, $dim(K_{1,1} \times K_2) = 2$ as $K_{1,1} \times K_2 \cong C_4$. For $n = 2$, $W = \{x_1, y_1\}$ is easily verified to be a smallest minimal resolving set and thus $dim(K_{1,2} \times K_2) = 2$.

We show that $dim(K_{1,3} \times K_2) = 3$. As seen in Theorem 2.4, $n - 1 \leq dim(K_{1,n} \times K_2)$, since $\{x_i, y_i\} \cap W = \emptyset$ for at most one index i and any resolving set W. Therefore, without loss of generality, due to symmetries among the vertex adjacency relations, the only possible resolving sets of order 2 are $W_1 = \{x_1, y_2\}$ and $W_2 = \{x_1, x_2\}$. However, $r(y \mid W_1) = r(x_2 \mid W_1)$ and $r(y \mid W_2) = r(x_3 \mid W_2)$. Thus, $dim(K_{1,3} \times K_2) \geq 3$. It is easy to see that $W = \{y, x_1, x_2\}$ is a resolving set.

To show that $dim(K_{1,4} \times K_2) = 4$, we use an argument similar to that in the last paragraph to determine the only candidates for resolving sets of order 3 are $W_1 = \{x_1, x_2, x_3\}$ and $W_2 = \{x_1, x_2, y_3\}$. However, $r(y \mid W_1) = r(x_4 \mid W_1)$ and $r(y \mid W_2) = r(x_3 \mid W_2)$. Thus $dim(K_{1,4} \times K_2) \geq 4$. It can be easily verified that $W = \{y, x_1, x_2, x_3\}$ is a resolving set.

Definition 2.2. We call the vertices which are adjacent with at least one pendent vertex *star vertices* and the subset of all such vertices as *star subset*. Also, the number of pendent edges incident on v is called the *star degree* of v, denoted by $sdeg(v)$. Thus, if u is not a star vertex, then $sdeg(u) = 0$. Clearly, if $d(v)$ denotes the degree of a star vertex v, then $sdeg(v) \leq d(v)$ and the equality holds in star graphs. Also, a tree that is not a star graph has at least two star vertices.

Example 2.1. *In Figure 2, the bold vertices u and v are the star vertices. So the set $X = \{u, v\}$ is a star subset.*

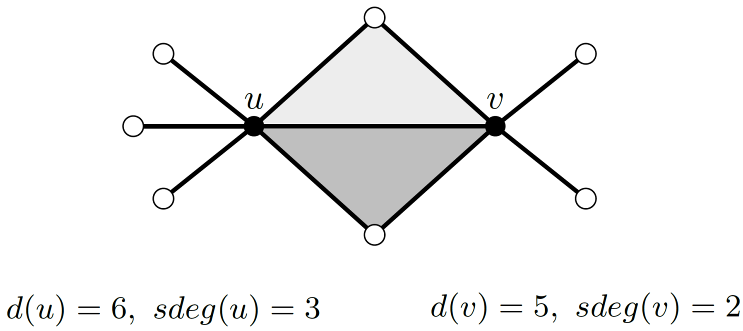

Figure 2: Star vertices and star subset.

Theorem 2.5. *Let G be a graph of order n with vertex set $V(G)$ with a star subset $X = \{x_1, x_2, \dots, x_p\}$ such that the star degree of x_i is $k_i \geq 2$, for all $1 \leq i \leq p$. Then $k - p \leq dim^+(G) \leq n - p$, where $k = k_1 + k_2 + ... + k_p$.*

Proof. For $1 \leq i \leq p$, choose a vertex x_i and let $v_1, v_2, \dots, v_{k_i}$, $k_i > 1$ be pendent vertices incident on x_i. Then v_m is distance similar to v_j for each $1 \leq m \leq k_i$ and $1 \leq j \leq k_i$. Thus, by Lemma 2.1, a subset of at least $k_i - 1$ of the vertices $\{v_1, v_2, \dots, v_{k_i}\}$ is contained in any minimal resolving set. Therefore, any resolving set has cardinality greater than or equal to $k_1 + k_2 + \cdots + k_p - p = k - p$. From Theorem 2.1, there exists a minimal resolving set containing no cut vertices. Thus, there exists a minimal resolving set containing no vertex from the set X. Therefore, it follows that $dim^+(G) \leq n - p$. Hence we conclude that $k - p \leq dim^+(G) \leq n - p$. □

Theorem 2.6. *Let T be a tree having no vertex of star degree 1. Then $dim(T) = dim^+(T) = k - p$, where k is the sum of star degrees of all star vertices and p is the number of star vertices.*

Proof. Since the pendent vertices incident on a given vertex are distance similar, therefore by Lemma 2.1, for each star vertex, we can skip at most one pendent vertex incident on it. Let x be a star vertex and $\{v_1, \dots, v_m\}$ be the set of pendant vertices adjacent to x, and W_0 be a resolving set with $\{v_1, \dots, v_m\} \subseteq W_0$. Let $W_1 = W_0 - \{v_1\}$. If $a, b \notin W_1$, then $r(a|W_0) \neq r(b|W_0)$ implies that $r(a|W_1) \neq r(b|W_1)$ as the only difference between these vectors is coordinate v_1, which can be identical to coordinates $v_2, \ldots, v_m$. If T is a star graph, then it is clear W_1 is a resolving set. Suppose $c \notin W_0$ and T is not a star graph. If $c = x$, $r(c|W_1) \neq r(v_1|W_1)$ as $d(x, v_2) = 1$ and $d(v_1, v_2) = 2$. If $c \neq x$ and $d(c, v_2) \neq d(v_1, v_2)$, then $r(c|W_1) \neq r(v_1|W_1)$. If $c \neq x$ and $d(c, v_2) = d(v_1, v_2) = 2$, then since T is not a star graph, there is some other star vertex y of T with $x - c - \cdots - y$ a path in T (including the possibility that $c = y$). Further, there is at least one pendant vertex w adjacent to y with $w \in W_1$. Since T is a tree and thus there is one and only one path between any pair of vertices, this implies that $d(v_1, w) = 1 + d(x, w) \geq 2 + d(c, w)$. However, this implies that $d(v_1, w) \neq d(c, w)$ and thus $r(c|W_1) \neq r(v_1|W_1)$. Hence, W_1 is a resolving set. Therefore, any minimal resolving set will only contain exactly $m-1$ of the m pendant vertices adjacent to any vertex x with star degree m.

Let W be the set consisting of all the pendant vertices of T. We will show that W is a resolving set. The result holds trivially if T is a star graph. So assume that T is not a star graph. Given any two distinct vertices $u, v \in v(T) - W$, there is a star vertex x such that $u - v - \cdots - x$ is a path in T (allowing the possibility that $v = x$) since the induced subgraph on $v(T) - W$ is still a connected tree. Let w be a pendant vertex adjacent to x. Then, since there is one and only one path from u to w in T, $d(u, w) = d(u, v) + d(v, w) > d(v, w)$. Hence, $r(u|W) \neq r(v|W)$.

Now, for each star vertex y of T, remove one pendant vertex adjacent to y from W to create a new set W^*. Then $|W^*| = k - p$. An argument similar to that in the first paragraph of the proof will show that W^* is a minimal resolving set of T. Since Theorem 2.1 implies that any resolving set of T contains a subset analogous to W^* (that is, containing all but one of the pendant vertices adjacent to each star vertex), we have $dim(T) = dim^+(T) = k - p$.

□

Example 2.2. Figure 3 (A) gives an example of a tree in which we construct its basis using Theorem 2.6. The bold vertices form a basis and the representations are given by taking the bold vertices in order of their numbering. Also, note that P_n is a tree, but it has star vertices of degree 1. If $n \geq 4$, then $dim(P_n) = 1$, but $dim^+(P_n) = 2$.

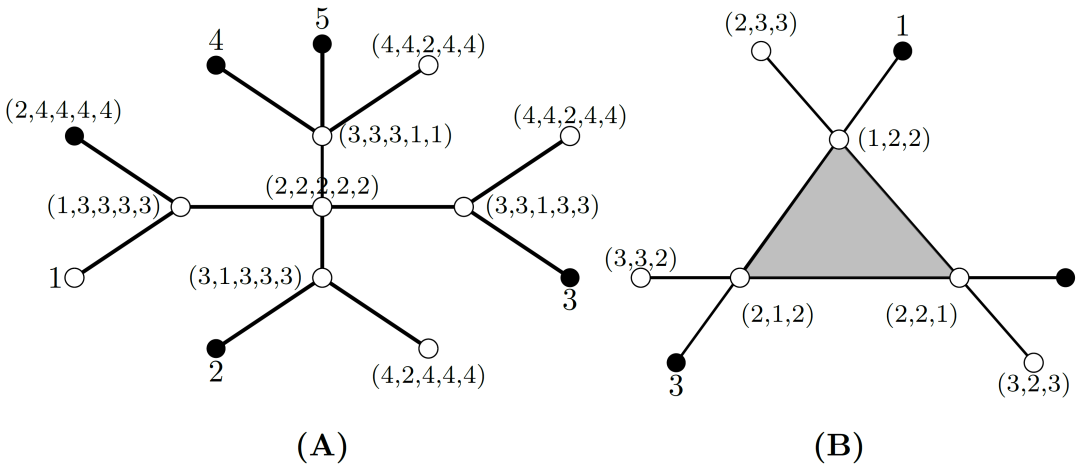

Figure 3: Graphs used in Example 2.2. The graphs given here are the examples for the bounds given above to be best possible. The upper bound holds in the graph (A) and lower bound holds in graph (B). Notice that in graph (A), $k = 9$, $p = 2$ and upper dimension is $k - 2p = 9 - 4 = 5$.

Acknowledgments

The authors are grateful to the anonymous referees for their useful suggestions which improved the presentation of the paper. This research is supported by Council of Scientific and Industrial Research (CSIR), New Delhi, India.

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