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Recurrences of Stirling and Lah numbers via second kind Bell polynomials*

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Abstract

In the paper, by virtue of several explicit formulas for special values and a recurrence of the Bell polynomials of the second kind, the authors derive several recurrences for the Stirling numbers of the first and second kinds, for 1-associate Stirling numbers of the second kind, for the Lah numbers, and for the binomial coefficients.

Keywords: recurrence; Bell polynomial of the second kind; Stirling number; 1-associate Stirling number of the second kind; Lah number.

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1. Motivations

There are two sequences of mathematical notions named in honor of Eric Temple Bell in combinatorial mathematics. The first sequence include the Bell numbers and polynomials, denoted by B_n and $B_n(x)$ respectively. The numbers B_n are also known as the exponential numbers, while the polynomial $B_n(x)$ are also called the exponential polynomials as well as the Touchard polynomials. They can be generated by

$$e^{e^t-1}=\sum_{k=0}^{\infty}\frac{B_n}{n!}t^n \quad\text{and}\quad e^{x(e^t-1)}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}.$$

These numbers and polynomials can be combinatorially interpreted, see [9, 28, 29], for example. In the paper [24], these numbers and polynomials have been generalized to multivariate exponential polynomials, denoted by $Q_{m,n}(\boldsymbol{x}_m)$, by

$$\exp(x_1[\exp(x_2[\exp(\cdots x_{m-1}[\exp(x_m[\exp(t)-1])-1]\cdots)-1])-1])=\sum_{n=0}^{\infty}Q_{m,n}(\boldsymbol{x}_m)\frac{t^n}{n!}.$$

For some recent work by the first author and his coauthors on the Bell numbers and polynomials, please refer to the papers [4, 10, 14, 16, 18–20, 22, 26] and closely related references therein.

Relative to multivariate exponential polynomials, a new notion, multivariate logarithmic polynomials, denoted by $L_{m,n}(\boldsymbol{x}_m)$, was introduced as

$$\ln(1+x_1\ln(1+x_2\ln(1+\cdots+x_{m-1}\ln(1+x_m\ln(1+t))\cdots)))=\sum_{n=0}^{\infty}L_{m,n}(\boldsymbol{x}_m)\frac{t^n}{n!}$$

and was investigated in [12, 13].

The study of combinatorial identities constitutes an extremely vast field of research in which, for a long time, it was believed that it was impossible to create a classification, see the monograph [28]. The article [2] can be considered as one of the interesting studies in this area. On the other hand, the (multivariate) Bell numbers and polynomials not only are important in number theory and combinatorics, but also can be applied in other fields [15, 24].

The second sequence of notions named after Eric Temple Bell are mainly related to the Bell polynomials of the second kind. These polynomials are also known as partial (exponential) Bell polynomials, incomplete (exponential) Bell polynomials, and complete (exponential) Bell polynomials. The partial Bell polynomials, or say, the Bell polynomials of the second

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kind, can be defined by

$$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})=\sum_{\substack{1\le i\le n-k+1\\ \ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n-k+1} i\ell_i=n\\ \sum_{i=1}^{n-k+1}\ell_i=k}}\frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i},$$

and can be generated by

$$\frac{1}{k!}\left(\sum_{i=1}^{\infty}x_i\frac{t^i}{i!}\right)^k=\sum_{n=k}^{\infty}\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})\frac{t^n}{n!},\quad k\ge 0, \tag{1}$$

where x_j for $j\ge 1$ are real or complex independent variables. See [1, Definition 11.2] and [3, p. 134, Theorem A]. One of the most important and classical applications of $\mathrm{B}_{n,k}$ is the famous Faà di Bruno formula which states that the n-th derivative of the composite function $f(h(t))$ can be computed in terms of $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ by

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}f\circ h(t)=\sum_{k=0}^{n}f^{(k)}(h(t))\,\mathrm{B}_{n,k}\big(h'(t),h''(t),\dots,h^{(n-k+1)}(t)\big), \tag{2}$$

where f and h are both n-time differentiable functions. See [1, Theorem 11.4] and [3, p. 139, Theorem C]. The sum

$$\mathrm{B}_n(x_1,x_2,\dots,x_n)=\sum_{k=1}^{n}\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$$

is called the n-th complete (exponential) Bell polynomials which can be generated by

$$\exp\left(\sum_{i=1}^{\infty}x_i\frac{t^i}{i!}\right)=\sum_{n=0}^{\infty}\mathrm{B}_n(x_1,x_2,\dots,x_n)\frac{t^n}{n!}.$$

The Bell polynomials of the second kind $\mathrm{B}_{n,k}$ satisfy the recurrence relation

$$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})=\sum_{\ell=1}^{n-k+1}\binom{n-1}{\ell-1}x_\ell\,\mathrm{B}_{n-\ell,k-1}(x_1,x_2,\dots,x_{n-\ell-k+2}), \tag{3}$$

where $\mathrm{B}_{0,0}=1$, $\mathrm{B}_{n,0}=0$ for $n\ge 1$, and $\mathrm{B}_{0,k}=0$ for $k\ge 1$. See [8, Proposition 1].

In recent articles [17, 21, 23, 25, 27], explicit formulas for special values of the Bell polynomials of the second kind have been constructed in correspondence to numeric values of the variables. Since the recurrence formula (3) has been highlighted in the paper [9], the combined use of some formulas for special values of the Bell polynomials of the second kind in [25] and the recurrence relation (3) allows us to construct several combinatorial identities.

We can not be sure that some of these identities can be found in the *mare magnum* of literature, but we just want to highlight the method by which these identities have been found, a method which permits an extension to a wide class of relationships.

2. An alternative formula and an open problem

In analytic combinatorics [1,3], the Stirling numbers of the second kind, denoted by $S(n,k)$ for $n\ge k\ge 0$, can be computed by

$$S(n,k)=\frac{1}{k!}\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}\ell^n$$

and can be generated by

$$\frac{(e^x-1)^k}{k!}=\sum_{n=k}^{\infty}S(n,k)\frac{x^n}{n!}.$$

In [7, p. 303, eq. (1.2)], the r-associate Stirling numbers of the second kind, denoted by $S(n,k;r)$, were defined by

$$\left(e^x-\sum_{i=0}^{r}\frac{x^i}{i!}\right)^k=\left(\sum_{j=r+1}^{\infty}\frac{x^j}{j!}\right)^k=k!\sum_{n=(r+1)k}^{\infty}S(n,k;r)\frac{x^n}{n!}.$$

In [6, p. 978, eq. (2.3)], the formula

$$\mathrm{B}_{n,k}(0,0,3,4,\dots,x_{n-k+1})=\frac{n!}{(n-k)!}S(n-k,k;1) \tag{4}$$

was claimed.

We now give an alternative formula for $\mathrm{B}_{n,k}(0,0,3,4,\dots,x_{n-k+1})$ in term of the Stirling numbers of the second kind $S(n,k)$.

Theorem 2.1. *For $n \ge k+2 \ge 2$, the Bell polynomials of the second kind satisfy*

$$\mathrm{B}_{n,k}(0,0,3,4,\dots,x_{n-k+1}) = n!\sum_{\ell=0}^{k}\frac{(-1)^\ell}{\ell!}\frac{S(n-k-\ell,k-\ell)}{(n-k-\ell)!}. \tag{5}$$

Proof. In [5, Theorem 1], the formula

$$\mathrm{B}_{n,k}(0,1,\dots,1) = \sum_{\ell=0}^{k}(-1)^\ell\binom{n}{\ell}S(n-\ell,k-\ell) \tag{6}$$

was established. See also [25, Section 1.8].

Applying $x_1 = x_2 = 0$ and $x_k = k$ for $k \ge 3$ to the formula (1) yields

$$\sum_{n=k+2}^{\infty}\mathrm{B}_{n,k}(0,0,3,4,\dots,x_{n-k+1})\frac{t^n}{n!} = \frac{1}{k!}\left[\sum_{i=3}^{\infty}\frac{t^i}{(i-1)!}\right]^k = \frac{1}{k!}[(e^t-t-1)t]^k.$$

This means that, for $n \ge k+2 \ge 2$, by the Leibniz theorem for differentiation of a product, by the Faà di Bruno formula (2), and by (6), we have

$$\begin{aligned}
\mathrm{B}_{n,k}(0,0,3,4,\dots,x_{n-k+1}) &= \frac{1}{k!}\lim_{t\to0}\frac{\mathrm{d}^n}{\mathrm{d}t^n}[(e^t-t-1)t]^k \\
&= \frac{1}{k!}\lim_{t\to0}\sum_{\ell=0}^{n}\binom{n}{\ell}\left[(e^t-t-1)^k\right]^{(\ell)}\left(t^k\right)^{(n-\ell)} \\
&= \frac{1}{k!}\lim_{t\to0}\sum_{\ell=0}^{n}\binom{n}{\ell}\left[(e^t-t-1)^k\right]^{(\ell)}\langle k\rangle_{n-\ell}t^{k-n+\ell} \\
&= \frac{1}{k!}\binom{n}{n-k}\langle k\rangle_k\lim_{t\to0}\left(\left[(e^t-t-1)^k\right]^{(n-k)}\right) \\
&= \binom{n}{n-k}\lim_{t\to0}\sum_{\ell=0}^{n-k}\left(u^k\right)^{(\ell)}\mathrm{B}_{n-k,\ell}\left(e^t-1,e^t,\dots,e^t\right) \\
&= \binom{n}{n-k}\lim_{t\to0}\sum_{\ell=0}^{n-k}\langle k\rangle_\ell u^{k-\ell}\,\mathrm{B}_{n-k,\ell}\left(e^t-1,e^t,\dots,e^t\right) \\
&= \binom{n}{n-k}\lim_{t\to0}\sum_{\ell=0}^{n-k}\langle k\rangle_\ell (e^t-t-1)^{k-\ell}\,\mathrm{B}_{n-k,\ell}\left(e^t-1,e^t,\dots,e^t\right) \\
&= \binom{n}{n-k}\langle k\rangle_k\lim_{t\to0}\mathrm{B}_{n-k,k}\left(e^t-1,e^t,\dots,e^t\right) \\
&= \binom{n}{n-k}k!\,\mathrm{B}_{n-k,k}(0,1,\dots,1) \\
&= \frac{n!}{(n-k)!}\sum_{\ell=0}^{k}(-1)^\ell\binom{n-k}{\ell}S(n-k-\ell,k-\ell),
\end{aligned}$$

where $u = u(t) = e^t - t - 1$. The formula (5) is thus proved. The proof of Theorem 2.1 is complete. □

Remark 2.1. *In [6, Section 2, eq. (2.2)], the closed formula*

$$\mathrm{B}_{n,k}(0,2,3,\dots,n-k+1) = \frac{n!}{(n-k)!}S(n-k,k) \tag{7}$$

was listed. Basing on the formulas (5) and (7), we pose an open problem: what is the general and explicit formula of

$$\mathrm{B}_{n,k}(\underbrace{0,\dots,0}_{s-1},s,s+1,\dots,n-k+1)$$

for $n \ge k+s-1 \ge s-1 \ge 3$ in terms of the Stirling numbers of the second kind $S(n,k)$?

3. Recurrences of Stirling and Lah numbers

In analytic combinatorics [1,3], the Stirling numbers of the first kind, denoted by $s(n,k)$, can be computed by

$$s(n,k) = (-1)^{n+k}(n-1)!\sum_{\ell_1=1}^{n-1}\frac{1}{\ell_1}\sum_{\ell_2=1}^{\ell_1-1}\frac{1}{\ell_2}\cdots\sum_{\ell_{k-2}=1}^{\ell_{k-3}-1}\frac{1}{\ell_{k-2}}\sum_{\ell_{k-1}=1}^{\ell_{k-2}-1}\frac{1}{\ell_{k-1}}$$

for $n\ge k\ge 1$, see [11], and can be generated by

$$\frac{[\ln(1+x)]^k}{k!}=\sum_{n=k}^{\infty}s(n,k)\frac{x^n}{n!}$$

for $n\ge k\ge 0$. The Lah numbers $L(n,k)$ for $n\ge k\ge 0$ can be computed by

$$L(n,k)=\binom{n-1}{k-1}\frac{n!}{k!}$$

and can also be generated [10] by

$$\left(e^{\pm1/x}\right)^{(n)}=(-1)^n e^{\pm1/x}\sum_{k=1}^{n}(\pm1)^k L(n,k)\frac{1}{x^{n+k}}.$$

Now we start out to derive several recurrences of the Stirling numbers of the first and second kinds $s(n,k)$ and $S(n,k)$, of 1-associate Stirling numbers of the second kind $S(n,k;1)$, and of the Lah numbers $L(n,k)$.

Theorem 3.1. *Let $n\ge k\ge 1$. Then*

1. *the Stirling numbers of the first kind $s(n,k)$ satisfy*

$$s(n,k)=(n-1)!\sum_{\ell=0}^{n-k}\frac{(-1)^\ell}{(n-\ell-1)!}s(n-\ell-1,k-1); \tag{8}$$

2. *the Stirling numbers of the second kind $S(n,k)$ satisfy*

$$S(n,k)=\sum_{\ell=0}^{n-k}\binom{n-1}{\ell}S(n-\ell-1,k-1) \tag{9}$$

and

$$S(n,k)=\frac{1}{n+k}\sum_{\ell=1}^{n}\binom{n}{\ell}(\ell+1)S(n-\ell,k-1); \tag{10}$$

3. *the 1-associate Stirling numbers of the second kind $S(n,k;1)$ satisfy*

$$S(n,k;1)=\frac{1}{n+k}\sum_{\ell=1}^{n+1}\ell\binom{n}{\ell-1}S(n-\ell+1,k-1;1); \tag{11}$$

4. *the Lah numbers $L(n,k)$ satisfy*

$$L(n,k)=\sum_{\ell=1}^{n-k+1}\ell!\binom{n-1}{\ell-1}L(n-\ell,k-1); \tag{12}$$

5. *for $n\ge k\ge 2$,*

$$\sum_{\ell=0}^{n}\binom{n}{\ell}\frac{\ell+1}{(k-1)^\ell}=\frac{(n+k)k^{n-1}}{(k-1)^n}. \tag{13}$$

Proof. In [3, p. 135], see also [25, Section 1.2], we find

$$\mathrm{B}_{n,k}(0!,1!,2!,\dots,(n-k)!)=(-1)^{n-k}s(n,k).$$

This means that

$$\mathrm{B}_{n-\ell-1,k-1}(0!,1!,2!,\dots,(n-\ell-k)!)=(-1)^{n-\ell-k}s(n-\ell-1,k-1).$$

Further utilizing the recurrence (3), we arrive at

$$(-1)^{n-k}s(n,k)=\sum_{\ell=0}^{n-k}\binom{n-1}{\ell}(-1)^{n-\ell-k}s(n-\ell-1,k-1)\ell!$$

which is equivalent to (8).

In [3, p. 135], it is given that

$$\mathrm{B}_{n,k}(1,1,\dots,1)=S(n,k).$$

See also [25, Section 1.1]. By applying the recurrence (3), we immediately find the recurrence (9).

In [6] and [25, Section 1.12], we find

$$\mathrm{B}_{n,k}(0,2,3,\dots,n-k+1)=\frac{n!}{(n-k)!}S(n-k,k).$$

This means that

$$\mathrm{B}_{n-\ell-1,k-1}(0,2,3,\dots,n-\ell-k+1)=\frac{(n-\ell-1)!}{(n-\ell-k)!}S(n-\ell-k,k-1).$$

Combining this with the recurrence (3), we can reveal that

$$\frac{n!}{(n-k)!}S(n-k,k)=\sum_{\ell=1}^{n-k}\binom{n-1}{\ell}\frac{(n-\ell-1)!}{(n-\ell-k)!}S(n-\ell-k,k-1)(\ell+1)$$

which can be further rearranged as

$$\frac{n!}{(n-k)!}S(n-k,k)=\sum_{\ell=1}^{n-k}\frac{(n-1)!(\ell+1)}{\ell!(n-k-\ell)!}S(n-\ell-k,k-1).$$

Therefore, we acquire

$$S(n-k,k)=\sum_{\ell=1}^{n-k}\binom{n-k}{\ell}\frac{\ell+1}{n}S(n-\ell-k,k-1).$$

The recurrence (10) is thus proved.

From (4), it follows that

$$\mathrm{B}_{n-\ell,k-1}(0,0,3,4,\dots,n-\ell-k+2)=\frac{(n-\ell)!}{(n-\ell-k+1)!}S(n-\ell-k+1,k-1;1).$$

Substituting this into the recurrence (3), we obtain

$$\frac{n!}{(n-k)!}S(n-k,k;1)=\sum_{\ell=1}^{n-k+1}\binom{n-1}{\ell-1}\ell\frac{(n-\ell)!}{(n-\ell-k+1)!}S(n-\ell-k+1,k-1;1)$$

which can be reformulated as (11).

In [25, Section 1.3], we find

$$\mathrm{B}_{n,k}(1!,2!,3!,\dots,(n-k+1)!)=L(n,k).$$

This mens that

$$\mathrm{B}_{n-\ell-1,k-1}(1!,2!,3!,\dots,(n-k-\ell+1)!)=L(n-\ell-1,k-1).$$

Substituting this into the recurrence (3), after simplification, we obtain the recurrence (12).

In [1, p. 451], [3, p. 135], and [25, Section 1.10], we find

$$\mathrm{B}_{n,k}(1,2,3,\dots,n-k+1)=\binom{n}{k}k^{n-k},\quad n\ge k\ge 1.$$

This means that

$$\mathrm{B}_{n-\ell-1,k-1}(1,2,3,\dots,n-\ell-k+1)=\binom{n-\ell-1}{k-1}(k-1)^{n-k-\ell}.$$

Substituting this into the recurrence (3), we obtain

$$\binom{n}{k}k^{n-k}=\sum_{\ell=0}^{n-k}(\ell+1)\binom{n-1}{\ell}\binom{n-\ell-1}{k-1}(k-1)^{n-k-\ell}$$

which can be reformulated as (13). The proof of Theorem 3.1 is complete. □

Remark 3.1. *The recurrence (12) can be reformulated as a combinatorial identity*

$$\sum_{\ell=0}^{n-k}(\ell+1)\binom{n-\ell-2}{k-2}=\binom{n}{k}.$$

4. Conclusion

In this paper, by virtue of several explicit formulas for special values and a recurrence of the Bell polynomials of the second kind $\mathrm{B}_{n,k}$, we derived several recurrences for the Stirling numbers of the first kind $s(n,k)$, for the Stirling numbers of the second kind $S(n,k)$, for 1-associate Stirling numbers of the second kind $S(n,k;1)$, for the Lah numbers $L(n,k)$, and for binomial coefficients $\binom{n}{k}$. The techniques used in this paper can be further applied to every special values for the Bell polynomials of the second kind $\mathrm{B}_{n,k}$. However, sometimes the derived results are quite complicate to evaluate.

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